Cartan pairs of Lie algebras*

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Abstract. *In this paper some author's results on the structure of certain pairs of Lie algebras are presented. For such pairs some interesting information on the relationship between representations, between coadjoint orbits and between completely prime primitive ideals in corresponding enveloping algebras can be obtained.*

Key words: *Lie algebra, Cartan subalgebra, symmetric pair, nonsymmetric pair, Cartan subspace, Cartan pair*

Sažetak. **Cartanovi parovi Liejevih algebri**. *U članku su prezentirani autorovi rezultati o strukturi nekih parova Liejevih algebri za koje se mogu dosta dobro proučavati veze medju reprezentacijama, medju koadjungiranim orbitama i medju potpuno prostim primitivnim idealima u pripadnim omotačkim algebrama.*

Ključne riječi: *Liejeva algebra, Cartanova podalgebra, simetričan par, nesimetričan par, Cartanov potprostor, Cartanov par*

1. Introduction

Let $\mathfrak{g}$ be a finite dimensional Lie algebra defined over a field k, $\mathrm{char}(k) = 0$. A pair $(\mathfrak{g}, \vartheta)$, where ϑ is an involutive automorphism of $\mathfrak{g}$, is called a symmetric Lie algebra. Denote

$$\mathfrak{g}_1 := \{x \in \mathfrak{g} : \vartheta x = x\} \quad \text{and} \quad \mathfrak{p} := \{x \in \mathfrak{g} : \vartheta x = -x\}.$$

Then $\mathfrak{g}_1$ is a Lie subalgebra of $\mathfrak{g}$, $[\mathfrak{g}_1, \mathfrak{p}] \subseteq \mathfrak{p}$, and

$$\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{p} \tag{1}$$

is a Killing-orthogonal decomposition. Moreover,

$$[\mathfrak{p}, \mathfrak{p}] \subseteq \mathfrak{g}_1. \tag{2}$$

Now we introduce the following terminology. A pair $(\mathfrak{g}, \mathfrak{g}_1)$ of Lie algebras is called *symmetric* if there exists an involutive automorphism ϑ of $\mathfrak{g}$ such that $\mathfrak{g}_1$ equals

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the $+1$-eigenspace of ϑ. Otherwise, $(\mathfrak{g}, \mathfrak{g}_1)$ is called *nonsymmetric*. A pair $(\mathfrak{g}, \mathfrak{g}_1)$ is called *semisimple* (resp. symmetric, resp. nonsymmetric) if $\mathfrak{g}$ is a semisimple Lie algebra. It is well known that semisimple symmetric pairs naturally occur in representation theory of semisimple (and reductive) Lie groups and Lie algebras. In particular, one is usually interested in the relationship between various objects which are closely related to $\mathfrak{g}$ and to $\mathfrak{g}_1$, respectively; e.g. $\mathfrak{g}$-modules and $\mathfrak{g}_1$-modules, completely prime primitive ideals in the enveloping algebras $\mathcal{U}(\mathfrak{g})$ and $\mathcal{U}(\mathfrak{g}_1)$, coadjoint orbits on the dual spaces $\mathfrak{g}^*$ and $\mathfrak{g}_1^*$, etc... Some interesting related results can be found in the papers [1], [3], [4], [5]. For some new ideas and techniques refer to [8].

In accordance with what we have said, it is natural to ask for possible nonsymmetric pairs of Lie algebras for which any useful information on the above mentioned relationships could be obtained.

2. On the structure of some pairs of Lie algebras

In this section we present some of the author's results from [6], [7]. (Let it be said that the starting point of our research was the paper of Levasseur and Smith [3]. Also, for the further investigation the paper [1] of R. Brylinski and Kostant was of a great importance.) First, we have some more notations. Let $\overline{k}$ denote the algebraic closure of k. Generally, if V is a k-vector space, then set $\overline{V} := V \otimes_k \overline{k}$. We say that a Lie algebra $\mathfrak{g}$ is *absolutely simple* if $\overline{\mathfrak{g}}$ is simple. Denote $\mathfrak{g}^\circ(x) := \bigcup_{n\geq 0} \ker(\mathrm{ad}x)^n$, the Fitting zero-component of an operator $\mathrm{ad}x \in L(\mathfrak{g})$. If $\mathfrak{h}$ is a subalgebra of $\mathfrak{g}$ and A a subset of $\mathfrak{g}$, then $C_{\mathfrak{h}}(A)$ denotes the centralizer of A in $\mathfrak{h}$. If $\mathfrak{g}$ is a reductive Lie algebra and $\mathfrak{h}$ a split Cartan subalgebra in $\mathfrak{g}$, $R(\mathfrak{g}, \mathfrak{h})$ denotes the root system of $\mathfrak{g}$ with respect to $\mathfrak{h}$.

Let now $(\mathfrak{g}, \mathfrak{g}_1)$ be a pair of Lie algebras and assume that $\mathfrak{g}_1$ is an absolutely simple subalgebra of $\mathfrak{g}$. By B we denote the Killing form of $\mathfrak{g}$. Also, let $\kappa : \mathfrak{g} \to \mathfrak{g}^*$ (resp. $\kappa_1 : \mathfrak{g}_1 \to \mathfrak{g}_1^*$) be the Killing homomorphism of $\mathfrak{g}$ (resp. $\mathfrak{g}_1$). Note that κ_1 is an isomorphism since $\mathfrak{g}_1$ is simple. Now we define the *homomorphism* π associated to the pair $(\mathfrak{g}, \mathfrak{g}_1)$ as the composition

$$\pi : \mathfrak{g} \longrightarrow \mathfrak{g}_1, \qquad \pi = \kappa_1^{-1} \circ r \circ \kappa,$$

where $r : \mathfrak{g}^* \to \mathfrak{g}_1^*$ denotes the restriction. Then we have the following.

Proposition 1. *The map π is a $\mathfrak{g}_1$-module homomorphism (for the adjoint representation), and for its kernel $\mathfrak{p}$ we have the B-orthogonal sum* (1). *Furthermore, $[\mathfrak{g}_1, \mathfrak{p}] \subseteq \mathfrak{p}$.*

It is also worthy to mention the following simple characterization of symmetric pairs.

Proposition 2. *A pair $(\mathfrak{g}, \mathfrak{g}_1)$ is symmetric if and only if there holds* (2).

With $\mathfrak{g}$, $\mathfrak{g}_1$ and $\mathfrak{p}$ as above we have the following definition which is well known for symmetric pairs (see, e.g., [2], Sect. 1.13).

Definition 1. *A* Cartan subspace *of a pair $(\mathfrak{g}, \mathfrak{g}_1)$ is a commutative subalgebra $\mathfrak{a}$ of $\mathfrak{g}$ which is reductive in $\mathfrak{g}$ and such that $\mathfrak{a} \subseteq \mathfrak{p}$ and $C_{\mathfrak{p}}(\mathfrak{a}) = \mathfrak{a}$.*

If $(\mathfrak{g}, \mathfrak{g}_1)$ is a semisimple symmetric pair and $\mathfrak{a}$ a Cartan subspace, then $\mathfrak{l} \oplus \mathfrak{a}$ is a Cartan subalgebra in $\mathfrak{g}$ for any Cartan subalgebra $\mathfrak{l}$ in $C_{\mathfrak{g}_1}(\mathfrak{a})$. (see, e.g., [2]). This is a very useful result if one wants to study the relationship between $\mathfrak{g}$-modules and $\mathfrak{g}_1$-modules. So, generally, it is interesting to have some information on Cartan subalgebras in $\mathfrak{g}_1$ and Cartan subalgebras in $\mathfrak{g}$ containing them. In the sequel we consider only those pairs $(\mathfrak{g}, \mathfrak{g}_1)$ which satisfy the equivalent conditions in the part (i) of the following first main theorem.

Theorem 1. *Let $\mathfrak{g}$ be a semisimple Lie algebra and $\mathfrak{g}_1$ a subalgebra of $\mathfrak{g}$ which is reductive in $\mathfrak{g}$. Let $\mathfrak{h}_1$ be a Cartan subalgebra in $\mathfrak{g}_1$ and $\mathfrak{c}_1$ (resp. $\mathfrak{c}$) a Cartan subalgebra in $\overline{\mathfrak{g}}_1$ (resp. $\overline{\mathfrak{g}}$) such that $\mathfrak{c}_1 \subseteq \mathfrak{c}$. Then*

(i) *The following two conditions are mutually equivalent:*

  (a) *there exists $x \in \mathfrak{h}_1$ such that $\mathfrak{g}^\circ(x)$ is a Cartan subalgebra in $\mathfrak{g}$ and also $\mathfrak{g}_1^\circ(x) = \mathfrak{h}_1$;*

  (b) *the restriction $\alpha_{|\mathfrak{c}_1}$ of any root $\alpha \in R(\overline{\mathfrak{g}}, \mathfrak{c})$ is nontrivial.*

(ii) *If a pair $(\mathfrak{g}, \mathfrak{g}_1)$ satisfies the equivalent conditions of* (i), *then for any Cartan subalgebra $\mathfrak{h}_1$ in $\mathfrak{g}_1$ there exists a unique Cartan subalgebra $\mathfrak{h}$ in $\mathfrak{g}$ such that $\mathfrak{h}_1 \subseteq \mathfrak{h}$ (cf. [1], Theorem 5.3]).*

(iii) *If $(\mathfrak{g}, \mathfrak{g}_1)$ is a symmetric pair (cf. [2], 1.13.3), then it satisfies the equivalent conditions of* (i).

Remark 1.

- *The assumption of the theorem that $\mathfrak{g}_1$ is reductive in $\mathfrak{g}$ cannot be replaced by saying that $\mathfrak{g}_1$ is a reductive subalgebra of $\mathfrak{g}$.*
- *For $\mathfrak{g}$ semisimple and $\mathfrak{g}_1$ reductive in $\mathfrak{g}$ we can always find Cartan subalgebras $\mathfrak{c}_1$ and $\mathfrak{c}$ as proposed in the theorem (see [7]).*

A pair $(\mathfrak{g}, \mathfrak{g}_1)$ of Lie algebras such that $\mathrm{rk}(\mathfrak{g}_1) = \mathrm{rk}(\mathfrak{g})$ is called *cuspidal*. From now on we consider only so-called *noncuspidal* pairs $(\mathfrak{g}, \mathfrak{g}_1)$; i.e. such that $\mathrm{rk}(\mathfrak{g}_1) < \mathrm{rk}(\mathfrak{g})$. We also assume that $\mathfrak{g}_1$ is absolutely simple and $(\mathfrak{g}, \mathfrak{g}_1)$ satisfies the equivalent conditions of (i) in *Theorem 1.* Then to any Cartan subalgebra $\mathfrak{h}_1$ in $\mathfrak{g}_1$ we associate a unique triple $(\mathfrak{h}_1, \mathfrak{h}, \mathfrak{a})$ where $\mathfrak{h} := \mathfrak{g}^{\mathfrak{h}_1}$ ($\mathfrak{h}_1$-invariants in $\mathfrak{g}$), and

$$\mathfrak{a} := \mathfrak{h} \cap \mathfrak{p}. \tag{3}$$

Note that then we have $\mathfrak{h} = \mathfrak{h}_1 \oplus \mathfrak{a}$. If k is an algebraically closed field, consider the subset R' of $R(\mathfrak{g}, \mathfrak{h})$ consisting of those roots which vanish on $\mathfrak{a}$. Then we introduce the following

Definition 2. *Let $\mathfrak{g}_1$ be the absolutely simple subalgebra of a semisimple Lie algebra $\mathfrak{g}$. Then $(\mathfrak{g}, \mathfrak{g}_1)$ is called a* Cartan pair *if:*

(I) it satisfies the equivalent conditions in (i) of Theorem 1*; and*

(II) *there exists a Cartan subalgebra* $\mathfrak{h}_1$ *of* $\overline{\mathfrak{g}}_1$ *such that the set* R'*, defined for the triple* $(\mathfrak{h}_1, \mathfrak{h}, \mathfrak{a})$ *and the root system* $R(\overline{\mathfrak{g}}, \mathfrak{h})$*, satisfies*

$$\alpha \in R' \Longrightarrow \overline{\mathfrak{g}}_\alpha \subseteq \overline{\mathfrak{g}}_1.$$

The following theorem is our second main result.

Theorem 2.

(i) *The definition of a Cartan pair does not depend on the choice of a Cartan subalgebra* $\mathfrak{h}_1$ *in* $\overline{\mathfrak{g}}_1$ *appearing in the above condition* (II).

(ii) *If* $(\mathfrak{g}, \mathfrak{g}_1)$ *is a Cartan pair, then the set* $\mathfrak{a}$ *defined by* (3) *is a Cartan subspace. Moreover,* $C_{\mathfrak{g}}(\mathfrak{a}) = C_{\mathfrak{g}_1}(\mathfrak{a}) \oplus \mathfrak{a}$.

(iii) *Let* $(\mathfrak{g}, \mathfrak{g}_1)$ *be a semisimple symmetric pair such that* $\mathfrak{g}_1$ *is absolutely simple. Then the following three conditions are mutually equivalent:*

(a) *for any Cartan subalgebra* $\mathfrak{h}_1$ *in* $\overline{\mathfrak{g}}_1$ *the set* $\mathfrak{a}$ *defined by* (3) *is a Cartan subspace of* $\overline{\mathfrak{g}}$*;*

(b) *for any Cartan subspace* $\mathfrak{c}$ *of* $\overline{\mathfrak{g}}$ *the pair* $(\overline{\mathfrak{g}}_1, C_{\overline{\mathfrak{g}}_1}(\mathfrak{c}))$ *is cuspidal;*

(c) $(\mathfrak{g}, \mathfrak{g}_1)$ *is a Cartan pair.*

Remark 2.

- *The notion of a Cartan pair is a covenient generalization of a semisimple symmetric pair. Such pairs have a good behavior on representations (see [7]).*
- *An example of a nonsymmetric Cartan pair and a list of properties which that pair possesses can be found in [6].*

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