On the first passage over the one-sided stochastic boundary*

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Abstract. *We present two methods on how to compute the distribution of an Itô diffusion at the first moment it becomes smaller than a function of its current maximum.*

Key words: *Itô diffusion, the first-passage time, one-sided stochastic boundary*

Sažetak. **O vremenu prvog prelaska preko jednostrane stohastičke granice.** *Pokazane su dvije metode kako izračunati distribuciju Itôve difuzije u prvom trenutku kada postane manja od funkcije tekućeg maksimuma.*

Ključne riječi: *Itôva difuzija, vrijeme prvog prelaska, jednostrana stohastička granica*

Let X_t denote the price of a share of a certain stock, and let S_t be the maximal price of that stock by the time $t \geq 0$. There is some interest in computing the distribution of the price at the first moment it becomes smaller than a function of the current maximum. Typical examples are the distribution of the price when it falls a units below the current maximum, or when it falls to a certain fraction of the current maximum.

A common model for stock prices (see, e.g. [2]) is a process that solves the stochastic differential equation

$$dX_t = \mu(X_t)dt + \sigma(X_t)dW_t, \quad X_0 = x, \tag{1}$$

where $W = (W_t; t \geq 0)$ is a standard one-dimensional Brownian motion, and $\sigma : \mathbb{R} \to (0, \infty)$, $\mu : \mathbb{R} \to \mathbb{R}$ are Lipschitz continuous functions. The process $X = (X_t; t \geq 0)$ is usually called an Itô diffusion. There exists a function s, the scale function of X, such that $s(X_t)$ is a local martingale. Explicitly,

$$s(u) = \int_c^x \exp\left(-\int_c^y \frac{2\mu(z)}{\sigma^2(z)}dz\right)dy.$$

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Let $S_t = \max_{0 \le r \le t} X_r$ be the maximum of X by the time t, and let $g : [0, \infty) \to \mathbb{R}$ be a monotone function. Let us introduce the stopping time

$$\tau = \inf\{t \ge 0; X_t \le g(S_t)\}.$$

Then τ is the first time that X becomes less than a function of its current maximum. The random variable S_τ is the maximum of X at time τ.

Theorem 1. *The distribution function F_{S_τ} of the random variable S_τ is given by*

$$F_{S_\tau}(u) = 1 - \exp\left(- \int_x^u \frac{ds(t)}{s(t) - s(g(t))} \right), \quad u \ge x. \tag{2}$$

Distribution of X_τ is easily obtained from F_{S_τ} and the fact that $X_\tau = g(S_\tau)$.

In order to prove the theorem one first assumes that X is a Brownian motion ($\mu = 0$, $\sigma = 1$).

There are two different approaches in calculating the distribution of S_τ in case X is a Brownian motion. The first approach relies on the first-order calculus for semimartingales ([1], [4]) and was in the same context exploited in [5]. The main ingredient is the fact that for a C^1-function H, the process $H(S_t) - (S_t - X_t)H'(S_t)$ is a martingale. The optional stopping theorem gives that

$$E[H(S_\tau)] = E[(S_\tau - X_\tau)H'(S_\tau)] \tag{3}$$

with $H(t) = \int_0^t h(u)du$, h a continuous, nonnegative function with compact support. Simple calculations imply that

$$E[H(S_\tau)] = \int_0^\infty (1 - F_{S_\tau}(u))h(u)du, \tag{4}$$

$$E[H(S_\tau - X_\tau)H'(S_\tau)] = \int_0^\infty (u - g(u))h(u)dF_{S_\tau}(u). \tag{5}$$

From (3),(4) and (5), one obtains the differential equation for F_{S_τ}: $(1 - F_{S_\tau}(u))du = (u - g(u))dF_{S_\tau}(u)$. It is easily seen that F_{S_τ} given in (2) (with $s(r) = r$, $x = 0$) solves this equation.

The second approach to calculate F_{S_τ} relies on the excursion theory, and can also be applied to Lévy processes with no positive jumps. For $u \ge 0$, let

$$T(u) = \inf\{t \ge 0; S_t > u\} = \inf\{t \ge 0; X_t > u\}.$$

Then $\{S_\tau > u\} = \{T(u) < \tau\}$. Hence, it suffices to compute $P(T(u) < \tau)$. The first passage time process $\{T(t); t \ge 0\}$ is an increasing Lévy process. For $t > 0$ such that $T(t-) < T(t)$, let $h_t = \sup\{(S - X)_{T(t-)+s}; 0 \le s < T(t) - T(t-)\}$ be the height of the excursion of the reflected process $S - X$ at the local time t. Then $\{(t, h_t); t > 0\}$ is a Poisson point process with characteristic measure $dt \times d\nu$, where $\nu(t, \infty) = 1/t$ (see, for example, [3]). The key observation in this approach (see [6]) is that $T(u) < \tau$ if and only if $H_t < t - g(t)$, for all $t \in [0, u]$. Let $\Lambda = \{(t, y) : y \ge t - g(t)\}$, and let

$$N_u^\Lambda = \sum_{0 < t \le u} 1_\Lambda(t, h_t)$$

be the number of points in Λ up to the (local) time u. Then $\{N_u^\Lambda = 0\} = \{h_t < t - g(t), \forall t \in [0,u]\}$. But N_u^Λ is a Poisson random variable with parameter $dt \times d\nu(\Lambda \cap [0,u] \times (0,\infty)) = \int_0^u \nu(t-g(t),\infty)dt$. Therefore,

$$P(N_u^\Lambda = 0) = \exp\left(-\int_0^u \nu(t-g(t),\infty)dt\right) = \exp\left(-\int_0^u \frac{dt}{t-g(t)}\right).$$

Once again, (2) easily follows from the preceding calculations.

Finally, in order to prove (2) for an Itô diffusion, it suffices to use the change of scale.

References

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