On regular almost convergence*

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Abstract. *In this paper possible regularity definitions for almost convergent double sequences are considered as generalizations of the regular convergence in the sense of G. H. Hardy and F. Móricz. Classes of almost convergent sequences with almost convergent rows and columns are characterized; also, a theorem on the principal limit and on row (as well as column) limits is proved.*

Key words: *almost convergence, double sequence, principal limit, regular convergence, row limit*

Sažetak. **O regularnoj skoro konvergenciji.** *U radu se ispituju moguće definicije regularnosti skoro konvergentnih dvostrukih nizova kao poopćenja regularne konvergencije u smislu G. H. Hardyja i F. Móricza. Karakteriziraju se klase skoro konvergentnih nizova koji imaju skoro konvergentne retke i stupce; također je dokazan i teorem o glavnom limesu i o limesima redaka (kao i stupaca).*

Ključne riječi: *skoro konvergencija, dvostruki nizovi, glavni limes, regularna konvergencija, limes retka*

1. Regular convergence

It seems that stronger convergence definitions for double sequences were initiated by the study of double series. Let

$$\sum_{i=1}^{\infty}\sum_{j=1}^{\infty} x_{ij} \tag{1}$$

be a double series of complex numbers. If this series converges absolutely, it can be summed by rows or by columns, and its sum by rows or by columns is equal to any of the accepted values of (1) as a double series. In particular, the row or

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column sum is equal to the sum of (1) in the sense of Pringsheim (see [1]), i. e. to the limit of the sequence of *partial sums*

$$s_{pq} = \sum_{i=1}^{p} \sum_{j=1}^{q} x_{ij}. \tag{2}$$

The limit is a usual one: a double sequence (x_{ij}), $i, j \in N$, converges to L (when both indexes tend to $+\infty$) if, for every $\epsilon > 0$, there exists $N_\epsilon \in N$ such that

$$|x_{ij} - L| < \epsilon \quad \text{if} \quad \min(i, j) \geq N_\epsilon. \tag{3}$$

The limit L is denoted by $\lim_{ij}$. A necessary and sufficient condition for the existence of $\lim_{ij}$ is the Cauchy condition: for every $\epsilon > 0$ there exists N_ϵ such that, for every $k, l \in N_0$,

$$|\Delta_{kl} x_{ij}| < \epsilon \quad \text{if} \quad \min(i, j) \geq N_\epsilon, \tag{4}$$

where

$$\Delta_{kl} x_{ij} = x_{ij} - x_{i+k, j+l}. \tag{5}$$

In case of double series, as well as in case of double sequences, convergence does not imply convergence of rows and columns. Various authors have studied nonabsolutely convergent double series with convergent rows and columns (cf. [4, p.88] and references to [1] there), but it was G. H. Hardy who connected such series to an alternative definition of convergence of double sequences. A sequence (x_{ij}) converges, after Hardy, *regularly* if there exist $L_j = \lim_i x_{ij}$ for every j, $L_i = \lim_j x_{ij}$ for every i, and $L = \lim_{ij} x_{ij}$ [4, *ibid.*]. In case of regularly convergent sequences the limit L is called the *principal limit*. Therefore,

$$\lim_{ij} x_{ij} = \lim_i \lim_j x_{ij} = \lim_j \lim_i x_{ij}. \tag{6}$$

Much later, and independently of Hardy, convergence of double series were studied by F. Móricz [6]. He considers *rectangular sums* which can be written as

$$s_{pq}^{mn} = \sum_{i=m}^{m+p-1} \sum_{j=n}^{n+q-1} x_{ij} \tag{7}$$

for $m, n, p, q \in N$. Móricz defines convergence in the *restricted* sense by the condition that for every $\epsilon > 0$ there exists $N_\epsilon \in N$ such that

$$|s_{pq}^{mn}| < \epsilon \quad \text{if} \quad \max(m, n) \geq N_\epsilon. \tag{8}$$

This definition corresponds to *Hardy's Lemma* δ [4, p.89], which is correct, although *Lemma* γ, used in the proof, does not characterize convergent series which have convergent rows and columns. On the other hand, Móricz shows [6, Theorem 1, p.136] that his "restricted convergence" is precisely the Hardy's regular convergence for series. Notice, that in [4] Hardy afterwards uses only *Lemma* δ, and besides *Lemma* γ gives no connection between sequence and series regularity.

If we wish to see what the Móricz definition is in terms of partial sums, we have to write s_{pq}^{mn} in terms of s_{ij}:

$$s_{pq}^{mn} = \begin{cases} s_{p+m-1,q+n-1} - s_{p+m-1,n-1} - s_{m-1,q+n-1} + s_{m-1,n-1} & \text{if } m,n \geq 2 \\ s_{p,q+n-1} - s_{p,n-1} & \text{if } m=1, n\geq 2 \\ s_{p+m-1,q} - s_{m-1,q} & \text{if } m\geq 2, n=1 \\ s_{pq} & \text{if } m=n=1. \end{cases} \tag{9}$$

It takes a more compact form if s_{ij} is extended to $N_0 \times N_0$ by $s_{i0} = s_{0j} = s_{00} = 0$ as it is done in [6, Remark 2, p.136] and suggested in [4, *Lemma* δ, p.89]. Namely, for extended partial sums we have

$$s_{pq}^{mn} = \Box_{pq} s_{m-1,n-1}, \tag{10}$$

where

$$\Box_{kl} s_{ij} = s_{ij} - s_{i+k,j} - s_{i,j+l} + s_{i+k,j+l}. \tag{11}$$

The Móricz definition in terms of partial sums requires that, for every $\epsilon > 0$, there exists N_ϵ such that, for every $p, q \in N_0$,

$$|\Box_{pq} s_{m-1,n-1}| < \epsilon \quad \text{if} \quad \max(m,n) \geq N_\epsilon. \tag{12}$$

The next example shows that (12) is not equivalent to the Hardy regular convergence of s_{ij} if the sequence is not extended by a nul-row and a nul-column, or if other additional conditions on the unextended sequence are not imposed:

Example 1. Let inside the frame be the sequence (s_{ij}) on $N \times N$, and let the whole scheme, with a frame of 0's, be the sequence (s_{ij}) extended to $N_0 \times N_0$.

For the unextended sequence (12) is satisfied, but nevertheless the sequence has not convergent rows.

The equivalence of Hardy and of Móricz regularity for sequences is obvious by a proposition which characterizes Hardy's regularity via the uniformity of convergence of rows and columns of (x_{ij}). Uniformity means that for every $\epsilon > 0$ there exist N'_ϵ, $N''_\epsilon \in N$ such that, for all j,

$$|x_{ij} - L_j| < \epsilon \quad \text{if} \quad i \geq N'_\epsilon, \tag{13}$$

and, for all i

$$|x_{ij} - L_i| < \epsilon \quad \text{if} \quad j \geq N''_\epsilon. \tag{14}$$

An equivalent statement is that, for every $\epsilon > 0$, there exist N'_ϵ, N''_ϵ such that, for all i and k,

$$|\triangle_{k0} x_{ij}| < \epsilon \quad \text{if} \quad i \geq N'_\epsilon \tag{15}$$

and, for all j and l,

$$|\triangle_{0l} x_{ij}| < \epsilon \quad \text{if} \quad j \geq N''_\epsilon \tag{16}$$

For multiple sequences a general theorem of this kind was proved by H. J. Hamilton; we write down the proof for double sequences, which has to be compared to the analogous theorem for almost convergent sequences (*Theorem 2* below):

Proposition 1. [3, Theorem (.003), p.34]. *The double sequence (x_{ij}) is regularly convergent if and only if the family of rows*

$$\{(x_{ij})_{i \in N} : j \in N\}$$

as well as the family of columns

$$\{(x_{ij})_{j \in N} : i \in N\}$$

converge uniformly.

Proof. Uniform convergence imply convergence of rows and columns, and for $N_\epsilon = \max(N'_\epsilon, N''_\epsilon)$, by (15)-(16)

$$|\triangle_{kl} x_{ij}| \leq |\triangle_{k0} x_{ij}| + |\triangle_{0l} x_{i+k,j}| \leq 2\epsilon \quad \text{if} \quad i, j \geq N_\epsilon, \tag{17}$$

which gives (4) and the existence of the principal limit.

If, on the contrary, the principal limit and all row and column limits exist, by (4) for $l = 0$ and every k we have

$$|\triangle_{k0} x_{ij}| < \epsilon \quad \text{if} \quad i, j \geq N_\epsilon. \tag{18}$$

The convergence of sequences from

$$\{(x_{ij})_{i \in N} : 1 \leq j < N_\epsilon\}$$

gives the existence of $N^0_\epsilon \in N$ such that (18) holds true for $i \geq N^0_\epsilon$, $1 \leq j < N_\epsilon$. With $N'_\epsilon = \max(N_\epsilon, N^0_\epsilon)$ and $i \geq N'_\epsilon$ we have the uniform convergence of rows; for columns similarly. □

Now, the Móricz condition (12) for $m = 1$ gives the uniformity of row limits (cf. (8) and (9)); similarly for columns. The opposite implication follows by

$$|\Box_{pq} s_{m-1,n-1}| \leq |\triangle_{p0} s_{m-1,n-1}| + |\triangle_{p0} s_{m-1,n+q-1}| \tag{19}$$

$$|\Box_{pq} s_{m-1,n-1}| \leq |\triangle_{0q} s_{m-1,n-1}| + |\triangle_{0q} s_{m+p-1,n-1}| \tag{20}$$

and relations analogous to (15)-(16).

What happens if in (3) instead of min we take max? That means that we ask that for every $\epsilon > 0$ there exists N_ϵ such that

$$|x_{ij} - L| < \epsilon \quad \text{if} \quad \max(i,j) \geq N_\epsilon. \tag{21}$$

This implies that L is a principal limit, but also that all rows and columns converge to L uniformly! Therefore, (x_{ij}) converges regularly, but moreover, all row as well as column limits are equal:

$$\lim_{ij} x_{ij} = \lim_i x_{ij} = \lim_j x_{ij}. \tag{22}$$

This is why [4, *Lemma* γ, p.89] does not characterize regular convergence: it states (21) for partial sums, which is too strong; regular convergence for sums is given by the weaker Móricz condition (12). We obtain the same class as by (21) by the Cauchy condition (4) with max instead of min: for every $\epsilon > 0$ let N_ϵ be such that

$$|\triangle_{kl} x_{ij}| < \epsilon \quad \text{if} \quad \max(i,j) \geq N_\epsilon. \tag{23}$$

Taking $l = 0$, and then $k = 0$, we have the regular convergence of (x_{ij}) by *Proposition 1.* But there are not j, l such that

$$L_j = \lim_i x_{ij} \neq \lim_i x_{i,j+l} = L_{j+l}. \tag{24}$$

In case that such j, l exist, for $\epsilon = |L_j - L_{j+l}|$ and $k = 0$ (23) gives

$$|\triangle_{0l} x_{ij}| < \frac{\epsilon}{2} \quad \text{if} \quad i \geq N_{\frac{\epsilon}{2}}, \tag{25}$$

which yields a contradiction (cf. the proof of *Theorem 3*). On the other hand, if (21) holds with $i \geq N_{\frac{\epsilon}{2}}$, we have $\max(i+k, j+l) \geq N_{\frac{\epsilon}{2}}$, for all k, j and l, and

$$|(x_{ij} - L) + (x_{i+k,j+l} - L)| \leq |x_{ij} - L| + |x_{i+k,j+l} - L| < \epsilon. \tag{26}$$

Similarly, with $j \geq N_{\frac{\epsilon}{2}}$, which gives (23).

Denoting various classes of double sequences we will retain as much as possible the notation in [3]; "rows" in case of double sequences means there rows and columns. The set of all double sequences of complex numbers is denoted by **s**, the set of bounded such sequences by **b**, and the set of convergent sequences (in the sense of (3)) by **c**; bounded and convergent sequences from **s** we denote by **bc**. Regularly convergent sequences from **s** are denoted by **rc**. Obviously

rc $\subset$ **c**, but, contrary to **c**, **rc** $\subset$ **b**. Sequences from **rc** with the principal limit equal to 0 are denoted by **rcn**. Sequences from **rc** with all row and column limits equal to 0 are denoted, as in [3], by **rcrn** ("regularly convergent row null"). Sequences from **rc** with equal row and column limits (i. e. sequences that satisfy (21)) we denote by **rcr**. This class was considered in [2] ("sequences convergent in the Hardy's sense"). Sequences from **rcr** with the principal limit equal to 0 are again the **rcrn** class.

2. Regular almost convergence

The usual definition of almost convergence involves uniform limits: a double sequence (x_{ij}) is almost convergent to L if arithmetical means

$$\sigma_{pq}^{mn} = \frac{1}{pq} s_{pq}^{mn} = \frac{1}{pq} \sum_{i=m}^{m+p-1} \sum_{j=n}^{n+q-1} x_{ij} \tag{27}$$

converge to L uniformly with respect to m and n. More precisely, for every $\epsilon > 0$ there exists $N_\epsilon \in N$ such that, for all $m, n \in N$,

$$|\sigma_{pq}^{mn} - L| < \epsilon \quad \text{if} \quad \min(p, q) \geq N_\epsilon. \tag{28}$$

An equivalent definition is given by the Cauchy property, which is (4) written for (27): for every $\epsilon > 0$ there exists N_ϵ such that, for all $m, n \in N$ and $k, l \in N_0$,

$$|\triangle_{kl} \sigma_{pq}^{mn}| < \epsilon \quad \text{if} \quad \min(p, q) \geq N_\epsilon. \tag{29}$$

The set of all double sequences for which (28) holds true is a linear space and is called the space of *almost convergent* sequences (cf. [5]); it is denoted by **ac**. It is known that **ac** $\subset$ **b**, and that **bc** $\subset$ **ac** [7]. The number L is called the *generalized principal limit* and is denoted by Lim_{ij} (if x is indexed by i, j); it is a linear form on **ac** and its restriction to **bc** is $\lim_{ij}$.

A class of almost convergent double sequences which can be considered analogous to the class of regularly convergent double sequences is a subclass of **ac** sequences with all rows and all columns almost convergent. We call this class a class of *regularly almost convergent* sequences and denote it by **rac**. If all generalized row limits $L_j = \mathrm{Lim}_i x_{ij}$ and column limits $L_i = \mathrm{Lim}_j x_{ij}$ are equal, the corresponding class will be denoted by **racr**, and if in addition L_j and L_i are 0, the class will be denoted by **racrn**. This is, on the other hand, the class of sequences from **racr** with the generalized principal limit equal to 0, as it follows by the following theorem:

Theorem 1.. *Let (x_{ij}) be an almost convergent double sequence with the generalized principal limit L, and let, moreover, every row $(x_{ij})_{i \in N}$ be almost convergent to L_j. Then the sequence $(L_j)_{j \in N}$ is almost convergent to L, i. e.*

$$\mathrm{Lim}_{ij} x_{ij} = \mathrm{Lim}_j \mathrm{Lim}_i x_{ij}.$$

Similarly for columns.

Proof. We have to prove that for every $\epsilon > 0$ there exists N''_ϵ such that, for all n,

$$\left| \frac{1}{q} \sum_{j=n}^{n+q+1} L_j - L \right| < \epsilon \quad \text{if} \quad q \geq N''_\epsilon. \tag{30}$$

Let us take $q \geq N_{\frac{\epsilon}{2}}$ with N_ϵ defined by (28). For every j there exists $N_\epsilon(j)$ such that, for all m,

$$\left| \frac{1}{p} \sum_{i=m}^{m+p-1} x_{ij} - L_j \right| < \epsilon \quad \text{if} \quad p \geq N_\epsilon(j). \tag{31}$$

Fix any n and take some p such that

$$p \geq \max(N_{\frac{\epsilon}{2}}, N_{\frac{\epsilon}{2}}(j = n), ..., N_{\frac{\epsilon}{2}}(j = n + q - 1)). \tag{32}$$

By (31) we have, for our n and every m,

$$\left| \frac{1}{pq} \sum_{i=m}^{m+p-1} \sum_{j=n}^{n+q-1} x_{ij} - \frac{1}{q} \sum_{j=n}^{n+q-1} L_j \right| \leq \frac{1}{q} \sum_{j=n}^{n+q-1} \left| \frac{1}{p} \sum_{j=m}^{m+p-1} x_{ij} - L_j \right| \leq \frac{\epsilon}{2}. \tag{33}$$

As $p, q \geq N_{\frac{\epsilon}{2}}$, by (28) and (33),

$$\left| \frac{1}{q} \sum_{j=n}^{n+q-1} L_j - L \right| \leq \left| \left(\frac{1}{pq} \sum_{i=m}^{m+p-1} \sum_{j=n}^{n+q-1} x_{ij} - L \right) - \left(\frac{1}{pq} \sum_{i=m}^{m+p-1} \sum_{j=n}^{n+q-1} x_{ij} - \frac{1}{q} \sum_{j=n}^{n+q-1} L_j \right) \right| < \epsilon. \tag{34}$$

Therefore, with $N''_\epsilon = N_{\frac{\epsilon}{2}}$ we have (30) for every n. Similarly for columns. □

The following examples show that, in spite of the analogy between **rc** and **rac**, for **rac** a statement similar to the statement of *Proposition 1* has no place.

Example 2. Let (x_{ij}) be

The number of 1's that exceeds the number of -1's or vice versa in any $p \times q$ block is bounded by $\max(p, q)$, and therefore $\text{Lim}_{ij} x_{ij} = 0$; also, rows and columns almost converge to 0. Therefore, the sequence is from **racrn**, and

$$\mathrm{Lim}_{ij} x_{ij} = \mathrm{Lim}_i \mathrm{Lim}_j x_{ij} = \mathrm{Lim}_j \mathrm{Lim}_i x_{ij} \tag{35}$$

holds (with outer limits reduced to lim). But there is no uniformity in j, i. e.

$$\left| \frac{1}{p} \sum_{i=m}^{m+p-1} x_{ij} - L_j \right| < \epsilon$$

for all m and with the same p: for every p there is a row with a p-long initial part of 1's (or -1's). Similarly holds for columns.

Example 3. Let (x_{ij}) be

Here again $\mathrm{Lim}_{ij} x_{ij} = 0$, rows and columns are convergent to 1 or -1, and (35) holds true (with inner generalized limits reduced to lim). The sequence is from **racn**, but again there are initial parts of rows and columns with alternating signs of 1's as long as you please.

On the other hand, it is easy to find examples of almost convergent double sequences with rows and columns uniformly almost convergent. It means that for every $\epsilon > 0$ there exists N'_ϵ such that, for all $j, m \in N$,

$$\left| \frac{1}{p} \sum_{i=m}^{m+p-1} x_{ij} - L_j \right| < \epsilon \quad \text{if} \quad p \geq N'_\epsilon; \tag{36}$$

and similarly, N''_ϵ such that, for all $i, n \in N$

$$\left| \frac{1}{q} \sum_{j=n}^{n+q-1} x_{ij} - L_i \right| < \epsilon \quad \text{if} \quad q \geq N''_\epsilon; \tag{37}$$

These conditions are equivalent to

$$\left| \frac{1}{p} \sum_{i=m}^{m+p-1} x_{ij} - \frac{1}{p+k} \sum_{i=m}^{m+p+k-1} x_{ij} \right| < \epsilon \quad \text{if} \quad p \geq N'_\epsilon, \tag{38}$$

resp. to

$$\left|\frac{1}{q}\sum_{j=n}^{n+q-1} x_{ij} - \frac{1}{q+l}\sum_{j=n}^{n+q+l-1} x_{ij}\right| < \epsilon \quad \text{if} \quad q \geq N''_\epsilon, \tag{39}$$

where i, j, m, n are from N and $k, l \in N_0$.

Example 4. Let (x_{ij}) be

The uniformity in this example is obvious, as well as the existence of the principal limit. In the general case of uniform almost convergent rows and columns, by (38) and (39) we have

$$|\triangle_{k0}\sigma_{pq}^{mn}| < \epsilon \quad \text{if} \quad p \geq N'_\epsilon; \tag{40}$$

$$|\triangle_{0l}\sigma_{pq}^{mn}| < \epsilon \quad \text{if} \quad q \geq N''_\epsilon$$

with $m, n, N'_\epsilon, N''_\epsilon$ as before; also, by

$$|\triangle_{kl}\sigma_{pq}^{mn}| \leq |\triangle_{k0}\sigma_{pq}^{mn}| + |\triangle_{0l}\sigma_{p+k,q}^{mn}| \tag{41}$$

(cf. (17)) with $p, q \geq \max(N'_\epsilon, N''_\epsilon) = N_\epsilon$ we have (29) i. e. the existence of $\text{Lim}_{ij} x_{ij}$.

The class of double sequences which satisfy (36) and (37) is therefore a subclass of **rac**. Because uniform almost convergence of rows and columns makes this class in many respects similar to the class **rc**, we call it *almost regularly convergent* and denote it by **arc**. As

$$|x_{ij} - L_j| < \epsilon \;\Rightarrow\; \left|\frac{1}{p}\sum_{i=m}^{m+p-1} x_{ij} - L_j\right| < \epsilon, \tag{42}$$

the uniform convergence of rows implies the uniform almost convergence of rows (and, similarly, for columns); therefore, **rc** $\subset$ **arc** $\subset$ **rac**.

Although arithmetical means can be given in a form which resembles Móricz rectangular sums (means are sums divided by the area of the rectangles), convergence of means is a generalization of the convergence of sequences: means are in a way connected to partial sums, and not to rectangular sums. Therefore, the next theorem is expected; but notice, that a Móricz-type condition gives the class with uniformly converging rows and columns and not the more general case of **rac**.

Theorem 2.. *A sequence* (x_{ij}) *is* **arc** *if and only if for every* $\epsilon > 0$ *there exists* $N_\epsilon \in N$ *such that, for every* $p, q, m, n \in N$ *and* $k, l \in N_0$,

$$|\square_{kl}\sigma_{pq}^{mn}| < \epsilon \quad \textit{if} \quad \max(p,q) \geq N_\epsilon, \tag{43}$$

where σ_{pq}^{mn} *is the sequence of arithmetical means extended by* $\sigma_{p0}^{mn} = \sigma_{0q}^{mn} = \sigma_{00}^{mn} = 0$.

Proof. For $q = 0$, $l = 1$ and $n = j$ (43) gives (38), which is the uniform almost convergence of rows; for columns similarly, with $p = 0$, $k = 1$ and $m = i$.

If, on the contrary, both families of rows and of columns are uniformly almost convergent, the factorization

$$\left(\frac{1}{q}\sum_{j=n}^{n+q-1} - \frac{1}{q+l}\sum_{j=n}^{n+q+l-1}\right)\left(\frac{1}{p}\sum_{i=m}^{m+p-1} - \frac{1}{p+k}\sum_{i=m}^{m+p+k-1}\right)x_{ij} \tag{44}$$

gives (43) in case of $p \geq N'_{\frac{\epsilon}{2}}$ and $q \geq 1$ with N'_ϵ given by (38); for $q = 0$, (40) and $p \geq N'_{\frac{\epsilon}{2}}$ with N'_ϵ given there give again the required inequality. In case of $q \geq N''_{\frac{\epsilon}{2}}$, (43) comes out by the factorization similar to (44), but with factors reversed in order. Therefore, we have (43) with $N_\epsilon = \max(N'_{\frac{\epsilon}{2}}, N''_{\frac{\epsilon}{2}})$. □

If all row and column limits, L_j and L_i, are equal, the class derived in such a way from **arc** i. e. the class **arcr** is easily characterized by a relation analogous to (21): for every $\epsilon > 0$ there exists N_ϵ such that, for every m, n,

$$|\sigma_{pq}^{mn} - L| < \epsilon \quad \text{if} \quad \max(p,q) \geq N_\epsilon. \tag{45}$$

For $q = 1$, $n = j$ and every m we have the uniform almost convergence of rows, and for $p = 1$, $m = i$ and every n for columns, while (36) and (37) with $L_j = L_i = L$ give (45). We put the counterpart of (23) in the following form:

Theorem 3.. *A sequence* (x_{ij}) *is* **arcr** *if and only if for every* $\epsilon > 0$ *there exists* $N_\epsilon \in N$ *such that, for every* $p, q, m, n \in N$ *and* $k, l \in N_0$,

$$|\triangle_{kl}\sigma_{pq}^{mn}| < \epsilon \quad \textit{if} \quad \max(p,q) \geq N_\epsilon. \tag{46}$$

Proof. By (46) with $q = 1$, $l = 0$, $n = j$ and every m, as well as with $p = 1$, $k = 0$, $m = i$ and every n we have uniform almost convergence of rows and columns. We have to prove, moreover, that for two rows the generalized limits are equal, i. e. that there is no j such that

$$L_j = \mathrm{Lim}_i x_{ij} \neq \mathrm{Lim}_i x_{i,j+1} = L_{j+1} \tag{47}$$

(and analogously for columns). In case that there exists such j, let $\epsilon = |L_j - L_{j+1}|$, and let us take $p \geq \max(N'_{\frac{\epsilon}{8}}, N_\epsilon)$, where N'_ϵ is given by (36) and N_ϵ by (46). Hence,

$$\left|\frac{1}{p}\sum_{i=m}^{m+p-1} x_{ij} - \frac{1}{p}\sum_{i=m}^{m+p-1} x_{i,j+1}\right| = \left|(\frac{1}{p}\sum_{i=m}^{m+p-1} x_{ij} - L_j) + (L_j - L_{j+1}) - (\frac{1}{p}\sum_{i=m}^{m+p-1} x_{i,j+1} - L_{j+1})\right|$$
$$\geq \epsilon - \frac{\epsilon}{8} - \frac{\epsilon}{8} = \frac{3\epsilon}{4}. \tag{48}$$

On the other hand, for $k = 0$ and $q = l = 1$, by (46) we have

$$|\triangle_{01}\sigma_{p1}^{mj}| = \left|\frac{1}{p}\sum_{i=m}^{m+p-1}\sum_{j=j}^{j} x_{ij} - \frac{1}{2p}\sum_{i=m}^{m+p-1}\sum_{j=j}^{j+1} x_{ij}\right| = \frac{1}{2}\left|\frac{1}{p}\sum_{i=m}^{m+p-1} x_{ij} - \frac{1}{p}\sum_{i=m}^{m+p-1} x_{i,j+1}\right| < \frac{\epsilon}{2} \tag{49}$$

which is a contradiction. The reverse is easy with $N_{\frac{\epsilon}{2}}$ from (45):

$$|(\sigma_{pq}^{mn} - L) - (\sigma_{p+k,q+l}^{mn} - L)| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon \quad \text{if} \quad \max(p,q) \geq N_{\frac{\epsilon}{2}}. \quad \square \tag{50}$$

Let us look at the end where the dichotomy of uniform and nonuniform convergence of rows and columns appears at first. The Hamilton theorem shows that this is not the case with regular convergence of sequences. Similarly, the Móricz condition (8) with $q = 1$, $n = j$ and $m \geq N_\epsilon$ gives the uniform convergence of rows for regularly convergent series; for columns similarly.

The simplest case with nonuniform row and column convergence is perhaps the Cesàro regular convergence: we have it, if in definitions of **rac**, **arc** etc. we put $m = n = 1$. *Examples 2* and *3* show that rows can converge nonuniformly, and *Example 4* is also an example of uniform convergence in case of the Cesàro regular convergence. We denote here only the general case, neglecting for the moment the subclasses with uniformly converging rows and columns; on the other hand, boundedness is an important and usually unavoidable condition in summability. Therefore, bounded Cesàro convergent sequences with Cesàro convergent rows and columns we denote by $\mathbf{brc}_C$, following the use we adapted in [2] ($\mathbf{bc}_C$ for bounded Cesàro convergent sequences, C stays for a double Cesàro matrix). If all Cesàro-limits of rows and columns are equal, the corresponding class is $\mathbf{brcr}_C$, etc. The inspection of the proof of *Theorem 1* with $n = 1$ gives at once for $\mathbf{brc}_C$ and Cesàro-limits C-lim

$$\text{C-lim}_{ij} x_{ij} = \text{C-lim}_j \text{C-lim}_i x_{ij} = \text{C-lim}_i \text{C-lim}_j x_{ij}. \tag{51}$$

Finally, let us list classes we considered; they are ordered in a diagram, where arrows $\rightarrow$ stay for inclusion $\supset$:

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