Wavelets: convergence almost everywhere*

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Abstract. *It has been proved in [**?**], using the Carleson-Hunt theorem on the pointwise convergence of Fourier series, that the wavelet inversion formula is valid pointwise for all L^p-functions, and also without restrictions on wavelets.*

Key words: *wavelets, (a.e.) convergence*

Sažetak. **Valići: konvergencija gotovo svuda.** *U radu [**?**] dokazana je konvergencija gotovo svuda za formulu inverzije općih valića i za L^p - funkcije. U dokazu je korišten vrlo netrivijalni Carleson-Hunt-ov teorem o točkovnoj konvergenciji Fourierovih redova.*

Ključne riječi: *valići, konvergencija gotovo svuda*

Consider a time-dependent incoming signal $f(t)$. We are interested in its *frequency content locally in time t*. A classical windowed Fourier transform is obtained by first choosing a "window" (time localization) $g(t)$ and then taking its Fourier transform

$$(\mathcal{F}^{win} f)(\omega, t) = \int f(s) g(s-t) \mathrm{e}^{-i\omega s} ds \, . \tag{1}$$

However, notice that the "window" has the same "width" regardless of frequency ω. If one wants to "zoom-in" on high-frequency phenomena with a very short "life-span", then one would prefer a "window" whose "width" is adjusted to the corresponding frequency (narrower "window" with higher frequency). Such is the idea of *wavelets*. Instead of taking a family of functions $\{g^{\omega,t}\}$ as in (**??**); where $g^{\omega,t}(s) = g(s-t)\mathrm{e}^{-i\omega s}$, one can apply dilates and translates of a single function ψ (known as a *mother wavelet*). In this way one obtains a family $\{\, \psi^{a,b} \; : \; a, b \in R, \; a \neq 0 \,\}$ of *wavelets*, where

$$\psi^{a,b}(s) = |\, a \,|^{-1/2} \psi \left(\frac{s-b}{a} \right) \, . \tag{2}$$

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The *continuous wavelet transform* is then defined by

$$(T^{wav} f)(a,b) = |\, a\,|^{-1/2} \int f(t)\bar{\psi}\left(\frac{t-b}{a}\right) dt\,. \tag{3}$$

In the early eighties similar ideas were applied in several papers by A. Grossmann and J. Morlet (see, in particular, [**?**]). Their work stems from the seizmic data analysis, and they coined the term wavelets. It is worth mentioning that many of the even earlier ideas, coming from various fields (A. P. Calderón's work in harmonic analysis, J. R. Klauder's work on coherent states in quantum physics, D. Gabor's theory of communication), can be interpreted in the same manner. It is beyond the scope of this article to go into the survey of the already enormous literature on wavelets. However, we recommend to the interested reader the literature survey in [**?**], and the recent books by I. Daubechies [**?**], E. Hernández and G. Weiss [**?**], and M. V. Wickerhauser [**?**].

One of the most important questions in wavelet analysis is how to recover the original function f from its wavelet transform. This is obtained by the inversion formula, which is easy to establish in the L^2 sense, but much more difficult in the almost everywhere sense. In the paper [**?**] by M.Rao, H.Šikić, and R.Song, the general inversion formula has been proved, using the celebrated Carleson-Hunt theorem on the pointwise convergence of Fourier series (see L.Carleson [**?**], and R.A.Hunt [**?**]). More precisely, the following result has been proved.

Let $\psi \in L^2(R)$ be a mother wavelet, i.e., $\psi \in L^1(R)$, $\|\psi\|_2 = 1$, and $\hat{\psi}(0) = 0$; where, for $f \in L^2(R)$, we denote by $\hat{f}$ its Fourier transform. By $\psi_a^0,\ a \in R, a \neq 0$ we denote wavelets, defined by

$$\psi_a^0(x) = |a|^{-1/2}\bar{\psi}\left(\frac{-x}{a}\right)\ , \tag{4}$$

where $\bar{z}$ denotes the complex conjugate of the complex number z. For $p \in (1, +\infty)$ and $f \in L^p(R)$ we define the continuous wavelet transform with respect to ψ by

$$(T^{wav} f)(a,b) = \left(f * \psi_a^0 \right)(b)\,, \tag{5}$$

where $a, b \in R$, $a \neq 0$, and $*$ is the convolution operator. The well-known Young theorem implies that $b \mapsto (T^{wav} f)(a,b)$ is in $L^p(R)$, for every $a \in R$, $a \neq 0$.

Theorem 1. *Let $1 < p, q < +\infty$, $1/p + 1/q = 1$. Suppose that ψ and φ are mother wavelets such that,*

$$\int_{-\infty}^{+\infty} \frac{|\hat{\psi}(\omega)| \cdot |\hat{\varphi}(\omega)|}{|\omega|} d\omega < +\infty \quad . \tag{6}$$

Then, for every $f \in L^p(R)$,

$$\lim_{\lambda \to 0+} \int_{|a| \geq \lambda} \frac{G(a,x)}{a^2} da = C_{\psi,\varphi} \cdot f(x) \quad (a.e.), \tag{7}$$

where $G(a,x)$ is defined by

$$G(a,x) = |a|^{-1/2} \int_{-\infty}^{+\infty} (T^{wav} f)(a,b) \cdot \varphi\left(\frac{x-b}{a}\right) db \quad , \tag{8}$$

where T^{wav} denotes the continuous wavelet transform with respect to ψ, and the constant $C_{\psi,\varphi}$ is given by

$$C_{\psi,\varphi} = \int_{-\infty}^{+\infty} \frac{\bar{\hat{\psi}}(\omega) \cdot \hat{\varphi}(\omega)}{|\omega|} d\omega < +\infty \quad . \tag{9}$$

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