The logarithmic mean is a mean

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Abstract. *The fact that the logarithmic mean of two positive numbers is a mean, that is, that it lies between those two numbers, is shown to have a number of consequences.*

Key words: *logarithmic mean, potential means, operator entropy*

Sažetak. Logaritamska sredina je sredina. *Činjenica da je logaritamska sredina dva pozitivna broja sredina, tj. da leži između ta dva broja ima niz primjena kako za potencijalne i generalizirane logaritamske sredine tako i u teoriji operatora.*

Ključne riječi: *logaritamska sredina, potencijalna sredina, operatorska entropija*

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1. Introduction

The logarithmic mean of positive numbers x, y is given by

$$L(x, y) = \frac{x - y}{\ln x - \ln y}$$

if $x \neq y$ and by $L(x, x) = x$ if $x = y$. It is a *mean*, that is, we have

$$\min(x, y) \leq L(x, y) \leq \max(x, y). \tag{1}$$

Here we shall show that this fact alone leads to a variety of interesting results.

Take $x \neq 1$. Setting $y = 1$ in (1) provides

$$\min(x, 1) \leq \frac{x - 1}{\ln x} \leq \max(x, 1),$$

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so that $\ln x \le x - 1 \le x \ln x$ for $x > 1$. Similarly, the same relation holds for $0 < x < 1$. In fact, equality occurs only for $x = 1$. So we have

$$\ln x \le x - 1 \le x \ln x \quad \text{for} \ \ x > 0. \tag{2}$$

Schaumberger [7] has shown that (2) provides a strikingly direct route to several results involving power means. We may use similar development to obtain results for integral means but with nonuniform rather than uniform weights.

In *Section 2* we derive a key inequality for potential means in this way. A discrete version is also given. In *Section 3* we note the existence of analogous results for other means which possess integral representations. Finally, in *Section 4*, we observe applications to the relative operator entropy of two positive operators on a Hilbert space.

2. Potential means

Let $f, w : [a, b] \to R$ be positive, integrable functions. The potential mean of order r of a function f with weight function w is given by

$$M_r(f, w) \ = \ \left\{ \frac{\displaystyle\int_a^b w(t) f(t)^r dt}{\displaystyle\int_a^b w(t) dt} \right\}^{1/r}, \quad r \neq 0,$$

$$M_0(f, w) \ = \ \exp \left\{ \frac{\displaystyle\int_a^b w(t) \ln f(t) dt}{\displaystyle\int_a^b w(t) dt} \right\}, \quad r = 0.$$

For convenience we write f for $f(t)$, w for $w(t)$, M for $M_r(f, w)$ $(r \neq 0)$ and M_0 for $M_0(f, w)$. Set $x = f^r / M^r$ in (2). We get after multiplication by wM^r that

$$rwM^r \ln \frac{f}{M} \le wf^r - wM^r \le rwf^r \ln \frac{f}{M}.$$

Integration gives

$$rM^r \int_a^b w \ln \frac{f}{M} \ dt \le 0 \le r \int_a^b wf^r \ln \frac{f}{M} \ dt,$$

that is, for $r > 0$,

$$M^r \int_a^b w \ln f \ dt - M^r \ln M \int_a^b w \ dt \le 0 \le \int_a^b wf^r \ln f \ dt - \ln M \int_a^b wf^r \ dt.$$

This is equivalent to

$$M^r \ln M_0 - M^r \ln M \le 0 \le \frac{\displaystyle\int_a^b wf^r \ln f \ dt}{\displaystyle\int_a^b w \ dt} - M^r \ln M$$

or

$$M^r \ln M_0 \le M^r \ln M \le \frac{\displaystyle\int_0^b w f^r \ln f \; dt}{\displaystyle\int_a^b w f^r \; dt} \cdot M^r$$

Therefore, we have

$$M_0 \le M \le M_0(f, wf^r),$$

that is, *in extenso*,

$$M_0(f, w) \le M_r(f, w) \le M_0(f, wf^r) \quad \text{for} \quad r > 0. \tag{3}$$

If $r < 0$, the inequalities are reversed.

Assuming all integrals exist, a consequence of (3) is that

$$\lim_{r \to 0} M_r(f, w) = M_0(f, w).$$

A similar development is available in the discrete case. If a, w are positive n–tuples, a potential mean of order r with weights w is given by

$$M_n^{[r]}(a, w) = \left\{ \frac{1}{W_n} \sum_{i=1}^{n} w_i a_i^r \right\}^{1/r}, \quad r \ne 0,$$

$$M_n^{[0]} = \left\{ \prod_{i=1}^{n} a_i^{w_i} \right\}^{1/W_n},$$

where $W_n = \sum_{i=1}^{n} w_i$. With the notation $wa^r = (w_1 a_1^r, w_2 a_2^r, \ldots, w_n a_n^r)$ we can obtain

$$M_n^{[0]}(a, w) \le M_n^{[r]}(a, w) \le M_n^{[0]}(a, wa^r),$$

which is a generalization of a result from [6] for the case of an unweighted mean. We have again that

$$\lim_{r \to 0} M_n^{[r]}(a, w) = M_n^{[0]}(a, w).$$

3. Analogues

We now consider the generalized logarithmic mean of order r of positive numbers x, y. For $x \ne y$ this is defined by

$$L_r(x, y) = \left\{ \frac{x^r - y^r}{r(x - y)} \right\}^{1/(r-1)}, \quad r \ne 0, 1,$$

$$L_0(x, y) = L(x, y),$$

$$L_1(x, y) = I(x, y) = \frac{1}{e} \left(x^x / y^y \right)^{1/(x-y)}$$

and for $x = y$ by $L_r(x,x) = x$.

These means have an integral representation

$$L_r(x,y) = M_{r-1}(e_1, e_0),$$

where $e_1(t) = t$, $e_0(t) = 1$ for all $t \in [\min(x,y), \max(x,y)]$. So, a consequence of (3) for $r > 1$ is that

$$I(x,y) \le L_r(x,y) \le I(x^r, y^r)^{1/r}. \tag{4}$$

Reverse inequalities apply for $r < 1$ ($r \neq 0$), while for $r = 0$ we have

$$G(x,y) \equiv \sqrt{xy} \le L(x,y) \le I(x,y).$$

We have immediately from (4) that

$$\lim_{r\to 1} L_r(x,y) = I(x,y).$$

In place of the integral potential mean of the function $f : [a,b] \to R$ we could consider more general potential means of functions $g : \Omega \to R$, where Ω is an arbitrary set. For such means, (3) follows in the same way. As special cases we can again consider, for example, the logarithmic means on n variables (see Pittenger [5]), the hypergeometric mean (see Brenner and Carlson [1]) and other means which have representations in the form of integral means (see [1]).

4. Operator theory

A further application lies in the theory of operators. Fujii and Kamei [2] introduced the notion of *the relative operator entropy* $S(A|B)$ for positive operators A, B on a Hilbert space. For A and B invertible, this is given by

$$S(A|B) \;=\; A^{1/2}\left(\ln A^{-1/2}BA^{-1/2}\right)A^{1/2}.$$

Fujii and Kamei [3] proved that

$$A - AB^{-1}A \;\le\; S(A|B) \;\le\; B - A. \tag{5}$$

We can derive this (see e.g. [6] pp. 269-273) as a consequence of (2), which may be re–expressed as

$$1 - 1/x \;\le \ln x \;\le\; x - 1 \quad \text{for all } \; x > 0. \tag{6}$$

Since $A^{-1/2}BA^{-1/2}$ is a positive operator, we may substitute it for x to obtain

$$I - A^{1/2}B^{-1}A^{1/2} \le \ln A^{-1/2}BA^{-1/2} \le A^{-1/2}BA^{-1/2} - I.$$

Pre– and postmultiplication by the positive operator $A^{1/2}$ now provide (5).

Recently Furuta [4] proved a generalization of this inequality, namely, if A and B are positive invertible operators, then for any positive number x_0 we have

$$(\ln x_0 - 1)A + Bx_0^{-1} \;\ge\; S(A|B) \;\ge\; (1 - \log x_0)A - AB^{-1}Ax_0^{-1}. \tag{7}$$

We show that this follows from (6). For on substituting x/x_0 for x in (6) we obtain

$$\ln x_0 + 1 - (x_0/x) \ \leq \ln x \ \leq \ \ln x_0 + (x/x_0) - 1 \quad \text{for all positive } \ x, x_0.$$

(This inequality is equivalent to (1) for $y = x_0$). Again we replace x by the positive operator $A^{-1/2}BA^{-1/2}$ to derive

$$(\ln x_0 + 1)I - x_0 A^{1/2}B^{-1}A^{1/2} \leq \ln A^{-1/2}BA^{-1/2} \leq (\ln x_0 - 1)I + A^{-1/2}BA^{-1/2}x_0^{-1}.$$

Pre– and postmultiplication by $A^{1/2}$ as before yields

$$(\ln x_0 + 1)A - x_0 AB^{-1}A \leq S(A|B) \leq (\ln x_0 - 1)A + Bx_0^{-1}. \tag{8}$$

We can replace x_0 by x_0^{-1} to give

$$(1 - \ln x_0)A - x_0^{-1}AB^{-1}A \leq S(A|B) \leq Bx_0^{-1} - (\ln x_0 + 1)A. \tag{9}$$

The first inequality in (9) taken with the second in (8) provides (7).

References

[1] J. L. Brenner, B. C. Carlson, *Homogeneous mean values: weights and asymptotics*, J. Math. Anal. Appl. **123**(1987), 265–280.

[2] J. I. Fujii, E. Kamei, *Relative operator entropy in noncommutative information theory*, Math. Japon. **34**(1989), 341–348.

[3] J. I. Fujii, E. Kamei, *Uhlmann's interpolational method for operator means*, Math. Japon. **34**(1989), 541–547.

[4] T. Furuta, *Furuta's inequality and its application to the relative operator entropy*, J. Operator Theory **30**(1993), 21–30.

[5] A. O. Pittenger, *The logarithmic mean in n variables*, Amer. Math. Monthly **92**(1985), 99–104.

[6] F. Riesz, B. Sz-Nagy, *Functional analysis*, Translated from the 2nd French edition by Leo F. Boron, Frederick Ungar Pub. Co., 1955.

[7] N. Schaumberger, *Extending a familiar inequality*, Pi Mu Epsilon Journal **10**(1992), 384–385.