# Linear matrix inequalities*

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**Abstract**. *The purpose of the paper is a popularisation of linear matrix inequalities.*

**Key words:** *linear matrix inequalities, convex optimization, system theory*

**Sažetak**. **Linearne matrične nejednadžbe**. *Namjera ovog članka je popularizacija linearnih matričnih nejednadžbi.*

**Ključne riječi:** *linearne matrične nejednadžbe, konveksna optimizacija, teorija sustava*

We use the notation $\sum^k$ for the set of symmetric matrices in $\mathbb{R}^{k\times k}$ and also use the notation $\sum_+^k$ for the set of positive – definite matrices in $\sum^k$. For $\mathbf{A}, \mathbf{B} \in \sum^k$ we write $\mathbf{A} > \mathbf{B}$ if $\mathbf{A} - \mathbf{B} \in \sum_+^k$.

A linear matrix inequality (LMI) is the formula

$$F(\mathbf{x}) > 0 \tag{1}$$

where $F : \mathbb{R}^m \to \sum^k$ is an affine mapping. A convex set $\sum_+^k$??? is the set of all solutions of (1) and the theory of LMI is based on the convex analysis.

The mapping $F$ has the form

$$F(\mathbf{x}) = A_0 + \sum_{i=1}^{m} x_i A_i,$$

for some $A_i \in \sum^k$, $i = 1, \ldots, m$, and for the variable $\mathbf{x} = [x_1, \ldots, x_m]^T \in \mathbb{R}^m$. Thus, LMI (1) is equivalent to

$$A_0 + \sum_{i=1}^{m} x_i A_i > 0. \tag{2}$$

The fact that $F(\mathbf{x}) \in \sum_+^k$ if and only if the leading principal minors of $F(x)$ are positive implies that (2) is equivalent to a set of polynomial inequalities in $\mathbf{x}$.

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Numerical methods for solving LMI, such as an ellipsoid algorithm and interior-point methods (method of centers, primar-dual methods, projective methods of Nemirovsky), are methods of convex optimization but with very important inprovements. It is possible to say that in the last ten years there exists numerical methods (and also software packages) which are extremely efficient in practice.

To illustrate applications of LMI we cite

**Theorem 1.** *For a matrix $\mathbf{A} \in \mathbb{R}^{n\times n}$ the following statements are equivalent*

1° *for every solution* $\mathbf{u} : [0,\infty) \to \mathbb{R}^n$ *of the differential equation* $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}$ *is* $\lim\limits_{t\to\infty} \mathbf{u}(t) = \mathbf{0}$*;*

2° *for every eigenvalue* $\nu$ *of* $\mathbf{A}$ *is* $\operatorname{Re}\nu < 0$*;*

3° *there exists* $\mathbf{P} \in \sum_+^n$ *such that* $-(\mathbf{A}^T\mathbf{P} + \mathbf{P}\mathbf{A}) \in \sum_+^n$.

The use of the implication $2° \Rightarrow 1°$ is classical–the famous Routh-Hurwitz or Nyquist criterion are both based on the implication, for some special matrix $\mathbf{A}$. The equivalence $1° \Rightarrow 3°$ is proposed by Lyapunov in 1890 and it is not difficult to see that 3° is a LMI. Indeed, if we use a basis $\{\mathbf{P}_1,\dots,\mathbf{P}_m\}$ in $\sum^n$, where $m = \dim\sum^n = \frac{1}{2}n(n+1)$, then $\mathbf{P} = \sum\limits_{i=1}^m x_i\mathbf{P}_i$ for $\mathbf{P}\in\sum^n$ and $\mathbf{P} > 0$, $\mathbf{A}^T\mathbf{P}+\mathbf{P}\mathbf{A} < 0$ if and only if $\mathbf{x} = [x_1,\dots,x_n]^T \in \mathbb{R}^n$ is a solution of (2) for $k = 2n$, $\mathbf{A}_0 = \mathbf{0}$ and

$$\mathbf{A}_i = \begin{bmatrix} \mathbf{P}_i & \mathbf{0} \\ \mathbf{0} & -\mathbf{A}^T\mathbf{P}_i - \mathbf{P}_i\mathbf{A} \end{bmatrix}$$

for $i = 1,\dots,m$.

Thanks to modern numerical methods it is possible now to use the implication $3° \Rightarrow 1°$ also in practice. Also, a wide variety of problems arising in system and control theory can be solved by using new efficient tool in numerical mathematics – softwares for linear matrix inequalities.

## References

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