Parallelogram spaces and their corresponding algebraic and geometric structures*

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Abstract. *In this paper a table with some mutually equivalent algebraic and geometric structures on the same set $\mathbb{Q}$ is given together with bibliographical references about defining these structures and about proofs of equivalencies existing for some pairs of these structures.*

Key words: *parallelogram space, heap, semiaffine space, Malcev function*

Sažetak. **Paralelogramski prostori i njima srodne algebarske i geometrijske strukture.** *U ovom radu dana je pregledna tablica u kojoj su predočene neke međusobno ekvivalentne algebarske i geometrijske strukture na istom skupu $\mathbb{Q}$ sa bibliografskim podacima o definiranju tih struktura i o dokazima ekvivalentnosti postojanja po dviju od tih struktura.*

Ključne riječi: *paralelogramski prostor, gruda, semiafini prostor, Malcevljeva funkcija*

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The elements of the enclosed table were arranged gradually like a puzzle. In ten "boxes" of that table there are various algebraic and geometric structures on the same non-empty set Q. A bi-directional arrow between the two "boxes" (i) and (j) denotes the statement that there exists a structure (i) on the set $\mathbb{Q}$ if and only if there exists the structure (j). The elements of the puzzle are chronologically arranged from (1) to (33). In the "box" of a particular structure a corresponding algebraic operation or a geometric relation or a family of transformations on the set $\mathbb{Q}$ is denoted.

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① The most popular, the oldest and the most important of all mentioned structures, a commutative group $(\mathbb{Q},+)$ is placed in the first "box". A lot of authors, e.g. Garver, Lorenzen, Whittaker, Furstenberg, Sholander, Morgado, Husain [6],[8],[9],[24],[5],[14],[10],[7], tried to reduce the number of axioms for the commutative group $(\mathbb{Q},+)$. It can be done by observing subtraction on the set $\mathbb{Q}$ instead of addition. Finally, Cardoso [3] and Morgado [10] (1963) noted that it was a new abstract structure which deserved a special name and special research. That is:

② a subtractive grupoid $(\mathbb{Q},-)$ with only two axioms and these are the identities $a-(a-b)=b$, $(a-c)-(b-c)=a-b$.

③ There exists a commutative group $(\mathbb{Q},+)$ if and only if there exists a subtractive grupoid $(\mathbb{Q},-)$, where subtraction in $(\mathbb{Q},+)$ is defined in a usual way by the formula $a-b=a+(-b)$, and vice versa, addition can be defined by subtraction by the formula $a+b=a-(0-b)$, where we use the fact that in the subtractive grupoid $(\mathbb{Q},-)$ there exists a selected element 0 (zero) such that identities $a-a=0$, $a-0=a$ hold.

④ (Faithful) homogeneous space $(\mathbb{Q},T)$ over a commutative group (T,o) was defined in 1951 by Bourbaki [2] as an ordered pair (Q,T), where T is a family of bijections $\tau:\mathbb{Q}\to\mathbb{Q}$ which operates on the set $\mathbb{Q}$ (strictly) transitively, and bijections τ are called transfers.

⑤ There exists a commutative group $(\mathbb{Q},+)$ if and only if there exists a faithful homogeneous space $(\mathbb{Q},T)$ over a commutative group (T,o), where transfers are defined by addition by the formula $\tau_{a,b}(x)=y\Leftrightarrow y=b-a+x$, and vice versa, addition is defined by transfers by the formula $a+b=c\Leftrightarrow\tau_{0,a}(b)=c$, where $0\in\mathbb{Q}$ is the selected element for which it could be shown that it is a neutral element for addition.

⑥ A laterally commutative heap $(\mathbb{Q},())$ was defined in 1953 by Vagner [15], where $():\mathbb{Q}^3\to\mathbb{Q}$ represents a ternary operation with properties $((abc)de)=(a(bcd)e)=(ab(cde))$ (associativity), $(abc)=(cba)$ (lateral commutativity), $(abb)=a$, $(bba)=a$ (() is said to be a Malcev function).

⑦ There exists a laterally commutative heap $(\mathbb{Q},())$ if and only if there exists a commutative group $(\mathbb{Q},+)$ (Vagner), where the ternary operation is defined by addition by the formula $(abc)=a-b+c$, and vice versa, addition is defined by the ternary operation by the formula $a+b=(a0b)$, where $0\in\mathbb{Q}$ is the selected element for which it could be shown that it is zero for $(\mathbb{Q},+)$.

⑧ Parallelogram space $(\mathbb{Q},Par)$ was defined by Ostermann and Schmidt in 1963, [11], where $Par\subset\mathbb{Q}^4$ is a quaternary relation whose elements are ordered quadruples of elements from $\mathbb{Q}$ (points), which are called parallelograms, and with properties: $(P1)$ $\forall a,b,c\in\mathbb{Q}$ $\exists!$ $d\in\mathbb{Q}$, so that $Par(a,b,c,d)$ $(P2)$ $Par(a,b,c,d)\Rightarrow Par(e,f,g,h)$, where (e,f,g,h) is any of cyclic permutations of (a,b,c,d) or of (d,c,b,a); $(P3)$ $Par(a,b,c,d)$, $Par(c,d,e,f)\Rightarrow Par(a,b,f,e)$.

⑨ There exists a parallelogram space $(\mathbb{Q}, Par)$ if and only if there exists a faithful homogeneous space $(\mathbb{Q}, T)$ over a commutative group (T, o) (Ostermann - Schmidt), where the formula $Par(a,b,c,d) \Leftrightarrow \tau_{a,b}(d) = c$ defines a parallelogram by means of a transfer, and vice versa, a transfer by means of a parallelogram.

⑩ There exists a parallelogram space $(\mathbb{Q}, Par)$ if and only if there exists a commutative group $(\mathbb{Q}, +)$ (Ostermann - Schmidt), where a parallelogram is defined by addition by the formula $Par(a,b,c,d) \Leftrightarrow a+c = b+d$, and vice versa, addition is defined by a parallelogram by the formula $a+b = c \Leftrightarrow Par(0,a,c,b)$, where $0 \in \mathbb{Q}$ is the selected element, later on zero in $(\mathbb{Q}, +)$.

⑪ There exists a parallelogram space $(\mathbb{Q}, Par)$ if and only if there exists a laterally commutative heap $(\mathbb{Q}, ())$ (Vakarelov, 1969-70, [16]), where by the definition $Par(a,b,c,d) \Leftrightarrow (abc) = d$ is used in both directions.

⑫ Semiaffine space (Bos-Wolff 1978 [1], Wolff 1978 [25]) is an ordered pair $(\mathbb{Q}, ())$ with the following properties: $(abc) = (cba)$, $(abb) = a$, $((abc)bd) = (ab(cbd))$, $((abc)db) = (adc)$.

⑬ There exists a semiaffine space $(\mathbb{Q}, ())$ if and only if there exists a parallelogram space $(\mathbb{Q}, Par)$ (Wolff), where again $Par(a,b,c,d) \Leftrightarrow (abc) = d$.

⑭ There exists a semiaffine space $(\mathbb{Q}, ())$ if and only if there exists a commutative group $(\mathbb{Q}, +)$ (Wolff), where again $(abc) = a - b + c$ and $a + b = (a0b)$ with the selected element $0 \in \mathbb{Q}$.

⑮ Medial Malcev function $(\mathbb{Q}, ())$ (Duda, 1982, [4]) has the following properties: $(abb) = a$, $(bba) = a$, $((abc)(def)(ghi)) = ((adg)(beh)(cfi))$ (mediality).

⑯ There exists a medial Malcev function $()$ on $\mathbb{Q}$ if and only if there exists a parallelogram space $(\mathbb{Q}, Par)$ (Duda), where again $Par(a,b,c,d) \Leftrightarrow (abc) = d$.

⑰ A TST-space $(\mathbb{Q}, S)$ (Polonijo, 1985, [12]) is an ordered pair $(\mathbb{Q}, S)$, where S represents a family of involutory bijections $\sigma : \mathbb{Q} \to \mathbb{Q}$, called symmetries, for which the property $\sigma_1, \sigma_2, \sigma_3 \in S \Rightarrow \sigma_1 o \sigma_2 o \sigma_3 \in S$ holds and which operate transitively on the set $\mathbb{Q}$.

⑱ There exists a TST-space $(\mathbb{Q}, S)$ if and only if there exists a parallelogram space $(\mathbb{Q}, Par)$ (Polonijo), where the formula $Par(a,b,c,d) \Leftrightarrow \sigma_{a,c}(b) = d$ defines a parallelogram by symmetry, and vice versa, symmetry by a parallelogram.

⑲ There exists a TST-space $(\mathbb{Q}, S)$ if and only if there exists a commutative group $(\mathbb{Q}, +)$ (Polonijo), where symmetry is defined by addition by means of the formula $\sigma_{a,b}(c) = a + b - c$, and addition is defined by symmetries by the formula $\sigma_{a+b,0} = \sigma_{a,0} o \sigma_{0,0} o \sigma_{b,0}$, where 0 is zero.

⑳ There exists a parallelogram space $(\mathbb{Q}, Par)$ if and only if there exists a subtractive groupoid (Polonijo, 1986, [13]), where a parallelogram is defined by means of subtraction by the formula $Par(a,b,c,d) \Leftrightarrow a - b = d - c$, and vice versa, subtraction is defined by the parallelogram by means of the formula $a - b = c \Leftrightarrow Par(a, b, 0, c)$, where $0 \in \mathbb{Q}$ is the selected element, for which it could be shown that it is the right neutral element of $(\mathbb{Q}, -)$.

21 There exists a TST-space $(\mathbb{Q}, S)$ if and only if there exists a laterally commutative heap $(\mathbb{Q}, ())$ (Volenec, 1987, [17]), where the formula $\sigma_{a,b}(c) = (acb)$ defines symmetry by means of a ternary operation, and vice versa, a ternary operation $()$ by symmetries.

22 There exists a TST-space $(\mathbb{Q}, S)$ if and only if there exists a faithful homogeneous space $(\mathbb{Q}, T)$ (Volenec, 1987, [18]), where the formula $\tau_{a,b}(c) = d \Leftrightarrow \sigma_{a,d}(b) = c$ defines symmetries by means of transfers and transfers by means of symmetries.

23 In the paper [19] (Volenec, 1988) the structure $(\mathbb{Q}, ())$ with properties: $(abc) = (cba)$; $((abc)cb) = a$ or $(abb) = a$; $((abc)cd) = (abd)$ is considered.

24 There exists a structure $(\mathbb{Q}, ())$ from 23 if and only if there exists a parallelogram space $(\mathbb{Q}, Par)$ (Volenec, 1988, [19]), where again $Par(a, b, c, d) \Leftrightarrow (abc) = d$.

25 There exists a structure $(\mathbb{Q}, ())$ from 23 if and only if there exists a laterally commutative heap $(\mathbb{Q}, ())$ (Volenec, 1988, [19]).

26 There exists a structure $(\mathbb{Q}, ())$ from 23 if and only if there exists a semiaffine space $(\mathbb{Q}, ())$ (Volenec, 1988, [19]).

27 There exists a laterally commutative heap $(\mathbb{Q}, ())$ if and only if there exists a semiaffine space $(\mathbb{Q}, ())$ (Volenec, 1988, [19]).

28 There exists a laterally commutative heap $(\mathbb{Q}, ())$ if and only if there exists a medial Malcev function $()$ on the set $\mathbb{Q}$ (Volenec, 1988, [20]).

29 There exists a TST-space $(\mathbb{Q}, S)$ if and only if there exists a subtractive grupoid $(\mathbb{Q}, -)$ (Volenec, 1989, [21]), where the formula $\sigma_{a,b}(x) = a - (x - b)$ defines symmetries by means of subtraction, and the formula $a - b = \sigma_{a,0}(b)$ defines subtraction by means of symmetries, where $0 \in \mathbb{Q}$ is the selected element, for which it could be shown that it is the right neutral element of $(\mathbb{Q}, -)$.

30 There exists a laterally commutative heap $(\mathbb{Q}, ())$ if and only if there exists a subtractive grupoid $(\mathbb{Q}, -)$ (Volenec, 1990, [22]), where the formula $(abc) = a-(b-c)$ defines a ternary operation by means of subtraction, and the formula $a - b = (ab0)$ defines subtraction by means of a ternary operation $()$, where $0 \in \mathbb{Q}$ is the selected element, later on the right neutral element for $(\mathbb{Q}, -)$.

31 In the paper [23] (Volenec, 1991) a structure $(\mathbb{Q}, Par)$ is defined with the following properties: $(P1)$ $\forall a, b, c \in \mathbb{Q}$ $\exists!$ $d \in \mathbb{Q}$, so that $Par(a, b, c, d)$ $(P4)$ $\forall a, b \in \mathbb{Q}$ $Par(a, a, b, b)$, $Par(a, b, b, a)$; $(P5)$ $Par(a_{i1}, a_{i2}, a_{i3}, a_{i4})$ $(i = 1, 2, 3, 4)$, $Par(a_{1j}, a_{2j}, a_{3j}, a_{4j})$ $(j = 1, 2, 3) \Rightarrow Par(a_{14}, a_{24}, a_{34}, a_{44})$.

32 There exists a structure $(\mathbb{Q}, Par)$ from 31 if and only if there exists a medial Malcev function $(\mathbb{Q}, ())$, where $Par(a, b, c, d) \Leftrightarrow (abc) = d$. The statement is obvious.

33 There exists a structure $(\mathbb{Q}, Par)$ from 31 if and only if there exists a parallelogram space $(\mathbb{Q}, Par)$. (Volenec, 1991, [23]).

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