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Interval-valued upside potential and downside risk portfolio optimisation

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ABSTRACT
A novel interval optimisation approach is developed to include imprecise forecasts into the portfolio selection process for investors measuring upside potential and downside risk as deviations from a target return. Crisp scenarios are substituted by interval scenarios and the resulting interval optimisation problem is solved in a tractable manner by means of a bi-objective formulation exploiting a partial order relation between intervals. Four utility case studies involving assets from the F.T.S.E. M.I.B. Index are considered to illustrate how impreciseness can be efficiently handled in portfolio management.

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1. Introduction

Many authors have studied the relationship between the theory of financial markets and the rational behaviour of an individual. They have presented different mathematical models which take into account both the uncertainty arising from the investments in a certain number of assets (portfolio theory) and the subjective risk aversion of a single investor (utility theory).

It is well known that utility theory was born with an historical essay by Bernoulli (1738). In this milestone work, it is stated that the value of a good is better represented by its utility than by its price. Moreover, utility only depends upon individual characteristics (such as current wealth and risk perception) of the decision maker.

Von Neumann and Morgenstern (1947) initiated modern utility theory. In their famous representation theorem they proved that every preference relation satisfying some given axioms can be represented by a utility (or value) function, which turns out to be concave in the case of risk aversion (i.e., when the decision maker refuses a fair gamble with zero expected value). Friedman and Savage (1948) considerably improved the aforementioned result by introducing reverse S-shaped (first concave and then convex) utility functions. Markowitz (1952, 1959) initiated modern portfolio theory by introducing mean-variance and semi-variance analysis. He justified the adoption of a quadratic utility function by arguing that it is

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a reasonable approximation of rational investors' behaviour with risk–return tradeoffs. On the other hand, Bawa (1975) and Fishburn (1977) illustrated the fact that mean-lower partial moment models can implement Von Neumann and Morgenstern utility functions and, at the same time, can be easily related to first, second and third stochastic dominance. In these models the wealth of the investor is replaced by a return rate below a desirable target. A few years later Fishburn and Kochenberger (1979) and Holthausen (1981) extended these models by introducing upper and lower partial moment models (U.P.M.-L.P.M.).

Prospect theory (P.T.) was originally developed in order to solve the famous paradoxes of Allais (1953) and Ellsberg (1961). The first paradox threatened the independence axiom of Von Neumann and Morgenstern, while the second paradox explained that risk aversion can arise from ambiguous descriptions of uncertainty. Kahneman and Tversky (1979) started from U.P.M.-L.P.M. models in order to add the concept of a distorted probability, namely a nonlinear transformation of the probability scale which possibly overweights small probabilities and underweights moderate and high probabilities.

In our opinion, in recent decades the concept of risk aversion, which is naturally associated with utility theory and preference-based models, has not been appropriately considered. At the same time, the main role has been played by static equilibrium models, such as the widely used capital asset pricing model, and by risk measures such as Value at Risk and Conditional Value at Risk, which have been studied mainly from a technical point of view. Nawrocki and Viole (2014) recently suggested focusing attention again on utility theory and therefore on the individual investor at the core of behavioural finance. In recent years several authors have faced the problem of defining optimal portfolios that achieve the maximum of the utility functions in U.P.M.-L.P.M. models, and therefore several nonlinear programming problems have been developed with this aim. For example, Cumova and Nawrocki (2014) introduced an augmented Lagrangian algorithm that finds the best asset allocation on the basis of an approximation of the L.P.M. and U.P.M. measures.

In this paper, following You (2013), who considers a fixed underlying probability space, we introduce an extended model for the financial market in which securities have random interval payoffs. The uncertainties arise from the realisations of random variables better than from probabilities measures. According to the imprecise probabilities approach studied for example by Miranda, Couso & Gil (2005a, 2005b), a random interval whose two endpoints are random variables is here interpreted as an imprecise perception of a random variable. It is remarkable to observe that the application of random set theory in econometrics and finance is not new (the reader may refer to Molchanov and Molinari (2014) for a detailed exposition). Beresteanu, Molchanov, and Molinari (2011, 2012) applied random set theory to partially identified models, Diaye and Koshevoy (2014) to decision making under risk and Liu, Zhang, and Zhang (2013) to multi-period portfolio selection optimisation.

We perform portfolio selection by using imprecise forecasts as random intervals in line with Kaucic and Daris (2016). In this study, however, we focus on a wider range of risk–reward profiles, including P.T. investors as a special case. The family of utility functions we adopt is the three-parameter one proposed by Holthausen (1981). In particular, four types of strategies characterising the principal attitudes toward risk and reward are compared in terms of compositions, diversification and expected rates of return. Moreover, from a computational point of view, we provide a tractable formulation of the resulting interval optimisation problem by defining a bi-objective optimisation problem on the basis of a partial order relation for intervals and a ranking approach. A multi-objective evolutionary algorithm based on decomposition allows us to identify the solutions. Results illustrate the potential of the proposed model.

The paper is organised as follows. In the next section we provide the necessary background on interval analysis and random interval theory. Section 3 introduces the upside potential and downside risk framework. Our interval extension of the model to include incomplete/imprecise forecasts of asset rates of return by means of interval scenarios is developed in Section 4. Section 5 first describes the scenario-generation process and the optimisation solver, and then illustrates an empirical application to real data from the Italian F.T.S.E. M.I.B. Index involving four different investment strategies. Concluding remarks are given in Section 6.

2. Preliminaries on interval analysis and order relations

2.1. Basic operations with intervals

Uncertain, imprecise or incomplete information can be incorporated into the portfolio optimisation process by expressing data and/or parameters as intervals instead of single values. Thus, an adequate algebraic and probabilistic setting has to be defined in order to properly define the decision maker actions in this context (for a detailed exposition, the interested reader may consult Molchanov (2005), Corral, Gil, and Gil (2011) and the references therein).

Definition 2.1: An interval number, denoted as $\tilde{a}$, is a bounded and closed subset of R given by

$$\tilde{a} = [a^l, a^u] \overset{def}{=} \left\{x \in R \mid a^l \le x \le a^u\right\} \tag{1}$$

Where a^l, $a^u \in R$, with $a^l \le a^u$, are the lower and the upper bounds of $\tilde{a}$, respectively.

The set of all interval numbers on $\mathbb{R}$ is denoted as $\mathcal{K}_c(R)$.

Remark 1. This representation of an interval number $\tilde{a}$ is called endpoints (E.P.) form.

Remark 2. I f $a^l = a^u$ then $\tilde{a}$ reduces to a real number.

The sum of two interval numbers and the product of an interval number by a scalar are defined in terms of the corresponding Minkowski set-theoretic operations.

Definition 2.2: For every $\tilde{a} = [a^l, a^u]$, $\tilde{b} = [b^l, b^u]$ in $\mathcal{K}_c(R)$ and $\gamma \in R$, we have

$$\tilde{a} + \tilde{b} \overset{def}{=} \left\{a + b \mid a \in \tilde{a},\ b \in \tilde{b}\right\} = [a^l + b^l, a^u + b^u] \tag{2}$$

and

$$\gamma * \tilde{a} \overset{def}{=} \{\gamma a \mid a \in \tilde{a}\} = \begin{cases} [\gamma a^l, \gamma a^u] & \text{if } \gamma \ge 0 \\ [\gamma a^u, \gamma a^l] & \text{if } \gamma < 0. \end{cases} \tag{3}$$

From Eqns. (2) and (3), we have that the opposite of an interval number $\tilde{a}$ is $-\tilde{a} = [-a^u, -a^l]$. Thus, the difference of two intervals can be defined as $\tilde{b} - \tilde{a} = \tilde{b} + (-\tilde{a}) = [b^l - a^u, b^u - a^l]$. A serious problem related to this expression is that $\tilde{a} - \tilde{a} \ne [0, 0]$, where we assume $[0, 0] = \{0\}$ represents the neutral element for addition. This implies that the space $\mathcal{K}_c(R)$

does not contain inverse elements. An alternative definition of intervals difference that overcomes these limits has been proposed independently by Markov (1979), Stefanini and Bede (2009) and Chalco-Cano, Roman-Flores, and Jimenez-Gamero (2015).

Definition 2.3: For every $\tilde{a} = [a^l, a^u]$, $\tilde{b} = [b^l, b^u]$ in $\mathcal{K}_c(R)$, the generalised Hukuhara difference (gH-difference) $\tilde{a}$ and $\tilde{b}$ is defined as

$$\tilde{a} -_{gH} b = [\min\{a^l - b^l, a^u - b^u\}, \max\{a^l - b^l, a^u - b^u\}]. \tag{4}$$

In particular, $\tilde{a} -_{gH} \tilde{a} = \{0\}$.

A second characterisation of an interval number is the following:

Definition 2.4: An interval number $\tilde{a} \in \mathcal{K}_c(R)$ is said to be in midpoint-radius (M.R.) form if it is encoded as the following vector of R^2

$$\tilde{a} = (a^c, a^w) \overset{def}{=} \left(\frac{a^u + a^l}{2}, \frac{a^u - a^l}{2} \right) \tag{5}$$

where a^c denotes the centre of the interval and $a^w$is its half-width.

By means of Eqn. (5), we can easily move from E.P. to M.R. encoding and vice versa. With an abuse of notation, for every $\tilde{a} \in \mathcal{K}_c(R)$, it holds that

$$(a^c, a^w) = \{x \in R \mid a^c - a^w \leq x \leq a^c + a^w\} = [a^l, a^u].$$

From these observations it emerges that the former encoding is suitable to introduce algebraic properties of intervals while the latter can be used to exhibit and explicitly manipulate the uncertainty in interval numbers.

In particular, the gH-difference of two intervals can also be expressed as in M.R. form as

$$\tilde{a} -_{gH} \tilde{b} = (a^c - b^c, |a^w - b^w|). \tag{6}$$

where $\tilde{a} = (a^c, a^w)$ and $\tilde{b} = (b^c, b^w)$.

2.2. Random intervals

We can characterise random intervals by exploiting the E.P. encoding of interval numbers as follows.

Definition 2.5: Let $(\Omega, \mathfrak{F}, P)$ be a probability space. A multi-valued mapping $\Gamma{:}\Omega \to \mathcal{K}_c(R)$, given by $\Gamma(\omega) = [\inf \Gamma(\omega), \sup \Gamma(\omega)]$, where $\inf \Gamma,\ \sup \Gamma{:}\Omega \to R$ are two real-valued functions such that $\inf \Gamma \leq \sup \Gamma$ almost surely, is said a random interval if $\inf \Gamma$ and $\sup \Gamma$ are real-valued random variables.

A notion associated to the concept of random interval is the following.

Definition 2.6: Let $\Gamma{:}\Omega \to \mathcal{K}_c(R)$ be a random interval. A random variable $X{:}\Omega \to R$ is said a (measurable) selection of Γ if X is measurable and $X(\omega) \in \Gamma(\omega)$ for all $\omega \in \Omega$.

The set of all measurable selections of Γ is denoted by $\mathcal{S}(\Gamma)$.

We assume that a random interval Γ represents an incomplete knowledge about the outcomes of a given random variable X. Thus, all the information we have available is that X is a measurable selection of Γ. Accordingly, let $\mathbb{E}(X)$ represent the Lebesgue expectation of a random variable X, the random interval corresponding to an imprecise/incomplete perception of $X(\omega)$ for all $\omega \in \Omega$ is the so-called Aumann expectation and is defined as follows.

Definition 2.7: Let $(\Omega, \mathfrak{F}, P)$ be a probability space and $\Gamma:\Omega \to \mathcal{K}_c(R)$ be a random interval such that all its selections are integrable, i.e. $X \in L^1(\Omega, \mathfrak{F}, P)$ for all $X \in \mathcal{S}(\Gamma)$. The interval number $\mathbb{E}(\Gamma)$ defined as

$$\mathbb{E}(\Gamma) = \left[\mathbb{E}(\inf \Gamma), \mathbb{E}(\sup \Gamma)\right]$$

where $\inf \Gamma$ and $\sup \Gamma$ are the two random variables specified in Definition 2.5, is called the expected (or mean) value of Γ in Aumann's sense.

Remark 3. In the definition of $\mathbb{E}(\Gamma)$, the set of all measurable selections $\mathcal{S}(\Gamma)$ is replaced by the subset of all integrable selections.

The Aumann expectation is coherent with interval arithmetic and inherits many valuable probabilistic and statistical properties from expectation of a real-valued random variable, such as the satisfaction of the strong law of large numbers. The next proposition, in particular, summarises some results that will be used in the next sections to formalise the notion of expected interval (rate of) return and other related notions.

Proposition 2.8: Let $(\Omega, \mathfrak{F}, P)$ be a probability space. The Aumann mean of a random interval satisfies the following properties:

(i) if Γ is a random interval such that $\Gamma(\Omega) = \{\tilde{a}_1, \dots, \tilde{a}_n\}$ and $\{\Omega_i\}_{i=1}^n$ is a partition of Ω, with $\Omega_i = \Gamma^{-1}(\tilde{a}_i)$, $i = 1, \dots, n$, then

$$\mathbb{E}(\Gamma) = \sum_{i=1}^{n} P(\Omega_i) * \tilde{a}_i$$

(ii) for every $\alpha, \beta \in \mathbb{R}$, $\tilde{a} \in \mathcal{K}_c(R)$ and Γ, Υ random intervals, then

$$\mathbb{E}(\alpha * \Gamma + \beta * \Upsilon + \tilde{a}) = \alpha * \mathbb{E}(\Gamma) + \beta * \mathbb{E}(\Upsilon) + \tilde{a}.$$

Remark 4. We have limited the presentation to the $\mathcal{K}_c(R)$ space, omitting the exposition for the general n-dimensional case, in order to avoid useless cumbersome notations since the results are almost the same.

2.3. Interval extension of a point-valued function

Before formally stating the procedure, the following preliminary definitions are necessary. An n-dimensional interval vector is a subset of R^n given by the Cartesian product of n interval numbers. Through the paper, it will be denoted in bold as $\tilde{\mathbf{a}} = \tilde{a}_1 \times \dots \times \tilde{a}_n$ and the associated space will be indicated with $\mathcal{K}_c(R)^n$.

The exposition specialises the arguments in Hickey, Ju, and Van Emden (2001) and Moore, Kearfott, and Cloud (2009) to the continuous case, since only this type of function will be handled in the next sections. Noting that a continuous point-valued function f maps compact sets into compact sets, we can state the following definition for a multi-valued mapping extending a real-valued function.

Definition 2.9: Let $f:R^n \to R$ be a continuous point-valued function. The natural interval extension of f is the multi-valued mapping $\hat{f}:\mathcal{K}_c(R)^n \to \mathcal{K}_c(R)$ given by

$$\hat{f}(\tilde{\mathbf{x}}) \stackrel{def}{=} \begin{cases} \{f(\mathbf{x}) \mid \mathbf{x} \in \tilde{\mathbf{x}} \cap \mathrm{dom}(f)\}, & \text{if } \tilde{\mathbf{x}} \cap \mathrm{dom}(f) \neq \emptyset \\ \emptyset, & \text{otherwise} \end{cases} \tag{7}$$

where $\mathrm{dom}(f)$ is the domain of f.

The natural interval extension can be straightforwardly computed in the following case.

Lemma 2.10: Let $f:\mathbb{R} \to \mathbb{R}$ be any monotone continuous point-valued function and assume $\tilde{x} \cap dom(f) = [x^l, x^u]$ is non-empty, then it holds

$$\hat{f}(\tilde{x}) = \begin{cases} [f(x^l), f(x^u)] & \text{if } f \text{ is increasing} \\ [f(x^u), f(x^l)] & \text{if } f \text{ is decreasing.} \end{cases}$$

The proof of this result is an immediate consequence of Eqn. (7) and of the Weierstrass' theorem.

In particular, we are interested in evaluating the natural extension of the power function with rational exponents $f(x) = kx^{\frac{r}{s}}$, where $k \in R\backslash\{0\}$ and r, s are coprime positive integers. Without loss of generality, we assume $\tilde{x} \cap \mathrm{dom}(f) = \tilde{x}$ and define the r-th power of an interval $\tilde{x} = [x^l, x^u]$ as

$$\tilde{x}^r \stackrel{def}{=} \begin{cases} \left[(x^l)^r, (x^u)^r\right] & \text{if } r \text{ is odd or } x^l \geq 0 \\ \left[(x^u)^r, (x^l)^r\right] & \text{if } r \text{ is even and } x^u \leq 0 \\ \left[0, \max\left\{(x^l)^r, (x^u)^r\right\}\right] & \text{if } r \text{ is even and } x^l \leq 0 \leq x^u \end{cases} \tag{8}$$

and its s-th root as

$$\tilde{x}^{\frac{1}{s}} \stackrel{def}{=} \begin{cases} \left[(x^l)^{\frac{1}{s}}, (x^u)^{\frac{1}{s}}\right] & \text{if } s \text{ is odd or } x^l \geq 0 \\ \left[0, (x^u)^{\frac{1}{s}}\right] & \text{if } s \text{ is even and } x^l \leq 0 \leq x^u \\ \emptyset & \text{if } s \text{ is even and } x^u < 0. \end{cases} \tag{9}$$

Accordingly, the natural extension of the power function with rational exponents may be obtained by combining Eqns. (8) and (9) with Eqn. (3) as follows

$$\hat{f}(\tilde{x}) = k * \tilde{x}^{\frac{r}{s}} \stackrel{def}{=} k * (\tilde{x}^r)^{\frac{1}{s}}. \tag{10}$$

for a given $k \in R\backslash\{0\}$.

2.4. Order relations for interval numbers

Mathematical programming involving interval numbers can be considered as optimisation problems with uncertain or imprecise information in the objective function coefficients and/or constraints. Thereby, the preference relations for interval numbers play an important role in selecting the best alternative. In this paper, we compare intervals according to the preference relation $\leq_{cw}$ proposed in Ishibuchi and Tanaka (1990).

For maximisation problems, this order relation can be represented in the following form.

Definition 2.11: Let $\tilde{a} = (a^c, a^w)$ and $\tilde{b} = (b^c, b^w)$ be two interval numbers in $\mathcal{K}_c(R)$, then

$$\tilde{a} \leq_{cw} \tilde{b} \Leftrightarrow a^c \leq b^c \ \wedge \ a^w \geq b^w. \tag{11}$$

Furthermore,

$$\tilde{a} <_{cw} \tilde{b} \Leftrightarrow \tilde{a} \leq_{cw} \tilde{b} \ \wedge \ \tilde{a} \neq \tilde{b} \tag{12}$$

defines the strict order relation on $\mathcal{K}_c(R)$.

For minimisation problems, the order relation $\leq_{cw}$ becomes as follows.

Definition 2.12: Let $\tilde{a} = (a^c, a^w)$ and $\tilde{b} = (b^c, b^w)$ be two interval numbers in $\mathcal{K}_c(R)$, then

$$\tilde{a} \leq_{cw} \tilde{b} \Leftrightarrow a^c \leq b^c \ \wedge \ a^w \leq b^w \tag{13}$$

and

$$\tilde{a} <_{cw} \tilde{b} \Leftrightarrow \tilde{a} \leq_{cw} \tilde{b} \ \wedge \ \tilde{a} \neq \tilde{b}. \tag{14}$$

Remark 5. The Ishibuchi and Tanaka's order relation compares two intervals in terms of the corresponding centres and uncertainties. The larger the width of an interval, the greater the uncertainty. In particular, if the problem is the maximisation of an interval utility, the greater the centre value of the objective function and the lesser its uncertainty will be better. Conversely, for portfolio optimisation problems aiming at minimising interval risks, less centre value and less uncertainty in the objective function are preferred.

The $\leq_{cw}$ order relation is a partial order on $\mathcal{K}_c(R)$ since it is not difficult to prove that it satisfies the following properties:

(i) reflexivity: for all $\tilde{a} \in \mathcal{K}_c(R)$, $\tilde{a} \leq_{cw} \tilde{a}$;
(ii) antisymmetry: for all $\tilde{a}, \tilde{b} \in \mathcal{K}_c(R)$, if $\tilde{a} \leq_{cw} \tilde{b}$ and $\tilde{b} \leq_{cw} \tilde{a}$ then $\tilde{a} = \tilde{b}$;
(iii) transitivity: for all $\tilde{a}, \tilde{b}, \tilde{c} \in \mathcal{K}_c(R)$, $\tilde{a} \leq_{cw} \tilde{b}$ and $\tilde{b} \leq_{cw} \tilde{c}$ then $\tilde{a} \leq_{cw} \tilde{c}$;

However, not all intervals are comparable with respect to $\leq_{cw}$. Consider for instance $A = [-1, 3]$, $B = [2, 4]$ and $C = [1, 5]$. For maximisation problems, A and B are incomparable since $a^c < b^c$ and $a^w < b^w$. At the same time, for minimisation problems, A and C are incomparable since $a^c < c^c$ and $a^w > c^w$.

An interval-ranking approach will be also used in this paper, the so-called acceptability index developed in Sengupta, Pal, and Chakraborty (2001).

Definition 2.13: Let $\tilde{a} = (a^c, a^w)$ and $\tilde{b} = (b^c, b^w)$ be two interval numbers in $\mathcal{K}_c(R)$, then the acceptability function $\mathcal{A}: \mathcal{K}_c(R) \times \mathcal{K}_c(R) \to R$ is defined as

$$\mathcal{A}(\tilde{a} \leq \tilde{b}) = \frac{b^c - a^c}{b^w + a^w} \tag{15}$$

where $b^w + a^w \neq 0$.

The real number $\mathcal{A}(\tilde{a} \leq \tilde{b})$ represents the grade of acceptability of the sentence '$\tilde{a}$ is inferior to $\tilde{b}$' or, equivalently, '$\tilde{b}$ is superior to $\tilde{a}$'. The terms 'inferior to' and 'superior to' are analogous to the terms 'smaller' and 'greater' for real numbers.

Remark 6. If $\tilde{a}$ and $\tilde{b}$ are degenerate intervals, i.e., both have zero radius, the $\leq$ order for reals is used.

From the definition, the following cases can occur:

$$\mathcal{A}\left(\tilde{a} \leq \tilde{b}\right) \begin{cases} < 0, & \text{if } a^c > b^c \\ = 0, & \text{if } a^c = b^c \\ \in (0, 1), & \text{if } a^c < b^c \text{ and } a^u > b^l \\ \geq 1, & \text{if } a^c < b^c \text{ and } a^u \leq b^l. \end{cases}$$

In the first two cases, the sentence '$\tilde{a}$ is inferior to $\tilde{b}$' is not accepted. If $0 < \mathcal{A}\left(\tilde{a} \leq \tilde{b}\right) < 1$, then the decision maker accepts that '$\tilde{a}$ is inferior to $\tilde{b}$' with grade of satisfaction ranging from zero to one (excluding zero and one). If $\mathcal{A}\left(\tilde{a} \leq \tilde{b}\right) \geq 1$, then the decision maker is completely satisfied with this sentence and accepts that $\tilde{a} \leq \tilde{b}$ is true.

Remark 7. This ranking index is unable to discriminate between $\tilde{a} \leq \tilde{b}$ and $\tilde{b} \leq \tilde{a}$ when $a^c = b^c$ and $b^w + a^w \neq 0$. In this situation, indeed, $\mathcal{A}\left(\tilde{a} \leq \tilde{b}\right) = \mathcal{A}\left(\tilde{b} \leq \tilde{a}\right) = 0$.

3. Upside potential and downside risk framework

The financial market is modelled by a probability space $(\Omega, \mathfrak{F}, P)$ and consists of n risky assets, indexed from 1 to n. Agents allocate their wealth over a one-period investment horizon according to the following table of scenarios:

$$\begin{pmatrix} \mathbf{r}_1 & \dots & \mathbf{r}_S \\ p_1 & \dots & p_S \end{pmatrix} \quad \text{with } \sum_{s=1}^{S} p_s = 1 \text{ and } p_s \geq 0 \ \forall s \tag{16}$$

where S represents the number of involved scenarios, $\mathbf{r}_s = \left(r_{1s}, \dots, r_{ns}\right)^T$ is the n-vector of rates of return for the s-th scenario and p_s is the associated probability of occurrence, $s = 1, \dots, S$. In this manner, the expected rate of return for the i-th security can be computed as the mean rate of return over the S scenarios, i.e., $\mathbb{E}(r_i) = \sum_{s=1}^{S} p_s r_{is}$. Let x_i be the weight of the i-th asset in the portfolio, we consider the following standard set of constraints for a portfolio to be feasible:

(i) $\sum_{i=1}^{n} x_i = 1$;

(ii) $x_i \geq 0$ for all i.

The set of portfolios $\mathbf{x} \in R^n$ satisfying (i)–(ii) is denoted by $\mathcal{X}$. Let the portfolio rate of return for a fixed $\mathbf{x} \in \mathcal{X}$ under the s-th scenario given by

$$r_s^p \stackrel{def}{=} \mathbf{x}^T \mathbf{r}_s = \sum_{i=1}^{n} x_i r_{is} \quad \text{for } i = 1, \dots, S, \tag{17}$$

it holds that the expected rate of return of the portfolio is

$$r_s^p \stackrel{def}{=} \mathbf{x}^T \mathbf{r}_s = \sum_{i=1}^{n} x_i r_{is} \quad \text{for } i = 1, \dots, S, \tag{18}$$

Market participants act the decisions related to their investments on the basis of the following assumptions:

(i) outcomes are evaluated in comparison to a certain benchmark rather than an absolute final wealth;
(ii) reactions toward probable gains and losses may be different.

The first characteristic is modelled by introducing a reference level of wealth, r^{ref}, which divides outcomes into gains and losses. To handle the different attitude toward risk and reward we consider the three-parameter family of piecewise power utility functions developed by Holthausen (1981), whose analytical expression is

$$u(\mathbf{x}|\mathbf{r}_s;r^{ref}) = \begin{cases} \left(\mathbf{x}^T\mathbf{r}_s - r^{ref}\right)^{\alpha}, & \text{if } \mathbf{x}^T\mathbf{r}_s \geq r^{ref} \\ -\gamma(r^{ref} - \mathbf{x}^T\mathbf{r}_s)^{\beta}, & \text{if } \mathbf{x}^T\mathbf{r}_s < r^{ref} \end{cases} \tag{19}$$

where r^{ref} is a reference rate of return for the investment, α denotes the reward parameter, β is the downside risk parameter and γ represents the loss aversion parameter, with $\gamma > 0$. Some examples of γ type of utility function are displayed in Figure 1.

Since the upside deviations from the reference level can be considered as potential benefits, we adopt the terminology proposed by Cumova and Nawrocki (2014), according to which we have the following investor behaviours. If $\alpha > 1$, then she/he is said upside potential seeking since the higher the returns above the target, the greater her/his utility; if $0 < \alpha < 1$, the agent is said upside potential averse; the case $\alpha = 1$ represents potential neutrality. Analogously, the following attitudes toward risks can be identified in terms of the β values for the downside deviations from the reference level. If $\beta > 1$, then the investor

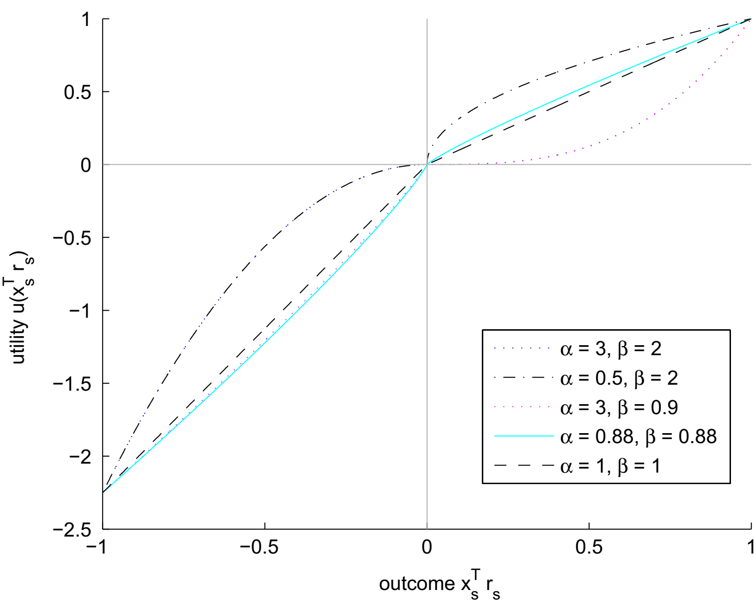

Figure 1. Plots of the utility function given by (19) for $\gamma = 2.25$ and various α, β values. Source: Calculated by the authors.

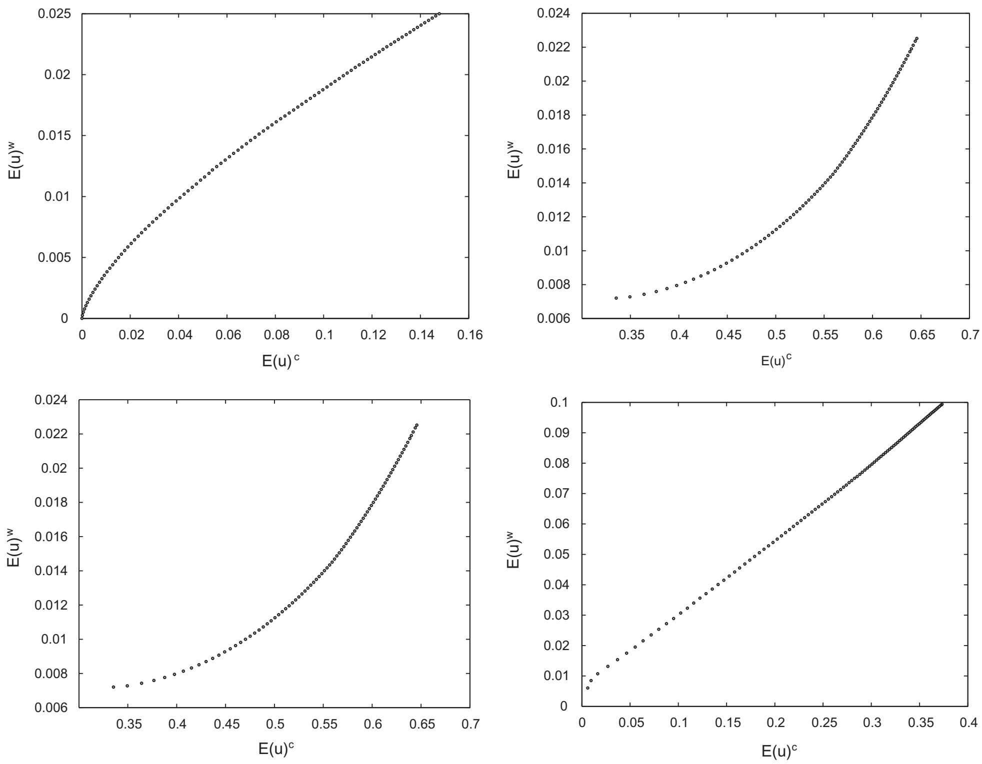

Figure 2. Plot of the non-dominated front found by M.O.E.A./D.-D.E. for each instance of Problem (28) related to investor behaviour. Source: Calculated by the authors.

is downside risk averse; if $0 < \beta < 1$, she/he is downside risk seeking; $\beta = 1$ indicates risk neutrality. The parameter γ controls the steepness of the utility function for losses: the greater γ and the steeper the curve.

Various risk–reward profiles can be represented by appropriately combining the values of these three parameters. The plots of some members of this class of utility functions are displayed in Figure 2, highlighting how the convexity–concavity shape changes by varying α and β according to the investor behaviour.

According to (19), investors are modelled as expected utility maximisers that formulate investment decisions according to the solution of the following nonlinear optimisation problem:

$$\begin{gathered} \max \mathbb{E}\left(u(\mathbf{x}|\mathbf{r}_1, \dots, \mathbf{r}_S; r^{ref})\right) = \sum_{s=1}^{S} p_s u\left(\mathbf{x}|\mathbf{r}_s; r^{ref}\right) \\ \text{s.t. } \mathbf{x} \in \mathcal{X} \subseteq R^n. \end{gathered} \tag{20}$$

4. The interval-based portfolio optimisation model

The market is the same as that described in the previous section with the same probability space $(\Omega, \mathfrak{F}, P)$. The novelty now is that rates of return are modelled by random intervals instead of random variables.

The interval counterparts of the financial quantities entering Problem (20) are to be established. More specifically, based on the treatment in Kaucic and Daris (2016), we can give the following definitions for the interval rate of return of a portfolio and for its expected value, respectively.

Definition 4.1: Let $\tilde{R}_i = \left[R_i^l, R_i^u\right]$ be the random interval rate of return of the i-th asset, $i = 1, \ldots, n$, with Aumann mean $\mathbb{E}(\tilde{R}_i) = \left[\mathbb{E}(R_i^l), \mathbb{E}(R_i^u)\right]$. The interval rate of return of the portfolio with weights $(x_1, \ldots, x_n)^T$ in the n-dimensional simplex $\mathcal{X}$ is the random interval

$$\tilde{R}^p = \left[R^{p,l}, R^{p,u}\right] \stackrel{def}{=} \sum_{i=1}^{n} x_i\, \tilde{R}_i = \left[\sum_{i=1}^{n} x_i R_i^l,\ \sum_{i=1}^{n} x_i R_i^u\right] \tag{21}$$

with Aumann mean given by

$$\mathbb{E}(\tilde{R}^p) = \left[\mathbb{E}\left(R^{p,l}\right), \mathbb{E}\left(R^{p,u}\right)\right] = \left[\sum_{i=1}^{n} x_i \mathbb{E}(R_i^l),\ \sum_{i=1}^{n} x_i \mathbb{E}(R_i^u)\right]. \tag{22}$$

Remark 8. If we remove the assumption of no short selling, (21) and (22) are not true in general due to (3).

We assume the investor operates her/his decisions on the basis of the following table of interval scenarios:

$$\begin{pmatrix} \tilde{\mathbf{r}}_1 & \cdots & \tilde{\mathbf{r}}_S \\ p_1 & \cdots & p_S \end{pmatrix} \quad \text{with} \ \sum_{s=1}^{S} p_s = 1 \text{ and } p_s \geq 0\ \forall s \tag{23}$$

where $\tilde{\mathbf{r}}_s = \left(\tilde{r}_{1s}, \ldots, \tilde{r}_{ns}\right)^T$ is the n-vector of interval rates of return for the s-th scenario, $s = 1, \ldots, S$, and p_s is the associated probability of occurrence for scenario s.

The portfolio interval rate of return under the s-th scenario can thus be defined as

$$\tilde{r}_s^p = \left[r_s^{p,l}, r_s^{p,u}\right] \stackrel{def}{=} \left[\sum_{i=1}^{n} x_i r_{is}^l,\ \sum_{i=1}^{n} x_i r_{is}^u\right]. \tag{24}$$

Similar to the case of random variables, it is easy to show that if $\mathbb{E}(R_i^l) = \sum_{s=1}^{S} p_s r_{is}^l$ and $\mathbb{E}(R_i^u) = \sum_{s=1}^{S} p_s r_{is}^u$, for $i = 1, \ldots, n$, the expected (in the Aumann's sense) interval rate of return of the portfolio in Eqn. (24) can be directly evaluated in terms of the scenarios as

$$\mathbb{E}(\tilde{R}^p) = \left[\sum_{s=1}^{S} p_s r_s^{p,l}, \sum_{s=1}^{S} p_s r_s^{p,u}\right]. \tag{25}$$

In this financial environment, the investor articulates her/his choices relative to an interval reference rate of return, $\tilde{r}^{ref} = [r^{ref,l}, r^{ref,u}]$, on the basis of the natural extension of the piecewise power function (19). Gains and losses are now defined in terms of the interval ranking (15): an interval rate of return is called a gain if it has a positive degree of acceptability to be preferred to the interval reference point; conversely, it represents a loss. Maintaining the same parameter setting of the utility function (19), its interval extension becomes

$$\hat{u}(\mathbf{x}|\,\tilde{\mathbf{r}}_s;\tilde{r}^{ref}) = \begin{cases} \left(\tilde{r}_s^p -_{gH} \tilde{r}^{ref}\right)^{\alpha}, & \text{if } \mathcal{A}\left(\tilde{r}^{ref} \preceq \tilde{r}_s^p\right) \geq 0 \\ -\gamma\left(\tilde{r}^{ref} -_{gH} \tilde{r}_s^p\right)^{\beta}, & \text{if } \mathcal{A}\left(\tilde{r}^{ref} \preceq \tilde{r}_s^p\right) < 0 \end{cases} \tag{26}$$

where $\tilde{r}_s^p$ is defined in (24).

Finally, the loss-averse investor that takes into account also imprecise forecasts for her/his investment decisions has to solve the following nonlinear interval-valued programming problem:

$$\begin{gathered} \max_{\leq_{cw}} \mathbb{E}\left(\hat{u}(\mathbf{x}|\,\tilde{\mathbf{r}}_1,\dots,\tilde{\mathbf{r}}_s;\tilde{r}^{ref})\right) = \sum_{s=1}^{S} p_s \hat{u}(\mathbf{x}|\,\tilde{\mathbf{r}}_s;\tilde{r}^{ref}) \\ \text{s.t. } \mathbf{x} \in \mathcal{X}. \end{gathered} \tag{27}$$

where 'max' is interpreted as the most preferred interval value for $\mathbb{E}\left(\hat{u}(\mathbf{x}|\,\tilde{\mathbf{r}}_1,\dots,\tilde{\mathbf{r}}_s;\tilde{r}^{ref})\right)$ with respect to the order relation (11).

Definition 4.2: A point $\mathbf{x}^* \in \mathcal{X}$ is an optimal solution of the interval optimisation problem (27) if there does not exist another point $\mathbf{x} \in \mathcal{X}$ such that $\mathbb{E}\left(\hat{u}(\mathbf{x}|\,\tilde{\mathbf{r}}_1,\dots,\tilde{\mathbf{r}}_s;\tilde{r}^{ref})\right) <_{cw} \mathbb{E}\left(\hat{u}(\mathbf{x}^*|\,\tilde{\mathbf{r}}_1,\dots,\tilde{\mathbf{r}}_s;\tilde{r}^{ref})\right)$.

Remark 9. If no imprecision is assumed in forecasts and reference point, Problem (27) reduces to Problem (20).

Next, we show that Problem (27) is equivalent to the bi-objective optimisation problem

$$\begin{gathered} \min\left(-\mathbb{E}\left(\hat{u}(\mathbf{x}|\,\tilde{\mathbf{r}}_1,\dots,\tilde{\mathbf{r}}_s;\tilde{r}^{ref})\right)^c, \mathbb{E}\left(\hat{u}(\mathbf{x}|\,\tilde{\mathbf{r}}_1,\dots,\tilde{\mathbf{r}}_s;\tilde{r}^{ref})\right)^w\right) \\ \text{s.t. } \mathbf{x} \in \mathcal{X} \end{gathered} \tag{28}$$

where $\mathbb{E}\left(\hat{u}(\mathbf{x}|\,\tilde{\mathbf{r}}_1,\dots,\tilde{\mathbf{r}}_s;\tilde{r}^{ref})\right)^c$ and $\mathbb{E}\left(\hat{u}(\mathbf{x}|\,\tilde{\mathbf{r}}_1,\dots,\tilde{\mathbf{r}}_s;\tilde{r}^{ref})\right)^w$ are the centre and the radius of the expected interval utility $\mathbb{E}\left(\hat{u}(\mathbf{x}|\,\tilde{\mathbf{r}}_1,\dots,\tilde{\mathbf{r}}_s;\tilde{r}^{ref})\right)$, respectively.

Definition 4.2: A point $\overline{\mathbf{x}} \in \mathcal{X}$ is a Pareto optimal solution of the bi-objective optimisation problem (28) if there does not exist another point $\mathbf{x} \in \mathcal{X}$ such that $\mathbb{E}\left(\hat{u}(\mathbf{x}|\,\tilde{\mathbf{r}}_1,\dots,\tilde{\mathbf{r}}_s;\tilde{r}^{ref})\right)^c \geq \mathbb{E}\left(\hat{u}(\overline{\mathbf{x}}|\,\tilde{\mathbf{r}}_1,\dots,\tilde{\mathbf{r}}_s;\tilde{r}^{ref})\right)^c$ and $\mathbb{E}\left(\hat{u}(\mathbf{x}|\,\tilde{\mathbf{r}}_1,\dots,\tilde{\mathbf{r}}_s;\tilde{r}^{ref})\right)^w \leq \mathbb{E}\left(\hat{u}(\overline{\mathbf{x}}|\,\tilde{\mathbf{r}}_1,\dots,\tilde{\mathbf{r}}_s;\tilde{r}^{ref})\right)^w$ with at least one strict inequality.

Proposition 4.4: $\mathbf{x}^* \in \mathcal{X}$ is an optimal solution of Problem (27) if and only if $\mathbf{x}^*$ is a Pareto optimal solution of Problem (28).

Proof. This proof is a direct consequence of the order relation $\leq_{cw}$ in $\mathcal{K}_c(R)$ and Definitions 4.2 and 4.3.

5. Illustrative example

5.1. Interval-valued scenario-generation process

Since no assumption regarding the distribution of asset rates of return is made, we adopt the scenario-generation process implemented in Zhu, Ji, and Li (2015) and Kaucic and Daris (2016) for the generation of imprecise forecasts. In particular, standard point-valued scenarios are first generated through a sampling procedure based on principal component analysis (P.C.A.). It works as follows:

Step 1. retain a sufficient number of principal components (P.C.) for each asset in order to capture most of the variability of historical samples;
Step 2. for each P.C., divide its range into several subintervals and define a point-valued scenario as the midpoint of the corresponding subinterval with the associated probability given by the ratio of the number of samples within that subinterval to the total number of samples;
Step 3. joint scenarios are the Cartesian product of scenarios of individual P.C.s;
Step 4. generate the scenarios of asset rates of return on the basis of an inverse linear transformation derived by P.C.A.

In the experiments we have fixed in Step 1 a lower threshold for the represented historical variability equal to 90.

Interval-valued scenarios are then obtained by applying to the point-valued scenarios a perturbation method. More specifically, let $\mathbf{r}_s = (r_{1s}, \dots, r_{ns})^T$ denote the n-vector of asset rates of return under the s-th scenario, $s = 1, \dots, S$, then the corresponding perturbed interval-valued scenario is defined as

$$\tilde{\mathbf{r}}_s = \left[r_{1s} - 1.96\frac{\hat{\sigma}_1}{\sqrt{T}}, r_{1s} + 1.96\frac{\hat{\sigma}_1}{\sqrt{T}} \right] \times \dots \times \left[r_{ns} - 1.96\frac{\hat{\sigma}_n}{\sqrt{T}}, r_{ns} + 1.96\frac{\hat{\sigma}_n}{\sqrt{T}} \right] \tag{29}$$

where $\hat{\sigma}_i$ is the standard deviation of rates of return for the i-th asset estimated by the T historical samples used to generate the traditional scenarios. In the M.R. form it can be compactly rewritten as

$$\tilde{\mathbf{r}}_s = \left(\mathbf{r}_s, 1.96\frac{\hat{\boldsymbol{\sigma}}}{\sqrt{T}} \right) \tag{30}$$

with $\hat{\boldsymbol{\sigma}} = (\hat{\sigma}_1, \dots, \hat{\sigma}_n)^T$. In this case, the interval portfolio rate of return under the s-th scenario (24) becomes

$$\tilde{\mathbf{r}}_s^p = \left(\sum_{i=1}^{n} x_i r_{is}, \frac{1.96}{\sqrt{T}} \sum_{i=1}^{n} x_i \hat{\sigma}_i \right) \tag{31}$$

for all $\mathbf{x} \in \mathcal{X}$ and $s = 1, \dots, S$.

5.2. Optimisation solver

For detecting optimal solutions to Problem (28), due to non-differentiability and non-concavity of the objectives, we use the so-called multi-objective evolutionary algorithm based on decomposition with differential evolution (M.O.E.A./D.-D.E.), a novel and robust stochastic technique that combines traditional mathematical programming and evolutionary computing. A general framework of M.O.E.A./D. is proposed in Qingfu Zhang and Hui Li (2007), while the effectiveness of including the differential evolution operator for some multi-objective optimisation problems with complicated Pareto set shapes is shown in Hui Li and Qingfu Zhang (2009).

The first step in the M.O.E.A./D.-D.E. is decomposing the multi-objective problem into a number of scalar optimisation sub-problems through the Tchebycheff approach. These sub-problems are then optimised by exploiting the neighbourhood relationship among them. A population is composed by the best solutions found so far for each sub-problem. The DE/best/1/bin operator and polynomial mutation are used to generate new solutions. For each sub-problem i, choose its current solution $\mathbf{x}^i$ as the first mating parent $\mathbf{x}^{r_1}$, and randomly select two other different mating parents $\mathbf{x}^{r_2}$ and $\mathbf{x}^{r_3}$ from the neighbourhood pool. An intermediate solution $\mathbf{y}^i = (y_1^i, \dots, y_n^i)^T$ is produced as follows:

$$\mathbf{y}^i = \begin{cases} \mathbf{x}^{r_1} + F(\mathbf{x}^{r_2} - \mathbf{x}^{r_3}), & \text{if } rand \leq CR \\ \mathbf{x}^{r_1}, & \text{otherwise} \end{cases}$$

where $F \in (0, 1]$ is a constant, called scaling factor, which controls the amplification of the differential variation $\mathbf{x}^{r_2} - \mathbf{x}^{r_3}$, CR is a crossover probability and $rand$ is a uniform random number from $[0, 1]$. A new solution $\hat{\mathbf{x}}^i = (\hat{x}_1^i, \dots, \hat{x}_n^i)^T$ is generated by polynomial mutation as

$$\hat{x}_k^i = \begin{cases} y_k^i + \sigma_k, & \text{if } rand \leq p_m \\ y_k^i, & \text{otherwise} \end{cases}$$

with

$$\sigma_k = \begin{cases} (2 \times rand)^{\frac{1}{\eta+1}} - 1, & \text{if } rand < 0.5 \\ 1 - (2 - 2 \times rand)^{\frac{1}{\eta+1}} - 1, & \text{otherwise} \end{cases}$$

where η is a control parameter and p_m is the mutation probability. A repair procedure on each offspring is then applied to guarantee its feasibility as follows:

(1) Each vector $\hat{\mathbf{x}}^i$ is clamped by projecting it onto the interval $[0, 1]$:

$$\tilde{x}_k^i = \begin{cases} 0, & \text{if } x_k^i < 0 \\ 1, & \text{if } x_k^i > 1 \\ \hat{x}_k^i, & \text{otherwise} \end{cases}$$

with $i = 1, \dots, n$. In this manner, $\tilde{\mathbf{x}} = (\tilde{x}_1, \dots, \tilde{x}_n)^T$ satisfies short-selling constraints.

(2) The projected vector $\tilde{\mathbf{x}} \in [0, 1]^n$ is now normalised through the transformation

$$\tilde{\tilde{x}}_i = \frac{\tilde{x}_i}{\sum_{j=1}^{n} \tilde{x}_j}, \quad i = 1, \dots, n.$$

After this step, the individual $\tilde{\tilde{\mathbf{x}}} = (\tilde{\tilde{x}}_1, \dots, \tilde{\tilde{x}}_n)^T$ also verifies the budget constraint.

The parameter setting of the M.O.E.A./D.-D.E. used in our experiments is listed in Table 1.

Table 1. Parameter setting for the implemented M.O.E.A./D.-D.E.

Parameter name	Value
Generations	1000
Population size	100
Neighbourhood size	5
Scaling factor F	0.50
Crossover probability CR	0.50
Mutation probability p_m	1/31
Distribution parameter η	20

Source: Calculated by the authors.

Table 2. List of assets included in the investible universe.

	Name	Code
1	A2A S.p.A.	A2A.MI
2	Atlantia S.p.A.	ATL.MI
3	Banca Generali S.p.A.	BGN.MI
4	Banca Mediolanum	BMED.MI
5	Banca Pop Emilia Romagna	BPE.MI
6	Buzzi Unicem	BZU.MI
7	CNH Industrial NV	CNHI.MI
8	Davide Campari-Milano S.p.A.	CPR.MI
9	ENEL Ente Nazionale per l'Energ Elet S.p.A.	ENEL.MI
10	Eni S.p.A.	ENI.MI
11	EXOR S.p.A.	EXO.MI
12	Fiat Chrysler Automobiles NV	FCA.MI
13	Brembo S.p.A.	BRE.MI
14	Assicurazioni Generali	G.MI
15	Intesa Sanpaolo	ISP.MI
16	Leonardo S.p.A.	LDO.MI
17	Luxottica Group S.p.A.	LUX.MI
18	Mediaset	MS.MI
19	Mediobanca	MB.MI
20	Moncler S.p.A.	MONC.MI
21	Prysmian	PRY.MI
22	Recordati S.p.A.	REC.MI
23	Salvatore Ferragamo S.p.A.	SFER.MI
24	Snam S.p.A.	SRG.MI
25	STMicroelectronics NV	STM.MI
26	Terna S.p.A.	TRN.MI
27	Unione di Banche Italiane S.p.A.	UBI.MI
28	UniCredit S.p.A.	UCG.MI
29	Unipol Gruppo Finanziario S.p.A.	UNI.MI
30	UnipolSai S.p.A.	US.MI
31	YOOX NETAPORT	YNAP.MI

Source: Calculated by the authors on the basis of data from http://it.finance.yahoo.com.

5.3. Data description and interval scenarios

The experiments have been based on data relative to the F.T.S.E. M.I.B. Index, which captures approximately 80% of the domestic Italian market capitalisation by replicating the broad sector weights of the Italian stock market. The investment universe comprises the 31 assets reported in Table 2. The time series include weekly closing prices covering the period from 1 January 2014 to 26 December 2016 for a total of 157 observations. The quotations are taken from http://it.finance.yahoo.com.

The descriptive statistics for the time series of rates of return are given in Table 3 from column 2 to column 5. As can be noted from the second and third columns, all assets have

Table 3. Descriptive statistics and Jarque–Bera test statistics with corresponding *p*-values for the weekly rates of return series from 6 January 2014 to 31 December 2016 for a total of 156 observations.

	Mean	Std. Dev.	Skewness	Kurtosis	Jarque–Bera	*p*-value
A2A.MI	0.0031	0.0356	−0.2428	2.8211	1.7403	0.3548
ATL.MI	0.0023	0.0294	−0.0791	3.6427	2.8473	0.1795
BGN.MI	0.0010	0.0455	0.6277	6.0535	70.8459	0.0010*
BMED.MI	0.0014	0.0414	−0.3851	3.7487	7.4990	0.0284*
BPE.MI	0.0007	0.0662	0.2502	3.4315	2.8373	0.1806
BZU.MI	0.0044	0.0434	−0.0595	3.3552	0.9123	0.5000
CNHI.MI	0.0009	0.0428	−0.0162	3.6619	2.8541	0.1788
CPR.MI	0.0032	0.0305	−0.1739	3.1354	0.9056	0.5000
ENEL.MI	0.0023	0.0325	−0.4917	3.8891	11.4256	0.0115*
ENI.MI	−0.0001	0.0358	−0.0431	3.5567	2.0629	0.2934
EXO.MI	0.0029	0.0405	0.1520	7.0872	109.1864	0.0010*
FCA.MI	0.0058	0.0574	0.1196	4.2558	10.6236	0.0136*
BRE.MI	0.0077	0.0428	0.4469	4.9616	30.2049	0.0010*
G.MI	−0.0005	0.0366	0.0733	4.1330	8.4832	0.0219*
ISP.MI	0.0029	0.0459	0.0048	3.1043	0.0712	0.5000
LDO.MI	0.0067	0.0461	0.1883	3.2684	1.3903	0.4366
LUX.MI	0.0022	0.0305	−0.3544	4.3488	15.0906	0.0060*
MS.MI	0.0027	0.0573	0.7469	6.6260	99.9679	0.0010*
MB.MI	0.0025	0.0506	−0.2656	2.6656	2.5605	0.2135
MONC.MI	0.0009	0.0415	0.0248	3.8083	4.2630	0.0854
PRY.MI	0.0025	0.0377	−0.4871	3.9654	12.2274	0.0098
REC.MI	0.0068	0.0353	−0.0412	3.4094	1.1333	0.5000
SFER.MI	−0.0003	0.0464	0.1560	3.1112	0.7128	0.5000
SRG.MI	0.0001	0.0321	−1.9069	16.7622	1325.6337	0.0010*
STM.MI	0.0053	0.0501	−0.3706	4.0597	10.8710	0.0129*
TRN.MI	0.0015	0.0248	−0.1090	4.9078	23.9671	0.0018*
UBI.MI	0.0010	0.0094	−0.3988	3.3058	4.0737	0.0897
UCG.MI	−0.0026	0.0600	0.1483	3.4468	1.8695	0.3293
UNI.MI	−0.0003	0.0517	−0.1060	2.8682	0.4049	0.5000
US.MI	−0.0004	0.0407	0.0249	3.2256	0.3470	0.5000
YNAP.MI	0.0001	0.0591	0.7301	6.6002	98.1075	0.0010*

The symbol *indicates that the null hypothesis of normally distributed data can be rejected at the 5% level.
Source: Calculated by the authors on the basis of data from http://it.finance.yahoo.com/.

a mean value ranging from −26 basis points of U.C.G.M.I to 77 basis points of B.R.E.M.I, with a volatility between 0.0094 of U.B.I.M.I and 0.06642 of B.P.E.M.I. From an analysis of the skewness and kurtosis values reported in the fourth and fifth columns, respectively, it emerges that about half of the assets present negative skewness or kurtosis larger than three, indicating non-normal distributions and pronounced downside risk. These findings are then confirmed by the Jarque–Bera test statistics and the corresponding p-values reported in the last two columns of the same table.

On the basis of these data, we have obtained 15 interval-valued scenarios for each asset by means of the procedure described in subsection 5.1. In particular, to offer a larger degree of freedom in modelling uncertainty, we allow the possibility that intervals related to a given asset may be overlapping. Table 4 provides these imprecise forecasts for the three scenarios with the highest probability of occurrence.

5.4. Interval-based utility maximisation problems

We test the proposed model in terms of both flexibility and efficiency to represent investor attitudes toward risk and profit when impreciseness is included in the forecasts. In line with Cumova and Nawrocki (2014), the following risk–reward profiles have been considered:

Table 4. Interval scenarios with the highest probability of occurrence.

	Scenario 1	Scenario 2	Scenario 3
Probability	0.16	0.63	0.11
A2A.MI	[-0.0024, 0.0088]	[-0.0046, 0.0066]	[-0.0069, 0.0043]
ATL.MI	[0.0450, 0.0542]	[0.0791, 0.0884]	[0.1148, 0.1241]
BGN.MI	[0.0185, 0.0328]	[0.0369, 0.0513]	[0.0562, 0.0705]
BMED.MI	[0.0535, 0.0666]	[0.1265, 0.1396]	[0.2027, 0.2158]
BPE.MI	[-0.0413, −0.0205]	[-0.0665, −0.0456]	[-0.0927, −0.0718]
BZU.MI	[-0.0826, −0.0689]	[-0.0859, −0.0723]	[-0.0894, −0.0757]
CNHI.MI	[-0.0719, −0.0584]	[-0.0614, −0.0480]	[-0.0505, −0.0371]
CPR.MI	[-0.0893, −0.0797]	[-0.1009, −0.0913]	[-0.1129, −0.1033]
ENEL.MI	[-0.0569, −0.0467]	[-0.0710, −0.0607]	[-0.0857, −0.0754]
ENI.MI	[-0.0481, −0.0368]	[-0.0365, −0.0252]	[-0.0244, −0.0131]
EXO.MI	[0.0751, 0.0879]	[0.0677, 0.0804]	[0.0599, 0.0726]
FCA.MI	[-0.0768, −0.0587]	[-0.0521, −0.0340]	[-0.0263, −0.0082]
BRE.MI	[-0.0768, −0.0633]	[-0.0694, −0.0559]	[-0.0617, −0.0482]
G.MI	[0.0543, 0.0658]	[0.0929, 0.1044]	[0.1332, 0.1448]
ISP.MI	[0.0377, 0.0522]	[0.0078, 0.0223]	[-0.0233, −0.0089]
LDO.MI	[0.1739, 0.1884]	[0.1484, 0.1629]	[0.1218, 0.1363]
LUX.MI	[-0.1452, −0.1356]	[-0.1327, −0.1231]	[-0.1197, −0.1101]
MS.MI	[0.0826, 0.1006]	[-0.0061, 0.0119]	[-0.0988, −0.0807]
MB.MI	[0.3647, 0.3807]	[0.2813, 0.2972]	[0.1941, 0.2100]
MONC.MI	[0.2743, 0.2874]	[0.2146, 0.2277]	[0.1522, 0.1653]
PRY.MI	[-0.1521, −0.1402]	[-0.1299, −0.1180]	[-0.1067, −0.0948]
REC.MI	[0.1316, 0.1427]	[0.0494, 0.0605]	[-0.0365, −0.0253]
SFER.MI	[0.0203, 0.0349]	[-0.0013, 0.0133]	[-0.0238, −0.0092]
SRG.MI	[-0.0739, −0.0637]	[-0.0737, −0.0636]	[-0.0735, −0.0634]
STM.MI	[0.1316, 0.1473]	[0.2122, 0.2279]	[0.2964, 0.3121]
TRN.MI	[-0.0604, −0.0526]	[-0.0689, −0.0611]	[-0.0778, −0.0700]
UBI.MI	[-0.1732, 0.0223]	[-0.2317, −0.0362]	[-0.2928, −0.0972]
UCG.MI	[0.0576, 0.0765]	[-0.0255, −0.0066]	[-0.1122, −0.0934]
UNI.MI	[-0.0485, −0.0323]	[-0.0765, −0.0602]	[-0.1057, −0.0894]
US.MI	[-0.1307, −0.1179]	[0.0562, 0.0691]	[0.2514, 0.2642]
YNAP.MI	[-0.1115, −0.0928]	[-0.0481, −0.0295]	[0.0181, 0.0367]

Source: Calculated by the authors on the basis of data from. http://it.finance.yahoo.com/.

Table 5. Parameter setting for the interval utility function (26).

Investor profile	α	β	γ	$\tilde{r}^{ref}$
1	3	2	2	[0, 0.0001]
2	0.5	2	2	[0, 0.0001]
3	3	0.9	2	[0, 0.0001]
4	0.88	0.88	2.25	[0, 0.0001]

Source: Calculated by the authors.

(1) upside potential seeking and downside risk aversion;
(2) upside potential aversion and downside risk aversion;
(3) upside potential seeking and downside risk seeking;
(4) upside potential aversion and downside risk seeking.

The numerical comparisons of these investment behaviours have been done with the parameter settings for the interval utility function (26) reported in Table 5. The first parameterisation represents the most common situation where an agent wish to reduce downside risk and try to take profit from potential upside movements. The second case corresponds to an investor everywhere risk averse. The third investment profile mimics an investor who likes exposure to high profits and accepts exposure to low returns. Finally, the representative of the fourth category is an individual enacting her/his decision in accordance with P.T. principles but avoiding using subjective decision weights and maximising expected values instead of prospective values (see Kahneman and Tversky (1979) for an introduction to the theory and Kaucic and Daris (2016) for an implementation in portfolio optimisation with interval forecasts).

As a result of Proposition 4.4, the solution of Problem (27) is not unique but a set of non-dominated solutions calculated on the basis of Problem (28). Thus, an approximation set is generated by the M.O.E.A./D.-D.E. for each investor profile. The corresponding dotted representations in the objective space are displayed in Figure 2. Note that for all the considered risk–reward profiles, impreciseness, represented by the width of the expected interval utility, increases as expectations, given by the centre of the expected interval utility, grow upwards.

To discriminate among the non-dominated solutions, we use the diversification index (D.I.) proposed by Woerheide and Persson (1993) and defined as

$$\mathrm{DI} = 1 - \sum_{i=1}^{n} x_i^2$$

Table 6. Diversification values and corresponding optimal portfolio compositions for the investor risk–reward attitudes considered.

	Profile 1	Profile 2	Profile 3	Profile 4
DI	0.8044	0.5097	0.9452	0.9329
$\mathbb{E}(\tilde{R}^p)$	[0.1299, 0.1429]	[0.2355, 0.2500]	[0.0000, 0.0164]	[-0.0059, 0.0067]
A2A.MI	0.0004	0.0003	0.0330	0.0329
ATL.MI	0.0757	0.0000	0.0148	0.0232
BGN.MI	0.0006	0.0003	0.0315	0.0000
BMED.MI	0.1364	0.0049	0.0290	0.0202
BPE.MI	0.0011	0.0001	0.0447	0.0000
BZU.MI	0.0001	0.0000	0.0673	0.0086
CNHI.MI	0.0040	0.0000	0.0146	0.0429
CPR.MI	0.0002	0.0004	0.0263	0.1188
ENEL.MI	0.0000	0.0000	0.0469	0.0903
ENI.MI	0.0000	0.0000	0.0290	0.0449
EXO.MI	0.2988	0.0002	0.0252	0.1334
FCA.MI	0.0005	0.0002	0.0305	0.0283
BRE.MI	0.0003	0.0000	0.0402	0.0650
G.MI	0.0300	0.0000	0.0500	0.0814
ISP.MI	0.0001	0.0001	0.0142	0.0135
LDO.MI	0.0209	0.0001	0.0384	0.0294
LUX.MI	0.0036	0.0003	0.0323	0.0145
MS.MI	0.0017	0.0002	0.0135	0.0073
MB.MI	0.0415	0.4967	0.0137	0.0139
MONC.MI	0.2622	0.4936	0.0767	0.0174
PRY.MI	0.0011	0.0000	0.0051	0.0045
REC.MI	0.0008	0.0001	0.0102	0.0167
SFER.MI	0.0000	0.0000	0.0200	0.0282
SRG.MI	0.0008	0.0009	0.0124	0.0398
STM.MI	0.1002	0.0009	0.0047	0.0083
TRN.MI	0.0037	0.0003	0.0606	0.0096
UBI.MI	0.0000	0.0000	0.0152	0.0000
UCG.MI	0.0016	0.0000	0.0440	0.0136
UNI.MI	0.0011	0.0001	0.0598	0.0369
US.MI	0.0104	0.0004	0.0553	0.0000
YNAP.MI	0.0022	0.0001	0.0409	0.0565

Source: Calculated by the authors.

where x_i is the weight of the i-th asset in the portfolio. A D.I. value equal to 0 represents a portfolio with absolutely no diversification while the maximum diversification is attained at 1. Thus, greater values of D.I. are advisable.

Table 6 reports the assets weights in the optimal portfolios identified by maximising the D.I. along the approximation set of each investment behaviour. The portfolios associated with investors belonging to the third and fourth profiles are the most diversified, with a value of D.I. almost equal to 95\. Conversely, the investor with upside potential aversion and downside risk aversion would select the least diversified portfolio, with a D.I. of almost 51\ and a proportion of 98\ of the capital evenly distributed between two assets, namely M.B.M.I. and M.O.N.C.M.I. However, focusing on the Aumann mean rates of return, we can see that the first two profiles guarantee the best results, with expected values in [0.1299, 0.1429] and in [0.2355, 0.2500] respectively. These findings confirm the importance of taking into account diversification for reducing risks and providing protection against extreme events, but also suggest that diversification alone is not adequate to be used as a task in portfolio selection. It has to be considered in conjunction with other criteria, such as cardinality, or with another set of constraints for the decision variables that better describe the decision maker's investment plan.

In general, we can conclude that the developed approach is sufficiently flexible to be adapted to a wide range of utility functions and, at the same time, the implementation of the M.O.E.A./D. permits the handling of large-scale problems involving nonlinear objective functions with reduced computational costs.

6. Conclusions

Expected utility theory is a valuable paradigm for representing the behaviour of decision makers in the context of risk and uncertainty. We extend this approach for portfolio optimisation problems involving imprecise/incomplete information about asset rates of return in the form of random sets. Crisp scenarios are thus substituted by interval scenarios. Investors are assumed to allocate their portfolios for a single period investment horizon according to an objective function composed by an interval-valued piecewise power function with rational exponents. The resulting interval-valued nonlinear optimisation problem is converted into a bi-objective nonlinear programming problem using a partial order relation between intervals and an interval-ranking approach, exploiting the decision maker's disposition to uncertainty. Due to non-differentiability and non-concavity of the objectives, a multi-objective evolutionary algorithm based on decomposition is used to solve this problem in a computationally tractable manner.

The proposed methodology is illustrated in a numerical example involving 31 assets from the Italian stock market index. Four risk–reward profiles are compared in terms of compositions, diversification and expected rates of return. The strategy with upside potential aversion and downside risk aversion produces portfolios with a low degree of diversification but, at the same time, high expected rates of return. As diversification increases, however, the capability to produce profits reduces. A good compromise is given by the profile of an investor that is upside potential seeking and downside risk averse. Agents acting investments according to P.T. seem to be not competitive in this context.

These findings are promising and, in order to better highlight strengths and weaknesses of impreciseness in behavioural portfolio management, we plan to carry on further analysis involving other classes of utility functions and order relations with constraints on both portfolio weights and risks.

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