Croatian Operational Research Review
CRORR **8**(2017), 489–498

A Hostile model for network reliability analysis

Daniel Lena[1], Franco Robledo[1] and Pablo Romero[1,*]

[1] *Facultad de Ingenería, Universidad de la República, PC 11300 Montevideo, Uruguay*
E-mail: ⟨{*daniel.lena,frobledo,promero*}*@fing.edu.uy*⟩

Abstract. In reliability analysis, the goal is to determine the probability of consistent operation of a system. We introduce the Hostile model, where the system under study is a network, and all the components may fail (both sites and links), except for a distinguished subset of sites, called terminals. The Hostile model includes the Classical Reliability model as a particular case. As a corollary, the exact reliability evaluation of a network in the Hostile model belongs to the list of $\mathcal{NP}$-hard computational problems.
Traditional methods for the classical reliability model such as Crude Monte Carlo, Importance Sampling and Recursive Variance Reduction are here adapted for the Hostile model. The performance of these methods is finally discussed using real-life networks.

Key words: network reliability, Hostile model, computational complexity

Received: June 2, 2016; accepted: August 13, 2017; available online: November 30, 2017

DOI: 10.17535/crorr.2017.0031

1. Introduction

Historically, network design was mainly driven by service availability. A major example is the public switched telephone network or PSTN, where the design is focused on connectivity aspects and high availability. The progress of the World Wide Web and digitalization produced a great impact in network design [2], where both quality of service and robustness under failures are mandatory.
In classical reliability analysis, the goal is to communicate distinct network sites, called terminals, and the only components under failures are the links between terminals. These issues are studied mathematically using random graphs, and the Gilbert model of independent link failures to capture its entire structure [3]. Arnie Rosenthal fully characterized the computational complexity proving that the exact reliability evaluation under the classical model is at least as hard as the Steiner Tree Problem in undirected graphs [14]. Given that the Steiner Tree Problem belongs to the Karp list of the $\mathcal{NP}$-Complete problems [11], Rosenthal proved that the exact reliability evaluation is thus $\mathcal{NP}$-hard. The specific literature on reliability analysis offers either exact exponential-time reliability methods or approximation methods. Monte Carlo methods show the most promising results, at least under the classical reliability model [8]. Even though Crude Monte Carlo provides a point-wise approximation for elementary parameters of a system, its inaccuracy under highly robust

*Corresponding author.

systems had led to the development of alternative methods [15]. In this work, we examine and modify two hypotheses of the classical model. The first one is that sites may fail. In real-life, we can find relevant examples such as fibre-optic communication, or a military defence network, that are certainly vulnerable. The second hypothesis is the use of Crude Monte Carlo as an approximation method for classical reliability.

In this article, we study the reliability of a Hostile model, where non-terminal sites may fail as well. Additionally, we study the reliability using three different statistical methods for highly robust systems. We perform a fair comparison between Crude Monte Carlo, Importance Sampling (IS) and Recursive Variance Reduction method (RVR) for the reliability evaluation under the Hostile model. The reasons to choose IS and RVR is that they present high performance under highly robust scenarios. The first method, RVR, received a distinction (i.e., a paper award) for its originally concept of network reliability [5]. On the other hand, the main concept of IS is to modify the probabilities, in such a way that the probability of rare events is increased. As a result, a zero-error is ideally produced. The concept is extended to determine the probability of an arbitrary rare event. Here we extend the concept of Important Sampling for our new model, to find the unreliability of a highly reliable system (which is precisely a rare event). The document is organized in the following manner. The Hostile model is presented in Section 2, as a particular case of a more abstract setting, called Stochastic Binary Systems (SBS). Section 3 presents Crude Monte Carlo, and its main weakness that promotes the development of alternative methods. In Section 4, RVR and IS are introduced for the Hostile model, in strict analogy with the classical model. Section 5 presents a fair comparison between the three approximation methods considered in this paper. Concluding remarks and trends for future work are covered in Section 6.

2. Hostile model

In this section the Hostile model is defined as a particular case of an SBS. We will adopt the terminology from Michael Ball [1]. An SBS is a triad (S, ϕ, p), where $S = \{s_1, \dots, s_r\}$ is a ground set of components, $\phi : \{0,1\}^r \to \{0,1\}$ is a function called rule, or structure, that assigns 1 to every operational subset of S, or 0 otherwise, and $p = (p_1, \dots, p_r)$ is a vector with the corresponding operational probabilities of the components, called elementary reliabilities.

Consider a random vector $X = (X_1, \dots, X_r)$ with independent coordinates, where each X_i is governed by a Bernoulli probability law with success $P(X_i = 1) = p_i$. The reliability R of an SBS is the probability that the random system is operational:

$$R = P(\phi(X) = 1) = E(\phi(X)) \tag{1}$$

Observe that $\phi(X)$ is a binary random variable that describes whether the system is operational or not. A stochastic binary system is monotonous if $\phi(S_1) \leq \phi(S_2)$ and provided that $S_1 \subseteq S_2$. In other words, an operational system will be consistent with respect to the repair of additional components. In a monotone SBS, a cutset is a subset $S' \subseteq S$ such that the system fails when all the components from S' fail.

In order to simplify the understanding, we present a different SBS:

1. Classical model: we are given an undirected graph $G = (V, E)$ terminal set $K \subseteq V$ and elementary reliabilities $p : E \to [0, 1]$. The random graph $\mathcal{G} = (V, \mathcal{E})$ has a corresponding random link-set $\mathcal{E} \subseteq E$, governed by the probability law p in the links. The classical reliability $R(K, G, p)$ is the probability that the terminal set K is connected in the random graph $\mathcal{G}$.

2. Link-Sites model: it is precisely the previous model where $p : E \cup V \to [0, 1]$, and sites may fail as well in the random graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$. When a site fails, all the incident links fail as well.

3. Diameter-Constrained model: we consider the classical model with an additional constraint. The diameter-constrained reliability is the probability that every pair of sites from the terminal set K is connected by a path with d links or less. The positive integer d is called diameter.

4. Hostile model: a special case of the Link-Sites model, where $p(v) = 1$ for every terminal site $v \in K$.

The classical model is monotonous, since the addition of links never disconnects the terminals. A similar fact holds for the diameter-constrained model. Nevertheless, the Link-Site model is not monotonous.
A remarkable difference between Link-Edge and Hostile models is precisely monotonicity. The adjective hostile stands for the fact that failures are more aggressive in the hostile model than the classical one.
Even though there is rich literature on the classical model, the Link-Sites model has been previously considered as well. Jacques Carlier et. al. developed an exact reliability evaluation method for both classical and link-sites models and exploiting Rosenthal decomposition [6]. Other authors aim to maximize the reliability in the link-site model, subject to a bounded budget [7]. The reader is invited to appreciate a discussion on network reliability models in the book [10].

3. Crude Monte Carlo

Macroscopically, Crude Monte Carlo (CMC) is a noteworthy computational tool for the simulation of complex systems. By means of strong laws from statistics, a pointwise estimation of the fundamental parameters of a system is feasible [8]. Here we describe CMC for the reliability estimation of an arbitrary SBS. Then, we show the main weakness of CMC: it is not suitable for rare event simulation. This is called the Central Problem from Rare Event simulation [15].

Let $\mathcal{S} = (S, \phi, p)$ by an SBS. We are given an independent sample $X_1, \ldots, X_N$, where each random vector $X_j = (X_j^1, \ldots, X_j^r)$ describes the operational components, and follows the probability law given by the vector p. The reliability estimation for the CMC method is given by the following expression:

$$R_{CMC} = \frac{1}{N} \sum_{j=1}^{N} \phi(X_j) \tag{2}$$

An important remark is that the estimator R_{MCC} converges almost certainly to the correct reliability $R = E(\phi(X))$, as stated in the Strong Law of Large Numbers. Moreover, the estimator R_{MCC} is unbiased, since it is an average of identically distributed random variables. Therefore, its mean square error is precisely its variance.

Let us denote $\gamma = 1 - R$ the *unreliability* of the SBS. There is a trade-off between the rarity of the probability γ and the sample size N required to keep the error under control (below a fixed threshold). Consider the Limit Central Theorem to find a speed of convergence. As a preamble, it is convenient to consider the concepts of Relative Error (RE), Bounded Relative Error (BRE) and Vanishing Relative Error (VRE):

Definition 1 (Relative Error)**.** *Given a risk level α, the relative error of an estimator is the ratio between half the confidence interval at level α and γ. We will denote $RE(\gamma)$ to the relative error.*

Definition 2 (Bounded Relative Error)**.** *An estimator satisfies Bounded Relative Error (BRE) property if the Relative Error is bounded when γ tends to zero. Mathematically:*

$$\lim_{\gamma \to 0} RE(\gamma) = L, \tag{3}$$

for some real number L.

Definition 3 (Vanishing Relative Error)**.** *An estimator satisfies the Vanishing Relative Error (VRE) property if the Relative Error vanishes when γ tends to zero. Mathematically:*

$$\lim_{\gamma \to 0} RE(\gamma) = 0. \tag{4}$$

Clearly, VRE property is stronger than BRE; the converse is not true. In order to understand the main problem of Rare Event simulation we should find the limit of RE when γ tends to zero for CMC method. Let α be a fixed risk level (typically 10% or lower, in relation with the application). By means of the Central Limit Theorem, the average of independent, identically distributed random variables with finite mean can be approximated in distribution to a normal law, with finite mean γ and variance $\hat{\sigma}^2 = \gamma(1-\gamma)/N$. The radius of the confidence interval is $\hat{\sigma} z_{\alpha/2}$, being $z_{\alpha/2}$ such that $P(Z > z_{\alpha/2}) = z_{\alpha/2}$ for a standard normal variable. Thus:

$$\lim_{\gamma \to 0} RE_{CMC}(\gamma) = \lim_{\gamma \to 0} \frac{\sqrt{\gamma}\sqrt{(1-\gamma)}}{\sqrt{N}\gamma} z_{\alpha/2} = +\infty. \tag{5}$$

As a consequence, CMC does not satisfy even the most basic BRE property. This is the main problem in the Rare Event simulation. The following section calls for a study into promising alternatives to cope with this problem. One is called RVR, and it is specifically suitable for the context of network reliability analysis. On the other hand, Importance Sampling (IS) can be extended to general context of rare event analysis.

4. Alternative methods

4.1. Recursive Variance Reduction

This method has been originally proposed by Héctor Cancela and Mohammed El Khadiri for the classical network reliability model [5]. The key idea is to find a cutset $C = \{e_1, \ldots, e_t\} \subseteq S$, and consider mutually exhaustive disjoint events on the state of the links that belong to that cutset. If the cutset does not occur, we can consider the family of disjoint events, where B_i represents the event that link e_i is operational but links $e_j, j < i$ fail. If all links fail, the cutset occurs, and the event is denoted by B_0. We observe that the set $\{B_i\}_{0 \leq i \leq t}$ is a family of mutually exhaustive disjoint events. Consider the conditional unreliability Y_i, for event B_i. If V denotes the random variable that points the first element under operation and q_C denotes the probability of B_0, then the following estimator Z:

$$Z = q_C + (1 - q_C) \sum_{i=1}^{t} 1_{\{V=i\}} Y_i. \tag{6}$$

is unbiased for the unreliability. The random variable Z suggests a recursive method, where smaller subsystems are considered step by step, fixing the state of some links. The authors consider all links incident to a given site as a cutset. Remarkably, they prove mathematically that the variance of the RVR method is never greater than that of CMC; the equality holds for trivial networks. It is possible to find a cutset efficiently in every monotone SBS; in particular under the Hostile model. The reader is invited to consult the proof in [4]. As a consequence, the RVR method is suitable for the Hostile model, and its mean square error is lower than CMC as well.

4.2. Importance Sampling

Importance Sampling is specifically developed to address the rare event simulation problem. The key concept is a change of probability measure, in such a way that the probability of the rare event is increased. A lower probability is assign to the complementary event, and a sample on the new probability measure will succeed to get trials of the rare event from the original measure. In the most general context of measures, this change of measure is precisely Radon-Nikodim derivative with respect to an absolutely continuous positive measure [9]. Let X be the random vector that represents the network state, and let us denote $P(x) = P(X = x)$ the original probability measure. If ϕ denotes the structure that determines whether the system is operational or not, then the new probability law is $\hat{P}$:

$$\hat{P}(x) = (1 - \phi(x))P(x)/\gamma \tag{7}$$

We remark that the probability of operational configurations is null, and the rare event has a unit probability under $\hat{P}$. A sample is considered under the new measure, and an averaging takes place:

$$R_{IS} = 1 - \frac{1}{N} \sum_{j=1}^{N} (1 - \phi(X_j)) L(X_j), \tag{8}$$

being $L(x_j) = P(x)/\hat{P}(x)$ whenever $\hat{P}(x) > 0$, or 0 otherwise. Observe that since the rare event has unit probability, the new estimator has null variance. However, the determination of the law $\hat{P}$ requires a knowledge of γ, which is precisely our goal. L' Ecuyer et. al. developed the Importance Sampling method for the network unreliability evaluation under the classical model, and considered a recursive formula with approximate zero variance, called AZVIS (Approximate Zero Variance Importance Sampling). This method satisfies the VRE property when special cutsets are considered [12]. The idea is to build a sequence of probability measures $\{P_n\}_{n\in\mathbb{N}}$ such that P_n converges to $\hat{P}$. We invite the reader to consult the original article for further details [12].

5. Experimental analysis

In this section, we perform a fair comparison among the three methods previously considered, i.e., CMC, RVR and IS. The main parameter that jointly considers computational efficiency and accuracy is the efficiency gain:

$$\hat{W}_{\frac{CMC}{M}} = \frac{\hat{V}_{CMC}.t_{CMC}}{\hat{V}_M.t_M}, \tag{9}$$

where M here stands for RVR or IS. In order to highlight the performance of each method we consider real-life instances, such as Arpanet and our National Telephony transport network, ANTEL [13]. By historical reasons, we also include Dodecahedron as well. We distinguish those nodes with the highest degree as terminal sites. They represent roughly 40% of the possible sites. Figures 1, 2 and 3 present Dodecahedron, ArpaNet and ANTEL, respectively. In order to get a highly robust network, we consider a homogeneous system, where links present identical elementary reliabilities $p \in \{0.95, 0.97, 0.99\}$, and non-terminal sites have elementary reliabilities $u \in \{0.95, 0.97, 0.99\}$. As a consequence, we get 9 instances per graph, and we use $N = 10^4$ for the sampling size. Tables 1, 2 and 3 present the unreliability estimator $\hat{\gamma}$, variance estimator $\hat{V}$, CPU time and relative efficiency gain $\hat{W}$ with respect to CMC for each method and pair (p, u).
The reader can appreciate from Tables 1, 2 and 3 that RVR is computationally more efficient than IS in terms of CPU time. Nevertheless, the variance of IS is lower in most scenarios, especially when the unreliability is a rare event. The RVR method presents lower variance when the unreliability is higher than 10^{-4}. There is a match here with the original concept of IS, developed specifically for rare events only. We can observe that IS outperforms RVR in terms of efficiency gain for most rare events scenarios, particularly when the elementary unreliability becomes $u = 0.99$.

6. Conclusions

This paper has introduced the Hostile model, where both links and non-terminal sites may fail independently. The model is a special case of a monotone stochastic binary system. Since Crude Monte Carlo does not meet the Bounded Relative Error property, it is not suitable for evaluating the reliability of highly reliable

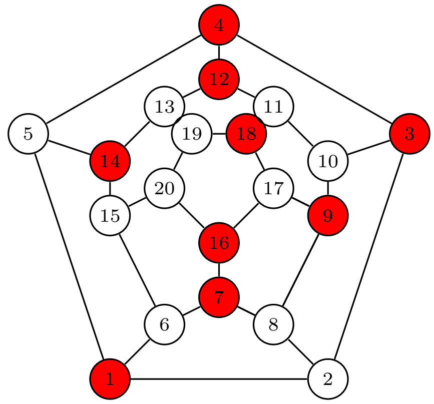

Figure 1: Dodecahedron

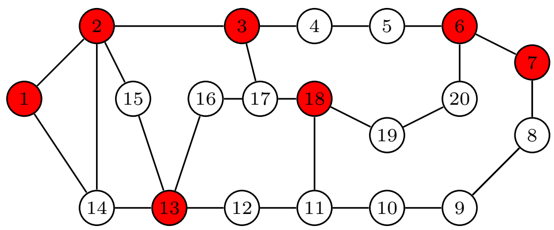

Figure 2: ArpaNet

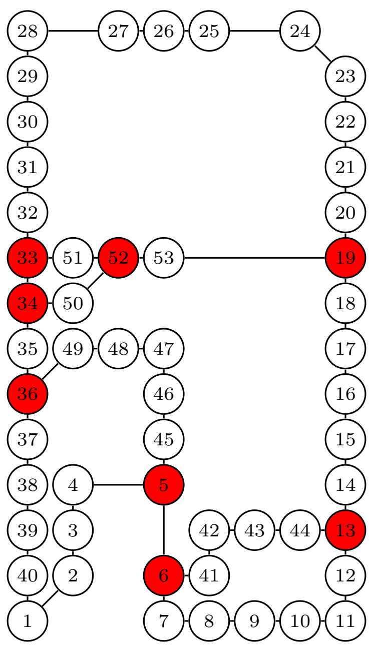

Figure 3: ANTEL

Dodecahedron							
M	p	u	N	$\hat{\gamma}$	$\hat{V}$	t	$\hat{W}$
RVR	0.95	0.95	10^4	0.0090535	$3.14E-07$	205.8	1.84
IS	0.95	0.95	10^4	0.0089496	$1.22E-07$	2609	37.4
RVR	0.95	0.97	10^4	0.004785	$1.07E-07$	306	2.48
IS	0.95	0.97	10^4	0.0047531	$3.31E-08$	3550.1	0.69
RVR	0.95	0.99	10^4	0.0019989	$2.19E-08$	358	5.41
IS	0.95	0.99	10^4	0.0020218	$1.14E-08$	3854.2	0.96
RVR	0.97	0.95	10^4	0.0033151	$1.04E-07$	189.5	2.7
IS	0.97	0.95	10^4	0.0032034	$5.06E-08$	2201.4	0.48
RVR	0.97	0.97	10^4	0.0012524	$2.54E-08$	197.2	5.19
IS	0.97	0.97	10^4	0.00183433	$1.72E-08$	2754.9	0.54
RVR	0.97	0.99	10^4	0.00047489	$2.30E-10$	321.1	151
IS	0.97	0.99	10^4	0.0006093	$9.69E-10$	3965	2.9
RVR	0.99	0.95	10^4	0.00087277	$2.95E-10$	137.8	465
IS	0.99	0.95	10^4	0.0009255	$6.77E-09$	1556.8	1.78
RVR	0.99	0.97	10^4	0.00025812	$9.55E-10$	166.2	71.5
IS	0.99	0.97	10^4	0.00026312	$1.61E-10$	1851.9	38
RVR	0.99	0.99	10^4	5.00E-05	$4.22E-11$	296.4	147
IS	0.99	0.99	10^4	3.49E-05	$2.99E-12$	3049.9	201

Table 1: Performance of RVR, IS and efficiency gain w.r.t. CMC in Dodecahedron

ArpaNet							
M	p	u	N	$\hat{\gamma}$	$\hat{V}$	t	$\hat{W}$
RVR	0.95	0.95	10^4	0.051351	$1.68E-06$	245	2.29
IS	0.95	0.95	10^4	0.036738	$1.60E-06$	761.1	0.75
RVR	0.95	0.97	10^4	0.044748	$1.17E-06$	214.4	2.78
IS	0.95	0.97	10^4	0.040738	$9.32E-07$	865.9	0.86
RVR	0.95	0.99	10^4	0.023175	$3.67E-07$	197.1	6.47
IS	0.95	0.99	10^4	0.023244	$5.98E-07$	839	0.93
RVR	0.97	0.95	10^4	0.023658	$3.13E-07$	196.1	8.37
IS	0.97	0.95	10^4	0.022267	$1.07E-06$	806.4	0.59
RVR	0.97	0.97	10^4	0.015214	$4.51E-07$	240.3	3.04
IS	0.97	0.97	10^4	0.01694	$3.73E-07$	752.8	1.17
RVR	0.97	0.99	10^4	0.0085	$7.66E-08$	232.1	9.27
IS	0.97	0.99	10^4	0.0085685	$4.28E-08$	865.2	4.45
RVR	0.99	0.95	10^4	0.0078039	$6.60E-08$	244	9.55
IS	0.99	0.95	10^4	0.0086097	$7.71E-08$	794.9	2.51
RVR	0.99	0.97	10^4	0.00349	$2.37E-08$	210.5	16.3
IS	0.99	0.97	10^4	0.0041413	$5.82E-08$	811.2	1.73
RVR	0.99	0.99	10^4	0.0011238	$5.46E-09$	213.3	20.9
IS	0.99	0.99	10^4	0.0013126	$1.06E-09$	813.7	28.2

Table 2: Performance of RVR, IS and efficiency gain w.r.t. CMC in ArpaNet

ANTEL							
M	p	u	N	$\hat{\gamma}$	$\hat{V}$	t	$\hat{W}$
RVR	0.95	0.95	10^4	0.41953	$1.32E-06$	369.8	6.52
IS	0.95	0.95	10^4	0.40769	$1.02E-06$	5713.9	0.29
RVR	0.95	0.97	10^4	0.31029	$9.05E-05$	472	0.68
IS	0.95	0.97	10^4	0.31433	$2.66E-05$	6193.1	0.73
RVR	0.95	0.99	10^4	0.22542	$7.50E-05$	441	1.41
IS	0.95	0.99	10^4	0.21542	$8.74E-05$	6499.5	0.36
RVR	0.97	0.95	10^4	0.29541	$1.26E-05$	384.5	85.2
IS	0.97	0.95	10^4	0.30109	$7.48E-05$	5724.3	0.60
RVR	0.97	0.97	10^4	0.19916	$9.76E-06$	427.2	10.5
IS	0.97	0.97	10^4	0.22906	$1.81E-05$	5245.2	0.20
RVR	0.97	0.99	10^4	0.10498	$3.87E-06$	495.1	24.5
IS	0.97	0.99	10^4	0.08054	$2.26E-06$	6352.7	0.49
RVR	0.99	0.95	10^4	0.14596	$8.11E-06$	369.8	24
IS	0.99	0.95	10^4	0.16962	$1.21E-06$	5670	2.85
RVR	0.99	0.97	10^4	0.07284	$1.47E-05$	400.2	12.7
IS	0.99	0.97	10^4	0.066295	$5.47E-06$	5688.3	1.18
RVR	0.99	0.99	10^4	0.022951	$1.70E-05$	598.5	5.69
IS	0.99	0.99	10^4	0.015605	$2.75E-06$	6074.5	2.30

Table 3: Performance of RVR, IS and efficiency gain w.r.t. CMC in ANTEL

systems, where unreliability is a rare event. As a consequence, we adapted two alternative methods for this model, i.e., the Importance Sampling and Recursive Variance Reduction methods. They are both competitive for rare event analysis. On one hand, Importance Sampling has been shown to be more suitable for rare event scenarios, when the elementary unreliability is reduced. On the other, Recursive Variance Reduction is computationally more efficient.
In future work, our intention is to study non-monotone stochastic binary systems (for instance, networks under arbitrary failures on the sites). Additionally, we would like to adapt and/or design new methods with performance better than Crude Monte Carlo in the non-monotone models as well.

7. Acknowledgements

This work is partially supported by Project 395 CSIC I+D *Sistemas Binarios Estocásticos Dinámicos*.

References

[1] Ball, M.O. (1986). Computational Complexity of Network Reliability Analysis: An Overview. IEEE Transactions on Reliability, 35(3), 230-239.

[2] Berners, T., Cailliau, R., Groff., J.F., and Pollerman, B. (1992). World-Wide Web: The Information Universe. Electronic Networking: Research, Applications and Policy, 1(2), 74-82.

[3] Bollobás, B. (2001). Random Graphs. Cambridge: Cambridge University Press.
[4] Canale, E., Cancela, H., Piccini, J., Robledo, F., Romero, P., Rubino, G. and Sartor, P. (2015). Recursive Variance Reduction method in stochastic monotone binary systems. Proceedings of the 7th International Workshop on Reliable Networks Design and Modeling (pp. 135-141).
[5] Cancela, H. and El Khadiri, M. (1995). A recursive variance-reduction algorithm for estimating communication-network reliability. IEEE Transactions on Reliability, 44(4), 595-602.
[6] Carlier, J, and Lucet, C. (1996). A decomposition algorithm for network reliability evaluation. Discrete Applied Mathematics, 65(1), 141-156.
[7] Dash, R.K., Barpanda, N.K., Tripaty, P.K. and Tripath, C.R. (2012). Network reliability optimization problem of interconnection network under node-edge failure model. Applied Soft Computing, 12(8), 2322-2328.
[8] Fishman, G. (1996). Monte Carlo: Concepts, Algorithms and Applications. Springer Verlag New York, USA.
[9] Folland, G.B. (1999). Real analysis: modern techniques and their applications. Pure and applied mathematics. Joh Wiley & Sons.
[10] Gertsbakh, I.B. and Shpungin, Y. (2009). Models of Network Reliability: Analysis, Combinatorics, and Monte Carlo. Florida: CRC Press Inc.
[11] Karp, R. M. (1972). Reducibility Among Combinatorial Problems. In Complexity of Computer Computations. London: Plenum Press (pp. 85-103).
[12] L'Ecuyer P., Rubino, G., Saggadi, S. and Tuffin, B. (2011). Approximate Zero-Variance Importance Sampling for Static Network Reliability Estimation. IEEE Transactions on Reliability, 60(3), 590-604.
[13] Risso, C. (2014). Using GRASP and GA to design resilient and cost-effective IP/MPLS networks. Ph.D. Thesis, University of the Republic, Uruguay.
[14] Rosenthal, A. (1977). Computing the Reliability of Complex Networks. SIAM Journal on Applied Mathematics, 32(2), 384-393.
[15] Rubino, G. and Tuffin, B. (2009). Rare event simulation using Monte Carlo methods. Hoboken, NJ: John Wiley & Sons.