Inequalities for the internal angle-bisectors of a triangle

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Abstract. *Several new inequalities of type* $\sum w_a^{-2m} \geq \alpha_m \cdot \sum a^{\pm m}$ *for angle-bisectors are proved. Certain algebraic cyclic inequalities are derived. Two conjectures and an open question are mentioned.*

Key words: *geometric inequalities, angle-bisectors, cyclic inequalities.*

Sažetak. *Dokazano je nekoliko novih nejednakosti tipa* $\sum w_a^{-2m} \geq \alpha_m \cdot \sum a^{\pm m}$ *za simetrale kutova. Izvedene su neke cikličke nejednakosti. Spomenute su dvije pretpostavke i jedno otvoreno pitanje.*

Ključne riječi: *geometrijske nejednakosti, simetrale kutova, cikličke nejednakosti*

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1. Introduction

In this note we extend methods from [2] in order to get some further inequalities linking the angle-bisectors and other elements of a (planar) triangle. We also state some conjectures.

As usual, for triangles w_a, w_b, w_c denote the internal angle-bisectors and a, b, c, s, r and R the sides, the semiperimeter, the inradius and the circumradius, resp..

All sums appearing are cyclic. (e.g. $\sum bc$ means $ab + bc + ca$, etc.) Throughout this note we let $m > 0$ be a real number.

2. Some lemmata

In this section we shall establish triangle-inequalities of type $\sum a^m \geq k_m \cdot \sum (bc)^m$, where $k_m := k_m(s, R)$ is "best possible". [Due to the fact that for, say, c fixed and

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$a = b \to c/2$ we get $R \to \infty$, there are factors k_m favourable not depending on R.] We firstly prove the algebraic

Lemma 1. *Let x, y and z be nonnegative real numbers such that $x+y+z \neq 0$. Then*

$$\frac{xy+yz+zx}{x+y+z} \leq \frac{x+y+z}{3} \tag{1}$$

Proof. We have

$$\begin{aligned}(1) \quad &\Leftrightarrow \quad 3(xy+yz+zx) \leq (x+y+z)^2 \Leftrightarrow \quad xy+yz+zx \leq x^2+y^2+z^2 \\ &\Leftrightarrow \quad (x-y)^2+(y-z)^2+(z-x)^2 \geq 0 \square\end{aligned}$$

a) k_m is of type K_m/s^m, K_m best possible constant. Then there holds the following

Lemma 2. $K_m = (3/2)^m$ *for* $0 < m \leq 1$.
I.e., if $0 < m \leq 1$ *then* $\sum a^m \geq [3/(2s)]^m \cdot \sum (bc)^m$.

Proof. Starting from $(1/3) \cdot \sum [3a/(2s)] = 1$ we get via the general means-inequality $(1/3) \cdot \sum [3a/(2s)]^m \leq 1$. Putting in *Lemma 1* $x = [3a/(2s)]^m$ etc. we get

$$\frac{\sum [3b/(2s)]^m [3c/(2s)]^m}{\sum [3a/(2s)]^m} \leq (1/3) \cdot \sum [3a/(2s)]^m \leq 1$$

and thus $[3/(2s)]^m \cdot \sum (bc)^m \leq \sum a^m$, as claimed. □

Remark 1. *It remains an open question to determine the value of K_m in case $m > 1$. Computer-searches indicate the following*

Conjecture 1.. *Let* $\mu := \ln 2/(\ln 3 - \ln 2) = 1.709511291\ldots$. *Then*

$$K_m = \begin{cases} (3/2)^m, & \text{if } 0 < m \leq \mu \\ 2, & \text{if } m > \mu \end{cases}$$

b) k_m is of type L_m/R^m, L_m best possible constant. Then we have the following

Lemma 3. $L_m = (1/\sqrt{3})^m$ *for* $0 < m \leq \tau$, *where* $\tau := (\ln 9 - \ln 4)/(\ln 4 - \ln 3) = 2.8188416793\ldots$ *I.e., if* $0 < m \leq \tau$ *then* $\sum a^m \geq [1/(R\sqrt{3})]^m \cdot \sum (bc)^m$

Proof. [1] has as item 5.28 that the inequality $(1/3) \cdot \sum [a/(R\sqrt{3})]^m \leq 1$ holds true if and only if $0 < m \leq \tau$. The rest of the proof of *Lemma 3* is similar as for *Lemma 2*. □

Remark 2. *Again it remains an open question to determine L_m in case $m > \tau$. For this we state the following*

Conjecture 2.. *Let* $\sigma := \ln 4/(\ln 4 - \ln 3) = 4.8188416793\ldots$. *Then*

$$L_m = \begin{cases} (1/\sqrt{3})^m, & \text{if } 0 < m \leq \sigma \\ 1/2^{m-1}, & \text{if } m > \sigma \end{cases}$$

It should be noted that due to [1], item 5.3, i.e. $2s \leq 3R\sqrt{3}$, for $m \leq 1$ *Lemma 2* always yields the better factor k_m than *Lemma 3* (in case of validity of *Conjecture 1* even for $m \leq \mu$). On the other hand, if both conjectures are valid, for $m > \mu$ it depends on the shape of a triangle whether *Lemma* 2 or 3 yields the better factor k_m.

c) An all-over lower estimation of the factor $k_m = k_m(s, R)$. We shall now prove

Lemma 4. *$k_m > \max\{1/s^m, 1/(2R)^m\}$ for all $m > 0$.*
I.e., if $m > 0$ then $\sum a^m > \max\{1/s^m, 1/(2R)^m\} \cdot \sum(bc)^m$

Proof. Due to $a, b, c \le 2R$ and $a, b, c < s$ we get $a, b, c \le \min(s, 2R) =: M$ with at least one side $< M$. Hence $\sum(bc)^m < M^m \cdot \sum a^m$ and the claimed inequality follows. □

Remark 3. *In view of* a) *and* b) Lemma 4 *deserves attention if $m > 1$.*

3. The results

A) Case $\mathbf{m \ge 1}$. We are now in the position to prove the following

Theorem 1. *Let $m \ge 1$. Then*

$$\sum w_a^{-2m} \ge \left[\frac{s^2 + (4R + r)^2}{12Rrs^3}\right]^m \cdot \sum a^m \tag{2}$$

Proof. We start from the well-known formulae $w_a = \frac{2bc\cos(A/2)}{b+c}$, etc. Since inequality (2) is symmetric with respect to a, b and c we may and do let $a \le b \le c$. Then

i) $1/c \le 1/b \le 1/a$ and further $1/b + 1/c \le 1/c + 1/a \le 1/a + 1/b$ and

ii) $A \le B \le C$ and thus $1/\cos(A/2) \le 1/\cos(B/2) \le 1/\cos(C/2)$.

Using Chebyshev's and the arithmetic-geometric inequalities we get

$$\begin{aligned}\sum w_a^{-2m} &= \sum \cos^{-2m}(A/2) \cdot \left[\tfrac{1}{2}\left(\tfrac{1}{b} + \tfrac{1}{c}\right)\right]^{2m} \ge \tfrac{1}{3} \sum \cos^{-2m}(A/2) \cdot \sum \left[\tfrac{1}{2}\left(\tfrac{1}{b} + \tfrac{1}{c}\right)\right]^{2m} \\ &\ge \tfrac{1}{3} \sum \cos^{-2m}(A/2) \cdot \sum \left[\tfrac{1}{bc}\right]^m\end{aligned} \tag{3}$$

The general means-inequality and [3], p. 35, yield

$$\frac{1}{3}\sum \cos^{-2m}(A/2) \ge \left[\frac{1}{3}\sum \cos^{-2}(A/2)\right]^m = \left[\frac{s^2 + (4R + r)^2}{3s^2}\right]^m$$

This inequality and the observation

$$\sum \left[\frac{1}{bc}\right]^m = \frac{\sum a^m}{(abc)^m} = \frac{\sum a^m}{(4Rrs)^m}$$

immediately via (3) lead to the claimed inequality (2). □

Applying *Lemma 3* we get from (2) the following

Corollary 1. *Let $1 \le m \le \tau$. Then*

$$\sum w_a^{-2m} \ge \left[\frac{s^2 + (4R + r)^2}{3s^2R\sqrt{3}}\right]^m \cdot \sum a^{-m}. \tag{4}$$

Using [1], items 5.5 and 5.12, i.e. $(4R + r)^2 \ge 3s^2$ and $Rr \le 2s^2/27$, resp., we get from (4) and (2) the further

Corollary 1'.

$$\sum w_a^{-2m} \geq \left[\frac{4}{3R\sqrt{3}}\right]^m \cdot \sum a^{-m}, \tag{5}$$

where $1 \leq m \leq \tau$
and

Corollary 2.

$$\sum w_a^{-2m} \geq \left[\frac{1}{3Rrs}\right]^m \cdot \sum a^m \geq \left[\frac{9}{2s^3}\right]^m \cdot \sum a^m, \tag{6}$$

where $m \geq 1$.

Remark 4. *For $m = 1$ we get from (2) and (6) the following improvement (interpolation) of [4], p. 217, item 11.1:*

$$\sum w_a^{-2} \geq \frac{s^2 + (4R+r)^2}{6Rrs^2} \geq \frac{2}{3Rr} \geq \frac{9}{s^2}.$$

B) Case $\mathbf{0 < m < 1}$ ($\mathbf{0 < m \leq 1}$). In this case there can hold only a (clearly) weaker inequality stated as

Theorem 2.

$$\sum w_a^{-2m} \geq \left[\frac{1}{4Rr^3s^5}\right]^{m/3} \cdot \sum a^m$$

Proof. Proceeding as for *Theorem 1* we get from (3) and [3], p. 35, via the arithmetic-geometric inequality

$$\sum w_a^{-2m} \geq \left[\frac{1}{\Pi\cos(A/2)}\right]^{\frac{2m}{3}} \cdot \sum\left[\frac{1}{bc}\right]^m = \left(\frac{4R}{s}\right)^{\frac{2m}{3}} \cdot \frac{\sum a^m}{(abc)^m},$$

and the claimed inequality follows. □

We now add some consequences of *Theorem 2*. Via *Lemma 2* we get

Corollary 3.

$$\sum w_a^{-2m} \geq \left(\frac{54R^2}{s^5}\right)^{\frac{m}{3}} \cdot \sum a^{-m} \tag{7}$$

Using once more $3R\sqrt{3} \geq 2s$ inequality (7) leads us to

Corollary 4.

$$\sum w_a^{-2m} \geq \left(\frac{2}{s}\right)^m \cdot \sum a^{-m}$$

Remark 5. *The above proofs immediately imply that equality holds in either of the inequalities if and only if $a = b = c$.*

4. Algebraic analoga of inequalities (6) and (7)

In this section we will apply the useful transformation $a = x_2 + x_3$, $b = x_3 + x_1$ and $c = x_1 + x_2$, where x_1, x_2 and x_3 are arbitrary positive real numbers (see e.g. [1], chapt. II). It always links geometric and algebraic (three-variable) inequalities.

Then upon normalizing $(s =)\ x_1 + x_2 + x_3 = 1$ short simplifications yield $w_a^{-1} = (1 + x_1)/(2\sqrt{x_1(1 - x_2)(1 - x_3)})$, etc. and $R = abc/4F = (1 - x_1)(1 - x_2)(1 - x_3)/(4\sqrt{x_1x_2x_3})$. Hence we get the following algebraic analoga of *Corollaries* 1' and 3

$$\sum_{k=1}^{3}\left[(1+x_k)^2\frac{1-x_k}{x_k}\right]^m \geq \left(\frac{4}{\sqrt{3}}\right)^m \cdot (\sqrt{x_1x_2x_3})^m \cdot \sum_{k=1}^{3}\frac{1}{(1-x_k)^m},$$

where $1 \leq m \leq \tau$,
and

$$\sum_{k=1}^{3}\left[(1-x_k)\frac{(1+x_k)^2}{x_k}\right]^m \geq 6^m \cdot \left\{\prod_{k=1}^{3}\left[\frac{1-x_k}{x_k}\right]\right\}^{\frac{m}{3}} \cdot \sum_{i<j}(1-x_i)^m(1-x_j)^m$$

where $0 < m < 1$ $(0 < m \leq 1)$.

Both inequalities lead to the following

Open Questions. Determine the respective sets A and B of all natural numbers n such that the n-dimensional analoga are valid with "$\geq$" throughout all $x_1, ..., x_n \in (0, 1)$, $x_1 + ... + x_n = 1$, and similarly "$\leq$", resp., and the respective constant factors are replaced by the suitable constants yielding equality at $x_1 = ... = x_n$ $(= 1/n)$.

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