Power matrix means and related inequalities*

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Abstract. *This survey paper contains recent results for power matrix means and related inequalities for convex functions, Hadamard product of matrices as well as some inequalities involving exponential function of matrices.*

Key words: *power matrix means, arithmetic – geometric matrix means inequality, Kantorovich inequality, convex functions, Hadamard product of matrices*

Sažetak. **Potencijalne sredine matrica i srodne nejednakosti**. *Ovaj ekspozitorni članak daje pregled najnovijih istraživanja za potencijalne sredine matrica, kao i srodnih nejednakosti za konveksne funkcije, Hadamardov produkt matrica i nejednakosti s eksponencijalnom funkcijom za matrice.*

Ključne riječi: *potencijalne sredine matrica, nejednakost između aritmetičke i geometrijske sredine matrica, konveksne funkcije, Hadamardov produkt matrica*

1. Introduction

The difficulty of establishing a (noncommutative) matrix inequality involving the geometric mean was discussed in 1978 by K.V. Bhagwat and R. Subramanian [10] who pointed out that the problem of defining a geometric mean for noncommutative operators "makes it difficult to establish the validity or otherwise of the classical inequalities involving the geometric mean". However, in a recent paper, Sagae and Tanabe [43] define a geometric mean and establish an AG-GM inequality for a finite number of positive definite matrices.

Here we make use of their definition of the geometric mean of positive definite matrices to establish matrix versions of numerous inequalities involving means.

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These include mixed mean matrix inequalities and conditions under which the matrix inequalities of [43] are reversed. Converses for the matrix convexity of the inverse function as well as some bounds for the exponential functions of matrices are also given, as well as related results for convex functions and Hadamard product of matrices.

2. General inequalities

Let w_i, C_i be positive numbers with $\sum_{i=1}^{r} w_i = 1$. Then the power means are defined by

$$M_r^{[t]}(C;w) = (w_1 C_1^t + \ldots + w_r C_r^t)^{1/t}, \quad t \neq 0 \tag{1}$$

and

$$M_r^{[0]}(C;w) = C_1^{w_1} \ldots C_r^{w_r}. \tag{2}$$

It is well-known that if $s \leq t$, then

$$M_r^{[s]}(C;w) \leq M_r^{[t]}(C;w). \tag{3}$$

Moreover, if $C_i, i = 1, \ldots, r$ are $n \times n$ positive definite matrices, we can use the definition of power means of order $t \neq 0$ given by (1).

The geometric mean $M_r^{[0]} \equiv G_r$ cannot be given by (2) since the product given by (2) need not be a positive definite matrix even if all C_i are positive definite.

Instead we shall use the definition of the geometric mean of r matrices recently given in [43]:

$$\begin{aligned} M_r^{[0]}(C;w) \quad \equiv \quad & (C_r^{1/2}(C_r^{-1/2}C_{r-1}^{1/2} \ldots (C_3^{-1/2}C_2^{1/2}(C_2^{-1/2}C_1C_2^{-1/2})^{u_1} \\ & C_2^{1/2}C_3^{-1/2})^{u_2} \ldots C_{r-1}^{1/2}C_r^{-1/2})^{u_{r-1}}C_r^{1/2} \end{aligned} \tag{4}$$

where $u_i = 1 - w_{i+1}\left(\sum_{k=1}^{i+1} w_k\right)^{-1}$ for $i = 1, \ldots, r-1$. We shall use the notation A_r for the arithmetic mean $M_r^{[1]}$ and H_r for the harmonic mean $M_r^{[-1]}$.

Sagae and Tanabe [43] proved that the inequalities between the harmonic, geometric and arithmetic means also hold in the matrix case, i.e.,

Theorem 2.1..

$$H_r \leq G_r \leq A_r \tag{5}$$

where $A \geq B$ means that $A - B$ is a positive semi-definite matrix. Equalities hold in (5) if and only if $C_1 = \ldots = C_r$.

If the weights w_i are not all positive, we can give reverse results. We begin with the following (see [2]):

Lemma 2.1.. *If $\alpha \in (0,1)$, then*

$$G_2 \equiv C_2^{1/2}(C_2^{-1/2}C_1C_2^{-1/2})^{\alpha}C_2^{1/2} \leq \alpha C_1 + (1-\alpha)C_2 \equiv A_2 \tag{6}$$

but if either $\alpha < 0$ or $\alpha > 1$, the reverse inequality, i.e.

$$G_2 \geq A_2 \tag{7}$$

is valid.

Proof. The following generalization of Bernoulli's inequality is well-known (see e.g. [23, p.34]), or [24, p.65].

For $-1 < x \neq 0$

$$(1+x)^\alpha > 1 + \alpha x \qquad \text{if} \qquad \alpha > 1 \quad \text{or} \qquad \alpha < 0$$

and

$$(1+x)^\alpha < 1 + \alpha x \quad \text{if} \qquad 0 < \alpha < 1. \tag{8}$$

For $1 + x = t$, this is equivalent to

$$t^\alpha > \alpha t + 1 - \alpha \quad \text{if} \quad \alpha > 1 \quad \text{or} \quad \alpha < 0 \tag{9}$$

and

$$t^\alpha < \alpha t + 1 - \alpha \quad \text{if} \quad 0 < \alpha < 1. \tag{10}$$

For $t = 1$, we have the equality.

If the positive definite matrix C has the representation $C = \Gamma D_\lambda \Gamma^*$ when Γ is unitary and $D_\lambda = \operatorname{diag}(\lambda_1, \ldots, \lambda_n)$ where $\lambda_1, \ldots, \lambda_n$ are characteristic roots of C, then from (9) and (10) it follows that

$$D_\lambda^\alpha > \alpha D_\lambda + (1-\alpha)I \qquad \text{if } \alpha > 1 \quad \text{or} \quad \alpha < 0$$

and

$$D_\lambda^\alpha < \alpha D_\lambda + (1-\alpha)I \qquad \text{if} \quad 0 < \alpha < 1.$$

Pre- and post-multiplication by Γ and Γ^* respectively, yields

$$C^\alpha > \alpha C + (1-\alpha)I \qquad \text{if} \quad \alpha > 1 \quad \text{or} \quad \alpha < 0 \tag{11}$$

and

$$C^\alpha < \alpha C + (1-\alpha)I \qquad \text{if} \quad 0 < \alpha < 1 \tag{12}$$

with equalities if and only if $C = I$.

If we now set $C = C_2^{-1/2} C_1 C_2^{-1/2}$, we obtain

$$(C_2^{-1/2} C_1 C_2^{-1/2})^\alpha > \alpha C_2^{-1/2} C_1 C_2^{-1/2} + (1-\alpha)I$$

if $\alpha > 1$ or $\alpha < 0$, and

$$(C_2^{-1/2} C_1 C_2^{-1/2})^\alpha < \alpha C_2^{-1/2} C_1 C_2^{-1/2} + (1-\alpha)I$$

for $0 < \alpha < 1$.

Pre- and post-multiplication by $C_2^{1/2}$ now yields (7) and (6), respectively. Equality holds if and only if $C_2^{-1/2} C_1 C_2^{-1/2} = I$, or equivalently $C_1 = C_2$. $\square$

Sagae and Tanabe [43] used (6) and mathematical induction to prove the inequality $G_r \leq A_r$ for positive weights. Similarly, we shall use (7).

Theorem 2.2.. *Let $w_i, i = 1, \ldots, r$ be real numbers such that $w_1 > 0$, $w_i < 0$, $i = 2, \ldots, r$; $w_1 + \ldots + w_r = 1$. Then*

$$A_r \leq G_r. \tag{13}$$

If we also have $w_1 C_1^{-1} + \ldots + w_r C_r^{-1} > 0$, then

$$G_r \leq H_r. \tag{14}$$

Equality holds in (13) and (14) if and only if $C_1 = \ldots = C_r$.

Proof. For $r = 2$, (2.11) is proved in *Lemma 2.1*, i.e., it is inequality (7). So, suppose (13) holds for $r - 1$.

Let A_{r-1} and G_{r-1} be weighted arithmetic and geometric means of matrices $C_1, \ldots, C_{r-1}$ with weights $\tilde{w}_i = w_i \left(\sum_{i=1}^{r-1} w_i \right)^{-1}$ for $i = 1, \ldots, r-1$.

Note that $\tilde{w}_1 > 0,\ \tilde{w}_i < 0, i = 2, \ldots, r-1;\ \tilde{w}_1 + \ldots + \tilde{w}_{r-1} = 1$, and $\tilde{u}_i = 1 - \tilde{w}_{i+1} \sum_{j=1}^{i+1} \tilde{w}_j = 1 - w_{i+1} \sum_{j=1}^{i+1} w_j = u_i$ for $i = 1, \ldots, r-2$ and $u_{r-1} = 1 - w_r (> 0)$. So, we have

$$\begin{aligned} A_r &= \sum_{i=1}^{r-1} w_i C_i + w_r C_r = (1 - w_r) A_{r-1} + w_r C_r \\ &\leq (1 - w_r) G_{r-1} + w_r C_r \qquad \text{(by the inductive hypothesis)} \\ &\leq C_r^{1/2} (C_r^{-1/2} G_{r-1} C_r^{-1/2})^{1 - w_r} C_r^{1/r} = G_r. \qquad \text{(by (7))} \end{aligned}$$

The equality $A_r = G_r$ holds only when all the equalities are valid simultaneously. Equality in the first inequality holds if $A_{r-1} = G_{r-1}$, i.e. $C_1 = C_2 = \ldots = C_{r-1}$ by induction for $r - 1$ and equality in the second inequality holds if $G_{r-1} = C_r$ by the conditions for equality for $r = 2$. Therefore, the equality $A_r = G_r$ holds if and only if $A_r = G_{r-1} = C_r$, i.e. $C_1 = C_2 = \ldots = C_r$. Now by the substitutions $C_i^{-1} \to C_i,\ i = 1, \ldots, r$, we get (14) from (13). $\square$

Remark.. *It is interesting that (13), i.e., the reverse arithmetic-geometric inequality is stronger than the same arithmetic-geometric means inequality in the sense that we can obtain the second inequality in (5) from (13) (see [2]).*

Power means inequality (3) for $t, s \neq 0$, in the case when $w_i = 1/r$ (in this case we shall write $M_r^{[t]}(C)$ for $M_r^{[t]}(C; w)$) was considered by Bhagwat and Subramanian [10]. They proved

1. If $s \leq t,\ s \notin (-1, 1),\ t \notin (-1, 1)$, then $M_r^{[t]}(C) \geq M_r^{[s]}(C)$.
2. For $t \geq 1/2$, $M_r^{[2t]}(C) \geq M_r^{[t]}(C)$.

Moreover, it can be shown that the following results hold (see [25], [26]):

Theorem 2.3.. *Inequality (3) holds if one of the following is valid:*
(a) $t \geq s,\ t \notin (-1,1),\ s \notin (-1,1)$*; or*
(b) $t \geq 1 \geq s \geq 1/2$*; or*
(c) $s \leq -1 \leq t \leq -1/2$.

Proof. First, we shall prove (3) if either (a); (d) $t \geq 1 \geq s \geq t/2$; or (e) $s \leq -1 \leq t \leq s/2$, holds. For this we need only Jensen's inequality for matrix convex functions

$$f\left(\sum_{k=1}^{r} w_k A_k\right) \leq \sum_{k=1}^{r} w_k f(A_k),\ w_k \geq 0,\ w_1 + \ldots + w_r = 1. \tag{15}$$

Further, it is known that the function $f(x) = x^p$ is matrix concave for $0 < p \leq 1$ and matrix convex for $1 \leq p \leq 2$ or $-1 \leq p < 0$; while the functions $g(x) = x^{1/t}$, for $t \geq 1$, and $h(x) = -x^{1/t}$, for $t \leq -1$ are matrix monotone. Using these facts and the substitutions $f(x) = x^{s/t}, A_i = C_i^t$ (or $f(x) = x^{t/s},\ A_i = C_i^s$) in (15), we get (3) with condition (a), (d) or (e). We shall only prove the case for $t \geq s \geq 1$.

The function $f(x) = x^{s/t}$ is matrix concave since $0 < s/t \leq 1$. With the substitution $A_i = C_i^t$, (15) becomes

$$\left(\sum_{k=1}^{r} w_k C_k^t\right)^{s/t} \geq \sum_{k=1}^{r} w_k C_k^s. \tag{16}$$

Since the function $g(x) = x^{1/s}$ is matrix monotone, (3) follows from (16).

Further, (d) for $t = 1$ gives (for $1 \geq s \geq 1/2$)

$$M_r^{[1]}(C;w) \geq M_r^{[s]}(C;w).$$

On the other hand, (a) for $s = 1$ gives

$$M_r^{[t]}(C;w) \geq M_r^{[1]}(C;w).$$

These inequalities give (3) if (b) holds. Similarly, using (e) and (a), we can prove that (3) holds if (c) is valid. □

3. Mixed means matrix inequalities

Recently, K. Kedlaya [19] has proved the following conjecture of F. Holland [13]:

Let $x_1, x_2, \ldots, x_n$ be positive real numbers. Then

$$\left(\prod_{j=1}^{n} \frac{x_1 + x_2 + \ldots x_j}{j}\right)^{1/n} \geq \frac{1}{n} \sum_{i=1}^{n} \sqrt[i]{x_1, x_2, \ldots, x_i} \tag{17}$$

with equality if and only if $x_1 = x_2 = \ldots = x_n$.

We shall consider related results for positive definite Hermitian matrices. The open problem is to give a generalization for positive definite matrices for arbitrary number of matrices. The cases of two matrices were considered in [27] and [33]. Let the arithmetic, geometric and harmonic means be defined respectively by

$$\begin{aligned} A\nabla_\lambda B &= \lambda A + (1-\lambda)B, \\ A\#_\lambda B &= A^{1/2}(A^{-1/2}BA^{-1/2})^{1-\lambda}A^{1/2}, \\ A!_\lambda B &= (\lambda A^{-1} + (1-\lambda)B^{-1})^{-1} \end{aligned}$$

where $\lambda \in (0,1)$.

The following are special cases of more general results obtained in [27]:

Let $\alpha, \lambda \in (0,1)$. Then

$$\begin{aligned} A\#_\alpha(A\nabla_\lambda B) &\geq A\nabla_\lambda(A\#_\alpha B), \\ A\#_\alpha(A!_\lambda B) &\leq A!_\lambda(A\#_\alpha B), \\ A!_\alpha(A\nabla_\lambda B) &\geq A\nabla_\lambda(A!_\alpha B). \end{aligned} \tag{18}$$

Moreover, using the idea of the method of proof from [19] (but instead of Hölder's inequality, the Minkowski type matrix inequality from [7] is used), the following mixed arithmetic-harmonic mean inequality was proved in [33]:

Theorem 3.1.. *Let $A_1, \ldots, A_n$ be positive definite $m \times m$ Hermitian matrices. Then*

$$\left[\frac{1}{n}\sum_{j=1}^{n}\left(\frac{A_1 + A_2 + \ldots + A_j}{j}\right)^{-1}\right]^{-1} \geq \frac{1}{n}\sum_{i=1}^{n}\left(\frac{1}{i}\sum_{k=1}^{i} A_k^{-1}\right)^{-1} \tag{19}$$

4. Reverse forms of a convex matrix inequality

Let A and B be two complex $n \times n$ Hermitian positive definite matrices, and let $0 \leq \lambda \leq 1$. Then

$$[\lambda A + (1-\lambda)B]^{-1} \leq \lambda A^{-1} + (1-\lambda)B^{-1}. \tag{20}$$

This result, i.e., matrix convexity of the inverse function is an old result that appears explicitly in the papers of J. Bendat and S. Sherman [9], C. Davis [11], M.H. Moore [37], I. Olkin and J. Pratt [38] and P. Whittle [44].

B. Mond and J. Pečarić [28] have proved the following reverse results:

Theorem 4.1.. *Let A and B be two complex Hermitian positive definite matrices, and let $0 \leq \lambda \leq 1$. Then*

$$[\lambda A + (1-\lambda)B]^{-1} \geq K(\lambda A^{-1} + (1-\lambda)B^{-1}) \tag{21}$$

where

$$K = 4 \min \frac{\mu_i}{(1+\mu_i)^2}, \tag{22}$$

and the μ_i are the solutions of the equation

$$\det(B - \mu A) = 0. \tag{23}$$

The constant K is best possible.

Theorem 4.2.. *Let A and B be positive definite Hermitian matrices. Then*

$$[\lambda A + (1-\lambda)B]^{-1} - (\lambda A^{-1} + (1-\lambda)B^{-1}) \geq \tilde{K} A^{-1} \tag{24}$$

where

$$\tilde{K} = \min \frac{(\sqrt{\mu}_i - 1)^2}{-\mu_i} \tag{25}$$

and the μ_i are the solutions of the equation (23). The constant $\tilde{K}$ is best possible.

Note that the matrix convexity of the Moore-Penrose (generalized) inverse was considered in the papers of D.G. Kaffes [16], A. Giovagnoli and H.P. Wynn [12] and D.G. Kaffe, T. Mathew, M.B. Rao and K. Subramanyam [17].

Let A and B be two complex Hermitian positive semi-definite matrices of the same order. The inequality

$$[\lambda A + (1-\lambda)B]^{+} \leq \lambda A^{+} + (1-\lambda)B^{+} \tag{26}$$

for every $0 \leq \lambda \leq 1$ holds if and only if $R(A) = R(B)$ where $R(A)$ is the range of A.

Similar results to those in *Theorems 4.1* and *4.2* can also be given (see [29]). A related result was also obtained by M. Alić, J. Pečarić and V. Volenec [5].

Theorem 4.3.. *Let A and B be two positive definite Hermitian matrices and let $\alpha \in (0,1)$. Then*

$$A \nabla_\alpha B \leq K(A \#_\alpha B) \tag{27}$$

where $A\#_\alpha B = B^{1/2}(B^{-1/2}AB^{-1/2})^\alpha B^{1/2}$ and

$$K = \frac{(\beta-1)\beta^{1/(\beta-1)}}{e \log \beta} \tag{28}$$

and $\beta = \max\{\lambda_1, \frac{1}{\lambda_n}\}, \lambda_1 \geq \ldots \lambda_n$ are the eigenvalues of $A^{-1/2}BA^{-1/2}$.

Remark.. *In the papers [28] and [29], we have direct proofs of* Theorems 4.1 *and* 4.2. *It was shown in [39] that such results can also be proved using the following general result [15]:*

Suppose X and Y are Hermitian matrices of the same size, with X positive definite and Y positive semi-definite. Then $X - Y$ is positive semi-definite if and only if $\rho(YX^{-1}) \leq 1$, where ρ is a spectral radius. For example, application *of this result with* $X = [\lambda A + (1-\lambda)B]^{-1} - (\lambda A^{-1} + (1-\lambda)B^{-1}),\ Y = \hat{K}A^{-1}$ *gives that the optimal value of* $\hat{K}$ *is over the entire range of admissible* λ*'s of*

$$\frac{1}{\lambda + (1-\lambda)\mu_i} - \lambda - (1-\lambda)\frac{1}{\mu_i}.$$

The minimum of this is $-\frac{(\sqrt{\mu_i}-1)^2}{\mu_i}$ *and, the best overall value of* $\tilde{K}$ *is* $\min \frac{(\sqrt{\mu_i}-1)^2}{-\mu_i}$, *i.e., the constant obtained in* Theorem 4.2.

5. A matrix version of the Kantorovich inequality and related results

Let w_i, a_i be positive numbers such that $0 < m \leq a_i \leq M,\ i = 1, \ldots, n$ and $\sum_{i=1}^{n} w_i = 1$. Then the well-known Kantorovich inequality holds

$$A_n(a,w) \leq \frac{(m+M)^2}{4mM} H_n(a,w). \tag{29}$$

There exist many matrix extensions of this inequality. Here we present some results of B. Mond and J. Pečarić [32].

Let A_j $(j = 1, \ldots, k)$ be positive definite Hermitian matrices of order n, with eigenvalues contained in the interval $[m, M]$ where $0 < m < M$ and let $U_j, j = 1, \ldots, k$ be $t \times n$ matrices such that

$$\sum_{j=1}^{k} U_j U_j^* = I.$$

We now consider the means

$$M_k^{[r]}(A;U) = \left(\sum_{j=1}^{k} U_j A_j^r U_j^* \right)^{1/r}. \quad (r \neq 0)$$

The following results are valid:

$$M_k^{[-1]}(A;U)^{-1} \leq \frac{(m+M)^2}{4mM} M_k^{[1]}(A;U)^{-1}, \tag{30}$$

$$M_k^{[1]}(A;U) - M_k^{[-1]}(A;U) \leq (\sqrt{M} - \sqrt{m})^2 I \tag{31}$$

$$M_k^{[2]}(A;U)^2 \leq \frac{(M+m)^2}{4mM} M_k^{[1]}(A;U)^2 \tag{32}$$

$$M_k^{[2]}(A;U) - M_k^{[1]}(A;U) \leq \frac{(M-m)}{4(M+m)} I. \tag{33}$$

For $k = 1$, (30) was proved by A.W. Marshall and I. Olkin [22]. (31)-(33) were proved by B. Mond and J. Pečarić [31]. Similarly, we can prove (see [39])

$$M_k^{[2]}(A;U)^2 - M_k^{[1]}(A;U)^2 \le \frac{(M-m)^2}{4} I. \tag{34}$$

As in [31], we can start with $mMI \le (m+M)A_j - A_j^2$ from which we get

$$mM \sum_{j=1}^{k} U_j U_j^* \le (m+M) \sum_{j=1}^{k} U_j A_j U_j^* - \sum_{j=1}^{k} U_j A_j^2 U_j^*,$$

i.e.

$$mMI \le (m+M) \sum_{j=1}^{k} U_j A_j U_j^* - \sum_{j=1}^{k} U_j A_j^2 U_j^*,$$

i.e.

$$\begin{aligned} M_k^{[2]}(A;U)^2 - M_k^{[1]}(A;U)^2 &\le (m+M)M_k^{[1]}(A;U) - mMI - M_k^{[1]}(A;U)^2 \\ &= \tfrac{(M-m)^2}{4} I - \{M_k^{[1]}(A;U) - \tfrac{M+m}{2} I\}^2 \le \tfrac{(M-m)^2}{4} I. \end{aligned}$$

Some further extensions are also obtained by B. Mond and J. Pečarić [32]. Namely, the following two theorems are proved:

Theorem 5.1.. *Let r, s be non-zero numbers with $s > r$ and either $s \notin (-1,1)$ or $r \notin (-1,1)$. Then*

$$M_k^{[s]}(A;U) - M_k^{[r]}(A;U) \le \triangle I \tag{35}$$

where $\triangle = [\theta M^s + (1-\theta)m^s]^{1/s} - [\theta M^r + (1-\theta)m^r]^{1/r}$ and where θ is more precisely defined.

Theorem 5.2.. *Let the conditions of* Theorem 5.1 *be satisfied. Then*

$$M_k^{[s]}(A;U) \le \tilde{\triangle} M_k^{[r]}(A;U) \tag{36}$$

where $\tilde{\triangle} = \left\{ \frac{r(\gamma^s - \gamma^r)}{(s-r)(\gamma^r - 1)} \right\}^{1/s} \left\{ \frac{s(\gamma^r - \gamma^s)}{(r-s)(\gamma^s - 1)} \right\}^{-1/r}$ and $\gamma = M/m$.

Remark.. *S. Liu and H. Neudecker and, independently, C-K Li and R. Mathias (see [32]) have noted that such matrix inequalities for several matrices are equivalent to the corresponding case of one matrix.*

Extensions of (30)-(34) to the case of an $n \times n$ positive semi-definite symmetric matrix were obtained in the papers of J.K. Baksalary and S. Puntanen [8] and J. Pečarić, S. Puntanen and G.P.H. Styan [42].

A^+ will denote the Moore- Penrose generalized inverse of the matrix A. P_A will denote the orthogonal projection on the range of A, i.e.,

$$P_A = A(A^*A)^+ A^*. \tag{37}$$

Theorem 5.3.. *(see [8], [42]). Let A be an $n \times n$ positive semi-definite Hermitian matrix with $r \leq n$ non-zero eigenvalues ordered by $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_r > 0$. Let U be an $n \times p$ matrix. Then*

$$U^* A^+ U \leq [(\lambda_1 + \lambda_r)/(\lambda_1 \lambda_r)] U^* P_A U - [1/(\lambda_1 \lambda_r)] U^* A U. \tag{38}$$

If A and U are such that $U^* P_A U$ is idempotent, then

$$U^* A^+ U \leq [(\lambda_1 + \lambda_r)^2/(4\lambda_1 \lambda_r)](U^* A U)^+, \tag{39}$$

$$U^* A U - (U^* A^+ U)^+ \leq (\sqrt{\lambda_1} - \sqrt{\lambda_r})^2 U^* P_A U, \tag{40}$$

$$U^* A^2 U \leq [(\lambda_1 + \lambda_r)^2/(4\lambda_1 \lambda_r)](U^* A U)^2, \tag{41}$$

$$(U^* A^2 U)^{1/2} - U^* A U \leq [(\lambda_1 - \lambda_r)^2/4(\lambda_1 + \lambda_r)] U^* P_A U, \tag{42}$$

$$U^* A^2 U - (U^* A U)^2 \leq \frac{1}{4}(\lambda_1 - \lambda_r)^2 U^* P_A U. \tag{43}$$

We note that the requirement that $U^* P_A U$ is idempotent is satisfied if $U^* U = I$.

6. Some improvements of the AG inequality

The following result was proved in [5]:

Theorem 6.1.. *Let A and B be two positive definite Hermitian matrices and let a be a real number. If $A > B, k \in N$, then*

$$\binom{a}{k+1}\{A\#_\alpha B - A\nabla_\alpha B\} \geq \binom{a}{k+1} \sum_{p=2}^{k} \binom{a}{p} B^{1/2}[B^{-1/2}(A-B)B^{-1/2}]^p B^{1/2} \tag{44}$$

but, if $A < B$, then

$$(-1)^{k+1}\binom{a}{k+1}\{A\#_\alpha B - A\nabla_\alpha B\} \geq$$

$$\geq (-1)^{k+1}\binom{a}{k+1} \sum_{p=2}^{k} \binom{a}{p} B^{1/2}[B^{-1/2}(A-B)B^{-1/2}]^p B^{1/2}. \tag{45}$$

A special case, $k = 2$ is the following:

Corollary 6.1.. *Let A and B be two positive definite Hermitian matrices and let a be a real number and let $A \geq B$. If $a \in (0,1) \cup (2, \infty)$, then*

$$A\nabla_\alpha B - A\#_\alpha B \leq \frac{a(1-a)}{2} B^{1/2}[B^{-1/2}(A-B)B^{-1/2}]^2 B^{1/2} \tag{46}$$

If $a \in (-\infty, 0) \cup (1, 2)$, the reverse inequality in (46) is valid. If $A \leq B$, we have reverse results.

Further generalizations are obtained in [1].

7. Some bounds involving exponential functions

Let A, G and H denote, respectively, the arithmetic, geometric and harmonic means of $n \times n$ positive definite matrices C_i. The following results were proved in [43]:

$$\begin{aligned} G\exp(G^{-1}A - I) &= \exp(AG^{-1} - I)G \geq (AG^{-1}A + G)/2 && (47)\\ &\geq A \geq G \geq H \geq 2(H^{-1}GH^{-1} + G^{-1})^{-1} && (48)\\ &\geq G\exp(I - H^{-1}G) = \exp(I - GH^{-1})G \end{aligned}$$

and

$$\begin{aligned} H\exp(H^{-1}A - I) &= \exp(AH^{-1} - I)H \geq (AH^{-1}A + H)/2 && (49)\\ &\geq A \geq G \geq H \\ &\geq 2(H^{-1}AH^{-1} + A^{-1})^{-1} \geq A\exp(I - H^{-1}A) && (50)\\ &= \exp(I - AH^{-1})A, \end{aligned}$$

where all the inequalities are strict unless $C_1 = \cdots = C_r$, in which case all the equalities hold.

In [2], it was noted that these results are a consequence of some general inequalities for exponential functions and of (5). Namely, the following result was proved in [3]:

Theorem 7.1.. *Let C and D be two positive definite Hermitian matrices such that $C \geq D$. Then for $r, k \in N$, we have*

$$\begin{aligned} &D\exp(D^{-1}C - I) = \exp(CD^{-1} - I)D \\ &\geq D\sum_{i=0}^{r}(D^{-1}C - I)^i/i! \geq \cdots \geq \frac{1}{2}(CD^{-1}C + D) \geq C \geq D && (51)\\ &\geq 2(D^{-1}CD^{-1} + C^{-1})^{-1} \geq \cdots \geq (\sum_{i=0}^{r}(CD^{-1} - I)^i/i!)^{-1}C \\ &\geq C\exp(I - D^{-1}C) = \exp(I - CD^{-1})C; \end{aligned}$$

$$\begin{aligned} C\sum_{i=0}^{2k-1}\frac{(C^{-1}D - I)^i}{i!} &\leq C\exp(C^{-1}D - I) = \exp(DC^{-1} - I)C && (52)\\ &\leq C\sum_{i=0}^{2k}\frac{(C^{-1}D - I)^i}{i!}, \end{aligned}$$

which, for $k = 1$, gives

$$D \leq C\exp(C^{-1}D - I) \leq \frac{1}{2}(C + DC^{-1}D); \qquad (53)$$

and

$$\begin{aligned}(\sum_{i=0}^{2k}(DC^{-1}-I)^i/i!)^{-1}D &\leq D\exp(I-C^{-1}D)=\exp(I-C^{-1}D)D\\ &\leq (\sum_{i=0}^{2k-1}(DC^{-1}-I)^i/i!)^{-1}D. \qquad (54)\end{aligned}$$

which, for $k=1$*, gives* $2(C^{-1}DC^{-1}+D^{-1})^{-1}\leq D\exp(I-C^{-1}D)\leq C$.

Now set instead of the couple (C,D), the couples (A_r,G_r), (G_r,H_r), (A_r,H_r), $(M_r^{[t]},M_r^{[s]})$ (from *Theorem 2.3*) or $(G_r,A_r,)$, (H_r,A_r), (H_r,G_r) from *Theorem 2.2* and we can obtain various extensions of (51) and (52) as well as many related results.

A corresponding result was also obtained in [4]. Namely, the result obtained for real numbers in [6] also holds in the matrix case. In fact, we have the following result for exponential matrix functions.

Theorem 7.2.. *Let* C *and* D *be two positive definite Hermitian matrices such that* $C\geq D$*. Then*

$$C-D\leq C\exp(C^{-1}D)-D\exp(-D^{-1}C)\leq \frac{3}{e}(C-D). \qquad (55)$$

The constant is best possible and equality holds if and only if $C=D$*.*

In fact, a matrix analogue of a result from [6] is a simple consequence of *Theorem 7.2* for the arithmetic mean A_r and the geometric mean G_r instead of C and D, respectively. Of course by using other results we can again obtain many similar theorems for matrix means.

8. Matrix inequalities for convex functions

If A is an $n\times n$ Hermitian matrix, then there exists a unitary matrix U such that

$$A=U^*[\lambda_1,\dots,\lambda_n]U$$

where $[\lambda_1,\dots,\lambda_n]$ is a diagonal matrix and the λ_i are the eigenvalues of A. $f(A)$ is then defined by

$$f(A)=U^*[f(\lambda_1),\dots,f(\lambda_n)]U.$$

If $F(t)=F(f(t),g(t))$, we will write $F(f(A),g(A))$ for the operator $F(A)$; while the function $F(A,B)$ denotes the matrix function of two variables when it is well-defined.

The following results were obtained in [40].

Theorem 8.1.. *Let* $A_j(j=1,\dots,k)$ *be Hermitian matrices of order* n *with eigenvalues in the interval* $[m,M]$*, and let* $U_j, j=1,\dots,k$ *be* $r\times n$ *matrices* *such that* $\sum_{j=1}^{k} U_j U_j^* = I$. *If* f *is a continuous convex function on* $[m, M]$, *then*

$$\sum_{j=1}^{k} U_j f(A_j) U_j^* \leq \frac{MI - \sum_{j=1}^{k} U_j A_j U_j^*}{M-m} f(m) + \frac{\sum_{j=1}^{k} U_j A_j U_j^* - mI}{M-m} f(M). \quad (56)$$

Proof. For a real valued convex function, we have [41, pp 1-2]

$$f(z) \leq \frac{M-z}{M-m} f(m) + \frac{z-m}{M-m} f(M) \ (z \in [m, M]). \quad (57)$$

Using this inequality, we can obtain the matrix inequality

$$f(A_j) \leq \frac{MI - A_j}{M-m} f(m) + \frac{A_j - mI}{M-m} f(M) \quad (58)$$

where $mI \leq A_j \leq MI (j = 1, \ldots, k)$. This inequality gives

$$U_j f(A_j) U_j^* \leq \frac{M U_j U_j^* - U_j A_j U_j^*}{M-m} f(m) + \frac{U_j A_j U_j^* - m U_j U_j^*}{M-m} f(M) \quad (59)$$

for $j = 1, \ldots, k$. Summing over j gives (56). □

Theorem 8.2.. *Assume that the conditions of* Theorem 8.1 *are satisfied. Let* J *be an interval such that* $J \supset f[m, M]$. *If* $F(u, v)$ *is a real valued continuous function defined on* $J \times J$ *and the matrix increasing in its first variable, then*

$$\begin{aligned} & F\left[\sum_{j=1}^{k} U_j f(A_j) U_j^*, f\left(\sum_{j=1}^{k} U_j A_j U_j^*\right)\right] \\ \leq \quad & \left\{\max_{x \in [m,M]} F\left[\frac{M-x}{M-m} f(m) + \frac{x-m}{M-m} f(M),\ \ f(x)\right]\right\} I \quad (60) \\ = \quad & \left\{\max_{\theta \in [0,1]} F\left[\theta f(m) + (1-\theta) f(M), f(\theta m + (1-\theta) M)\right]\right\} I \end{aligned}$$

Proof. By (56) and the matrix monotone character of $F(\cdot, y)$, we have

$$F\left[\sum_{j=1}^{k} U_j f(A_j) U_j^*, f\left(\sum_{j=1}^{k} U_j A_j U_j^*\right)\right] \leq F\left[\frac{MI - \tilde{A}}{M-m} f(m) + \frac{\tilde{A} - mI}{M-m} f(M), f(\tilde{A})\right] \quad (61)$$

where $\tilde{A} = \sum_{j=1}^{k} U_j A_j U_j^*$. Now, we again consider the inequality

$$F\left[\frac{M-x}{M-m}f(m) + \frac{x-m}{M-m}f(M), f(x)\right] \le K$$

where

$$K = \max_{m \le x \le M} F\left[\frac{M-x}{M-m}f(m) + \frac{x-m}{M-m}f(M), f(x)\right]$$

and, as in [32], we can get the matrix inequality

$$F\left[\frac{MI-\tilde{A}}{M-m}f(m) + \frac{\tilde{A}-mI}{M-m}f(M), f(\tilde{A})\right] \le KI \tag{62}$$

for matrices $\tilde{A}$ such that $mI \le \tilde{A} \le MI$. Now (61) and (62) give (61). Moreover, the second form of the right side of (61) follows from the change of variables $\theta = (M-x)/(M-m)$, so that $x = \theta m + (1-\theta)M$ with $0 \le \theta \le 1$. □

In the same way (or, more simply, by replacing F by $-F$ in the last theorem), we can prove the following:

Theorem 8.2'.. *Under the same hypotheses as in* Theorem 8.2*, except that F is a matrix decreasing in its first variable, we have*

$$\begin{aligned} & F\left[\sum_{j=1}^{k} U_j f(A_j) U_j^*, f\left(\sum_{j=1}^{k} U_j A_j U_j^*\right)\right] \\ \ge\ & \left\{\min_{x\in[m,M]} F\left[\frac{M-x}{M-m}f(m) + \frac{x-m}{M-m}f(M), f(x)\right]\right\} I \\ =\ & \left\{\min_{\theta\in[0,1]} F\left[\theta f(m) + (1-\theta)f(m), f(\theta m + (1-\theta)M)\right]\right\} I. \end{aligned} \tag{63}$$

By using the functions $F(u,v) = u - v$ and $F(u,v) = v^{-\frac{1}{2}}uv^{-\frac{1}{2}}$ which are matrices increasing in their first variables, we can obtain, as in [32], the following consequences of *Theorems 8.2* and *8.2'*.

Theorem 8.3.. *Let $f(x)$ be a strictly convex twice differentiable function on $J = [m,M]$ $(-\infty < m < M < \infty)$, and let the conditions of* Theorem 8.1 *be satisfied. Suppose that either (i) $f(x) > 0$ for all $x \in J$, or (ii) $f(x) < 0$ for all $x \in J$. Then*

$$\sum_{j=1}^{k} U_j f(A_j) U_j^* \le \lambda f\left(\sum_{j=1}^{k} U_j A_j U_j^*\right) \tag{64}$$

holds for some $\lambda > 1$ in case (i); or, $\lambda \in (0,1)$ in case (ii). More precisely, a value of λ (depending only on m, M, f) for (64) may be determined as follows:

Let $\mu = (f(M) - f(m))/(M - m)$. *If* $\mu = 0$, *let* $x = \bar{x}$ *be the unique solution of the equation* $f'(x) = 0, (m < \bar{x} < M)$; *then,* $\lambda = f(m)/f(\bar{x})$ *suffices for (64). If* $\mu \neq 0$, *let* $x = \bar{x}$ *be the unique solution in* (m, M) *of the equation*

$$\mu f(x) - f'(x)(f(m) + \mu(x - m)) = 0; \tag{65}$$

then $\lambda = \mu/f'(\bar{x})$ *suffices for (64).*

Theorem 8.4.. *Let the conditions of* Theorem 8.1 *be satisfied and let* f *be differentiable and* f' *strictly increasing on* J. *Then*

$$\sum_{j=1}^{k} U_j f(A_j) U_j^* \leq \lambda I + f\left(\sum_{j=1}^{k} U_j A_j U_j^*\right) \tag{66}$$

holds for some λ *satisfying* $0 < \lambda < (M - m)(\mu - f'(m))$ *where* μ *is defined as in* Theorem 8.3.

More precisely, λ may be determined for (66) as follows: let $x = \bar{x}$ be the unique solution of the equation $f'(x) = \mu$ $(m < \bar{x} < M)$, then

$$\lambda = f(m) - f(\bar{x}) + \mu(\bar{x} - m) \tag{67}$$

suffices in (66).

We now consider the special case when $\tilde{f}(t) = t^p, t > 0$. It is well-known that $\tilde{f}$ is convex if either $p < 0$ or $p > 1$, while it is concave if $0 < p < 1$ (i.e., $-\tilde{f}$ is convex if $0 < p < 1$). Thus, (64) gives for $p < 0$ or $p > 1$,

$$\sum_{j=1}^{k} U_j A_j^p U_j^* \leq \tilde{\lambda}\left(\sum_{j=1}^{k} U_j A_j U_j^*\right)^p \tag{68}$$

where

$$\tilde{\lambda} = \frac{\gamma^p - \gamma}{(p-1)(\gamma - 1)}\left\{\frac{p(\gamma - \gamma^p)}{(1-p)(\gamma^p - 1)}\right\}^{-p}, \ \gamma = M/m$$

while for $0 < p < 1$, we have the reverse inequality in (68).

Similar consequences of *Theorems 8.4* are, in the case $p < 0$ or $p \geq 1$,

$$\sum_{j=1}^{k} U_j A_j^p U_j^* \leq \tilde{\lambda} I + \left(\sum_{j=1}^{k} U_j A_j U_j^*\right)^p$$

where

$$\tilde{\lambda} = m^p - \left(\frac{1}{p}\frac{M^p - m^p}{M - m}\right)^{\frac{p}{p-1}} + \frac{M^p - m^p}{M - m}\left[\left(\frac{1}{p}\frac{M^p - m^p}{M - m}\right)^{\frac{1}{p-1}} - m\right]$$

where for $0 < p < 1$, we have the reverse inequality in (8.).

9. Inequalities for the Hadamard product of matrices

If A, B are positive semi-definite $n \times n$ Hermitian matrices, then $A^2 \circ B^2 - (A \circ B)^2$ is positive semidefinite, i.e., in inequality form, we have (see [14])

$$(A \circ B)^2 \leq A^2 \circ B^2 \tag{69}$$

where $A \circ B$ is the Hadamard product of matrices A and B.

From this inequality, we can also get (see [14]):

$$A \circ B \leq (A^2 \circ B^2)^{1/2} \tag{70}$$

and

$$A^{1/2} \circ B^{1/2} \leq (A \circ B)^{1/2}. \tag{71}$$

Some converse results have been obtained recently in [21]. We have

$$A^2 \circ B^2 - (A \circ B)^2 \leq \frac{1}{4}(M - m)^2 I, \tag{72}$$

and

$$(A^2 \circ B^2)^{1/2} \leq \frac{M + m}{2\sqrt{Mm}} A \circ B, \tag{73}$$

where A and B are positive definite Hermitian matrices, M and m are the largest and the smallest eigenvalues of $A \otimes B$ (the Kronecker product of A and B), respectively.

Some generalizations and related results will be given in this paper.

Theorem 9.1.. *Let A and B be positive definite $n \times n$ Hermitian matrices and let r and s be two nonzero integers such that $s > r$. Then*

$$(A^s \circ B^s)^{1/s} \geq (A^r \circ B^r)^{1/r} \tag{74}$$

Proof. The following result holds (see [36]):

Let A be an $n \times n$ positive definite Hermitian matrix and let V be an $n \times t$ matrix such that $V^* V = I$. Then

$$(V^* A^s V)^{1/s} \geq (V^* A^r V)^{1/r} \tag{75}$$

for all real r and s such that $s \notin (-1, 1)$ and $r \notin (-1, 1)$, $s > r$.

In our case, nonzero integers r and s satisfy these conditions. Further, instead of V, we use J, the selection matrix of order $n^2 \times n$ with the property that (see [20], [21])

$$A \circ B = J^t (A \otimes B) J \tag{76}$$

as well as the fact that for any integer p we have

$$(A \otimes B)^p = A^p \otimes B^p. \tag{77}$$

Thus (75) gives $(J^t (A \otimes B)^s J)^{1/s} \geq (J^t (A \otimes B)^r)^{1/r}$, and, from (77), $(J^t (A^s \otimes B^s) J)^{1/s} \geq (J^t (A^r \otimes B^r) J)^{1/r}$, which is, by (76), the inequality (74). □

Special cases. Some special cases of (74) are the following:

$$(A^{-1} \circ B^{-1})^{-1} \le A \circ B \tag{78}$$

or, equivalently,

$$(A \circ B)^{-1} \le A^{-1} \circ B^{-1}. \tag{79}$$

For the positive integer r,

$$A \circ B \le (A^r \circ B^r)^{1/r} \tag{80}$$

from which we can get

$$A^{1/r} \circ B^{1/r} \le (A \circ B)^{1/r}. \tag{81}$$

These last two results are extensions of (70) and (71).

Remark.. *Inequalities (69) and (79) can be obtained by using Jensen's inequality for the matrix convex functions, i.e., for matrix convex function f (see [18])*

$$f(V^*AV) \le V^* f(A) V. \tag{82}$$

Namely, using this result for the matrix convex function $f(t) = t^2$, we can get

$$\begin{aligned} (A \circ B)^2 &= (J^t(A \otimes B)J)^2 \le J^t(A \otimes B)^2 J \\ &= J^t(A^2 \otimes B^2)J = A^2 \circ B^2 \end{aligned}$$

which is (69). Similarly, we can use (82) for the matrix convex function $f(t) = t^{-1}$ to get (79).

Theorem 9.2.. *Let A and B be two positive definite $n \times n$ Hermitian matrices and let r and s be nonzero integers such that $r < s$. Then*

$$r(A^r \circ B^r - aA^s \circ B^s - bI) \ge 0 \tag{83}$$

where $a = (M^r - m^r)/(M^s - m^s)$, $\quad b = (M^s m^r - M^r m^s)/(M^s - m^s)$ and M and m are the largest and smallest eigenvalues of $A \otimes B$.

Proof. We have the matrix inequality (see [13]) $r(A^r - aA^s - bI) \ge 0$ i.e.,

$$r[(A \otimes B)^r - a(A \otimes B)^s - bI] \ge 0.$$

Therefore, from (77),

$$r[A^r \otimes B^r - a(A^s \otimes B^s) - bI] \ge 0.$$

Now pre- and post-multiplication by J^t and J, respectively, gives (83). □

Remark.. *We can also prove* Theorem 9.2 *by using* Theorem 1 *from [32].*

Theorem 9.3.. *Let the conditions of* Theorem 9.2 *be satisfied. Then*

$$(A^s \circ B^s)^{1/s} \le \tilde{\Delta}(A^r \circ B^r)^{1/r} \tag{84}$$

where

$$\tilde{\Delta} = \left\{ \frac{r(\gamma^s - \gamma^r)}{(s-r)(\gamma^r - 1)} \right\}^{1/s} \quad \left\{ \frac{s(\gamma^r - \gamma^s)}{(r-s)(\gamma^s - 1)} \right\}^{-1/r} \tag{85}$$

and $\quad \gamma = M/m$.

Proof. Let A be an $n\times n$ positive definite Hermitian matrix with eigenvalues contained in the interval $[m, M]$ where $0 < m < M$ and let V be an $n \times t$ matrix such that $V^*V = I$. If r, s are nonzero real numbers such that $s > r$ and either $s \notin (-1, 1)$ or $r \notin (-1, 1)$, then (see [32])

$$(V^* A^s V)^{1/s} \le \tilde{\Delta}(V^* A^r V)^{1/r} \tag{86}$$

where $\tilde{\Delta}$ is given by (85). Therefore, in our case, we have

$$(A^s \circ B^s)^{1/s} = (J^t(A^s \otimes B^s)J)^{1/s} = (J^t(A \otimes B)^s J)^{1/s}$$

$$\le \tilde{\Delta}(J^t(A \otimes B)^r J)^{1/r} = \tilde{\Delta}(J^t(A^r \otimes B^r)J)^{1/r} = \tilde{\Delta}(A^r \circ B^r)^{1/r}. \quad \square$$

Special cases.

1. For $s = 2$ and $r = 1$, we get (73).
2. For $s = 1,\ r = -1$, we get

$$A \circ B \le \frac{(m+M)^2}{4Mm}(A^{-1} \circ B^{-1})^{-1} \tag{87}$$

or, equivalently,

$$A^{-1} \circ B^{-1} \le \frac{(M+m)^2}{4Mm}(A \circ B)^{-1} \tag{88}$$

Theorem 9.4.. *Let the conditions of* Theorem 9.2 *be satisfied. Then*

$$(A^s \circ B^s)^{1/s} - (A^r \circ B^r)^{1/r} \le \Delta I \tag{89}$$

where

$$\Delta = \max_{\theta \in [0,1]} \left\{ [\theta M^s + (1-\theta)m^s]^{1/s} - [\theta M^r + (1-\theta)m^r]^{1/r} \right\}. \tag{90}$$

Proof. Let A be an $n\times n$ positive definite Hermitian matrix with eigenvalues contained in the interval $[m, M]$, where $0 < m < M$, and let V be an $n \times t$ matrix such that $V^*V = I$. If $r,\ s$ are nonzero real numbers such that $s > r$ and either $s \notin (-1, 1)$ or $r \notin (-1, 1)$, then (see [32])

$$(V^* A^s V)^{1/s} - (V^* A^r V)^{1/r} \le \Delta I \tag{91}$$

where Δ is given by (90). Thus, in our case, we have

$$\begin{aligned}(A^s \circ B^s)^{1/s} - (A^r \circ B^r)^{1/r} &= [J^t(A^s \otimes B^s)]^{1/s} - -[J^t(A^r \otimes B^r)J]^{1/r} \\ &= [J^t(A \otimes B)^s J]^{1/s} - [J^t(A \otimes B)^r J]^{1/r} \le \Delta I. \quad \square\end{aligned}$$

Special cases.

1. For $s=2,\ r=1$, we get

$$(A^2 \circ B^2)^{1/2} - A \circ B \leq \frac{(M-m)^2}{4(M+m)} I \tag{92}$$

2. For $s=1,\ r=-1$, we get

$$A \circ B - (A^{-1} \circ B^{-1})^{-1} \leq (\sqrt{M} - \sqrt{m})^2 I \tag{93}$$

We note that the eigenvalues of $A \otimes B$ are the n^2 products of the eigenvalues of A by the eigenvalues of B [15, p.245]. Thus, if the eigenvalues of A and B, respectively, are ordered by

$$\alpha_1 \geq \alpha_2 \geq \ldots \geq \alpha_n > 0;\ \beta_1 \geq \beta_2 \geq \ldots \geq \beta_n,$$

then in all previous results $M = \alpha_1\beta_1$ and $m = \alpha_n\beta_n$.

Thus (72), (73), (87), (86), (92) and (93) become

$$A^2 \circ B^2 - (A \circ B)^2 \leq \tfrac{1}{4}(\alpha_1\beta_1 - \alpha_n\beta_n)^2 I,$$

$$(A^2 \circ B^2)^{1/2} \leq \tfrac{(\alpha_1\beta_1 + \alpha_n\beta_n)}{2\sqrt{\alpha_1\beta_1\alpha_n\beta_n}} A \circ B,$$

$$A \circ B \leq \tfrac{(\alpha_1\beta_1 + \alpha_n\beta_n)^2}{4\alpha_1\beta_1\alpha_n\beta_n} (A^{-1} \circ B^{-1})^{-1},$$

$$A^{-1} \circ B^{-1} \leq \tfrac{(\alpha_1\beta_1 + \alpha_n\beta_n)^2}{4\alpha_1\beta_1\alpha_n\beta_n} (A \circ B)^{-1},$$

$$(A^2 \circ B^2)^{1/2} - A \circ B \leq \tfrac{(\alpha_1\beta_1 - \alpha_n\beta_n)^2}{4(\alpha_1\beta_1 + \alpha_n\beta_n)} I,$$

respectively and

$$A \circ B - (A^{-1} \circ B^{-1})^{-1} \leq (\sqrt{\alpha_1\beta_1} - \sqrt{\alpha_n\beta_n})^2 I.$$

Remark .. *Previous results are obtained in [34]. For some related results see [40], while some results for generalized inverses are given in [35].*

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