# Study of Gram matrices in Fock representation of multiparametric canonical commutation relations, extended Zagier's conjecture, hyperplane arrangements and quantum groups*

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**Abstract**. *In this Colloquium Lecture D. Svrtan reported on the joined research with S. Meljanac on the subject given in the title. By quite laborious mathematics it is explained how one can handle systems in which each Heisenberg commutation relation is deformed separately. For Hilbert space realizability a detailed determinant computations (extending Zagier's one – parametric formulas) are carried out. The inversion problem of the associated Gram matrices on Fock weight spaces is completely solved (Extended Zagier's conjecture) and a counterexample to the original Zagier's conjecture is presented in detail.*

**Key words:** *Multiparametric canonical commutation relations, deformed partial derivatives, lattice of subdivisions, deformed regular representation, quantum bilinear form, Zagier's conjecture.*

**Sažetak**. **Proučavanje Gramovih matrica u Fockovoj reprezentaciji višeparametarskih kanonskih komutacijskih relacija, proširena Zagierova hipoteza, aranžmani hiperravnina i kvantne grupe**. *Na ovom kolokviju D. Svrtan cjelovito je prikazao istraživanja u suradnji sa S. Meljancem o temama formuliranima u naslovu. S poprilično matematike objašnjeno je kako se mogu obrađivati sustavi u kojima je svaka Heisenbergova komutacijska relacija deformirana odvojeno. Za realizabilnost na Hilbertovu prostoru provedeno je detaljno računanje determinanata (koje proširuje Zagierove jednoparametarske formule). Problem inverzije pridruženih Gramovih matrica na Fockovim težinskim prostorima je potpuno*

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*riješen (proširena Zagierova hipoteza) i kontraprimjer za originalnu Zagierovu hipotezu je detaljno prikazan.*

**Ključne riječi:** *Multiparametarske kanonske komutacijske relacije, deformirane parcijalne derivacije, rešetka subdivizija, deformirana regularna reprezentacija, kvantna bilinearna forma, Zagierova hipoteza.*

## Introduction

Following Greenberg, Zagier, Božejko and Speicher and others we study a collection of operators $a(k)$ satisfying the "$q_{kl}$-canonical commutation relations"

$$a(k)a^{\dagger}(l) - q_{kl}a^{\dagger}(l)a(k) = \delta_{kl}$$

(corresponding for $q_{kl} = q$ to Greenberg (infinite) statistics, for $q = \pm 1$ to classical Bose and Fermi statistics). We show that $n! \times n!$ matrices $A_n(\{q_{kl}\})$ representing the scalar products of $n$-particle states is positive definite for all $n$ if $|q_{kl}| < 1$, all $k$,$l$, so that the above commutation relations have a Hilbert space realization in this case. This is achieved by explicit factorizations of $A_n(\{q_{kl}\})$ as a product of matrices of the form $(1 - QT)^{\pm 1}$, where Q is a diagonal matrix and T is a regular representation of a cyclic matrix. From such factorizations we obtain in *Theorem 1.9.2* (determinant formula) explicit formulas for the determinant of $A_n(\{q_{kl}\})$ in the generic case (which generalizes Zagier's 1-parametric formula). The problem of computing the inverse of $A_n(\{q_{kl}\})$ in its original form is computationally intractable (for $n = 4$ one has to invert a $24 \times 24$ symbolic matrix). Fortunately, by using another approach (originated by Božejko and Speicher ) we obtain in *Theorem* 2.2.6 a definite answer to that inversion problem in terms of maximal chains in so called subdivision lattices. Our algorithm in *Proposition* 2.2.15 for computing the entries of the inverse of $A_n(\{q_{kl}\})$ is very efficient. In particular for $n = 8$, when all $q_{kl} = q$, we found a counterexample to Zagier's conjecture concerning the form of the denominators of the entries in the inverse of $A_n(q)$. In *Corollary* 2.2.8 we formulate and prove Extended Zagier's Conjecture which turns to be the best possible in the multiparametric case and which implies in one parametric case an interesting extension of the original Zagier's Conjecture. By using a faster algorithm in *Proposition* 2.2.16 we obtain in *Theorem* 2.2.17 (inverse matrix entries) explicit formulas for the inverse of the matrices $A_n(\{q_{kl}\})$ in the generic case. Finally, there are applications of the results above to discriminant arrangements of hyperplanes and to contravariant forms of certain quantum groups.

## 1. Multiparametric quon algebras, Fock-like representation and determinants

### 1.1. $q_{ij}$-canonical commutation relations

Let $\mathbf{q} = \{q_{ij} : i, j \in I, \bar{q}_{ij} = q_{ji}\}$ be a hermitian family of complex numbers (parameters), where $I$ is a finite (or infinite) set of indices. Then by a *multiparametric quon algebra* $\mathcal{A} = \mathcal{A}^{(\mathbf{q})}$ we shall mean an associative (complex) algebra generated by $\{a_i, a_i^{\dagger}, i \in I\}$ subject to the following $q_{ij}$- canonical commutation relations

$$a_i a_j^{\dagger} = q_{ij} a_j^{\dagger} a_i + \delta_{ij}, \quad \text{for all } i, j \in I.$$

Shortly, we shall give an explicit Fock-like representation of the algebra $\mathcal{A}^{(\mathbf{q})}$ on the free associative algebra $\mathbf{f}$ (the algebra of noncommuting polynomials in the indeterminates $\theta_i, i \in I$) with $a_i$ acting as a generalized $q_{ij}$-deformed partial derivatives ${}_i\partial = {}_i^{\mathbf{q}}\partial$ w.r.t. variable $\theta_i$ (the i-th annihilation operator), and $a_i^\dagger$ as multiplication by $\theta_i$ (the $i$-th creation operator). Moreover, $a_i^\dagger$ will be adjoint to $a_i$ w.r.t. certain sesquilinear form $(\ ,\ )_{\mathbf{q}}$ on $\mathbf{f}$ which will be better described via certain canonical $\mathbf{q}$-deformed bialgebra structure on $\mathbf{f}$, generalizing the one used by Lusztig in his excellent treatment of quantum groups [Lus]. Then by explicit computation (which extends Zagier's method) of the determinant of $(\ ,\ )_{\mathbf{q}}$ we show that $(\ ,\ )_{\mathbf{q}}$ is positive definite provided the following condition on the parameters $q_{ij}$ holds true: $|q_{ij}| < 1, \quad$ for all $i, j \in I$ This condition ensures that all many-particle states $a_{i_1}^\dagger \cdots a_{i_r}^\dagger |0 >= \theta_{i_1} \cdots \theta_{i_r}$, $i_j \in I, r \geq 0$, are linearly independent, so we obtain a Hilbert space realization of the $q_{ij}$-canonical commutation relations. We first need some notations:

$\mathbf{N} = \{0, 1, 2, \ldots\}$ = the set of nonnegative integers, $\mathbf{C}$ = the set of complex numbers

$(\mathbf{N}[I], +)$ = the *weight monoid*, i.e. the set of all finite formal linear combinations $\nu = \sum_{i\in I} \nu_i i$, $\nu_i \in \mathbf{N}, i \in I$ with componentwise addition $\nu + \nu' = \sum_{i\in I}(\nu_i + \nu_i')i$

$|\nu| = \sum_{i\in I} \nu_i \in \mathbf{N}$ for $\nu = \sum_{i\in I} \nu_i i \in \mathbf{N}[I]$

$\beta : (\mathbf{N}[I], +) \times (\mathbf{N}[I], +) \longrightarrow (\mathbf{C}, \cdot)$, the bilinear form on $(\mathbf{N}[I], +)$ given by $i, j \mapsto q_{ij}$, i.e. for $\nu = \sum_{i\in I} \nu_i i$, $\nu' = \sum_{j\in I} \nu_j' j$, $\beta(\nu, \nu') = \prod_{ij} q_{ij}^{\nu_i \nu_j'}$.

## 1.2. The algebra f

We denote by $\mathbf{f}$ the free associative $\mathbf{C}$-algebra with generators $\theta_i (i \in I)$. For any weight $\nu = \sum_{i\in I} \nu_i i \in \mathbf{N}[I]$ we denote by $\mathbf{f}_\nu$ the corresponding *weight space*, i.e. the subspace of $\mathbf{f}$ spanned by monomials $\theta_{\mathbf{i}} = \theta_{i_1} \cdots \theta_{i_n}$ indexed by sequences $\mathbf{i} = i_1 \ldots i_n$ of weight $\nu$,$|\mathbf{i}| = \nu$ (this means that the number of occurrences of $i$ in $\mathbf{i}$ is equal to $\nu_i$, for all $i \in I$). Then each $\mathbf{f}_\nu$ is a finite dimensional complex vector space and we have a direct sum decomposition $\mathbf{f} = \bigoplus_\nu \mathbf{f}_\nu$, where $\nu$ runs over $\mathbf{N}[I]$. We have $\mathbf{f}_\nu \mathbf{f}_{\nu'} \subset \mathbf{f}_{\nu+\nu'}$, $1 \in \mathbf{f}_0$ and $\theta_i \in \mathbf{f}_{(i)}$. An element $x$ of $\mathbf{f}$ is said to be *homogeneous* if it belongs to $\mathbf{f}_\nu$ for some $\nu$. We then say that $x$ has *weight* $\nu$ and write $|x| = \nu$.
We consider the tensor product $\mathbf{f} \otimes \mathbf{f}$ with the following *$q_{ij}$-deformed multiplication*

$$(x_1 \otimes x_2)(x_1' \otimes x_2') = (\prod_{i,j} q_{ij}^{\nu_i \nu_j'}) x_1 x_1' \otimes x_2 x_2', \text{ if } x_2 \in \mathbf{f}_\nu,\ x_1' \in \mathbf{f}_{\nu'}$$

where $x_1, x_1', x_2, x_2' \in \mathbf{f}$ are homogeneous; this algebra is associative since $\beta(\nu, \nu')$ is bilinear. The following statement is easily verified: if $r = r_{\mathbf{q}}$ :

$\mathbf{f} \longrightarrow \mathbf{f} \otimes \mathbf{f}$ is the unique algebra homomorphism such that $r(\theta_i) = \theta_i \otimes 1 + 1 \otimes \theta_i$, for all $i$, then

$$r(\theta_i\theta_j) \;=\; r(\theta_i)r(\theta_j) \;=\; \theta_i\theta_j \otimes 1 + q_{ij}\theta_j \otimes \theta_i + \theta_i \otimes \theta_j + 1 \otimes \theta_i\theta_j$$

More generally, the value of $r$ on any monomial $\theta_{\mathbf{i}} = \theta_{i_1}\theta_{i_2}\cdots\theta_{i_n}$ is given by:

$$r(\theta_{\mathbf{i}}) = \sum_{k+l=n, g=(k,l)-shuffle} q_{\mathbf{i},g}\theta_{i_{g(1)}}\cdots\theta_{i_{g(k)}} \otimes \theta_{i_{g(k+1)}}\cdots\theta_{i_{g(k+l)}}$$

where $(k,l)-shuffle$ is a permutation $g \in S_{k+l}$ such that $g(1) < g(2) < \cdots < g(k)$ and $g(k+1) < g(k+2) < \cdots < g(k+l)$ and where for $g \in S_n$ we denote by $q_{\mathbf{i},g}$the quantity

$$q_{\mathbf{i},g} := \prod_{a<b, g(a)>g(b)} q_{i_a i_b}$$

## 1.3. The sesquilinear form $(\ ,\ )_{\mathbf{q}}$ on f

Note that $r$ maps $\mathbf{f}_\nu$ into $\bigoplus_{(\nu'+\nu''=\nu)} \mathbf{f}_{\nu'} \bigotimes \mathbf{f}_{\nu''}$. Then the linear maps $\mathbf{f}_{\nu'+\nu''} \longrightarrow \mathbf{f}_{\nu'} \bigotimes \mathbf{f}_{\nu''}$ defined by $r$ give, by passage to dual spaces, linear maps $\mathbf{f}^*_{\nu'} \bigotimes \mathbf{f}^*_{\nu''} \longrightarrow \mathbf{f}^*_{\nu'+\nu''}$. These define the structure of an associative algebra with 1 on $\bigoplus_\nu \mathbf{f}^*_\nu$. For any $i \in I$, let $\theta_i^* \in \mathbf{f}_i^*$ be the linear form given by $\theta_i^*(\theta_j) = \delta_{ij}$. Let $\Phi_{\mathbf{q}} : \mathbf{f} \longrightarrow \bigoplus_\nu \mathbf{f}^*_\nu$ be the unique conjugate-linear algebra homomorphism preserving 1, such that $\Phi_{\mathbf{q}}(\theta_i) = \theta_i^*$, for all $i$. For $x, y \in \mathbf{f}$, we set

$$(x,y)_{\mathbf{q}} = \Phi_{\mathbf{q}}(y)(x)$$

Then $(\ ,\ ) = (\ ,\ )_{\mathbf{q}}$ is a unique sesquilinear form on $\mathbf{f}$ such that **a)** $(\theta_i, \theta_j) = \delta_{ij}$, for all $i, j \in I$; **b)** $(x, y'y'') = (r(x), y' \bigotimes y'')$, for all $x, y', y'' \in \mathbf{f}$; **c)** $(xx', y'') = (x \bigotimes x', r(y''))$, for all $x, x', y'' \in \mathbf{f}$. Clearly, **d)** $(x,y) = 0$ if $x$ and $y$ are homogeneous with $|x| \neq |y|$. Thus the subspaces $\mathbf{f}_\nu, \mathbf{f}_{\nu'}$ are orthogonal w.r.t. $(\ ,\ )$ for $\nu \neq \nu'$.

## 1.4. The $q_{ij}$-deformed partial derivative maps ${}^{\mathbf{q}}_i\partial$ and ${}^{\mathbf{q}}\partial_i$

Let $i \in I$. Clearly, there exists a unique $\mathbf{C}$-linear map ${}_i\partial = {}^{\mathbf{q}}_i\partial : \mathbf{f} \longrightarrow \mathbf{f}$ such that ${}_i\partial(1) = 0$, ${}_i\partial(\theta_j) = \delta_{ij}$, for all $j$ and obeying the generalized Leibniz rule :

**a)** ${}_i\partial(xy) = {}_i\partial(x)y + \beta(i,|x|)x{}_i\partial(y) = {}_i\partial(x)y + \prod_j q_{ij}^{\nu_j} x{}_i\partial(y)$, if $x \in \mathbf{f}_\nu$
for all homogeneous $x$,$y$. If $x \in \mathbf{f}_\nu$ we have ${}_i\partial(x) \in \mathbf{f}_{\nu-i}$ if $\nu_i \geq 1$ and ${}_i\partial(x) = 0$ if $\nu_i = 0$; moreover, $r(x) = \theta_i \bigotimes {}_i\partial(x)$ + terms of other bihomogeneities. Similarly, we define a unique $\mathbf{C}$-linear map $\partial_i = {}^{\mathbf{q}}\partial_i : \mathbf{f} \longrightarrow \mathbf{f}$ such that $\partial_i(1) = 0$, $\partial_i(\theta_j) = \delta_{ij}$ for all $j$ and $\partial_i(xy) = \beta(|y|, i)\partial_i(x)y + x\partial_i(y)$ $(= (\prod_j q_{ji}^{\nu_j})\partial_i(x)y + x\partial_i(y)$, if $y \in \mathbf{f}_\nu)$ for all homogeneous $x$,$y$. From the definition we see that
**b)** $(\theta_i y, x) = (y, {}_i\partial(x)), (y\theta_i, x) = (y, \partial_i(x))$, for all $x, y$;i.e. the operator ${}_i\partial$

(resp. $\partial_i$) is the adjoint of the left (resp. right) multiplication by $\theta_i$. We shall need the following explicit formula for ${}_i\partial = {}_i^{\mathbf{q}}\partial : \mathbf{f} \longrightarrow \mathbf{f}$

**c)** ${}_i\partial(\theta_{j_1} \cdots \theta_{j_n}) = \sum_{(p:j_p=i)} q_{ij_1} \cdots q_{ij_{p-1}} \theta_{j_1} \cdots \hat{\theta}_{j_p} \cdots \theta_{j_n}$
where $\hat{}$ denotes omission of the factor $\theta_{j_p}$. This formula is obtained by iterating the recursive definition a) for ${}_i\partial$ or by using the general formula for $r$ in 1.2. A similar formula holds for $\partial_i$.

## 1.5. Fock representation of multiparametric quon algebra $\mathcal{A}^{(\mathbf{q})}$

Here we give a representation of the multiparametric quon algebra $\mathcal{A} = \mathcal{A}^{(\mathbf{q})}$ (defined in 1.1) on the underlying vector space of the free associative algebra $\mathbf{f}$.

**Proposition 1.5.1..** *For each $i \in I$ let $a_i^\dagger$ act on $\mathbf{f}$ as left multiplication by $\theta_i$ and let $a_i$ act as a linear map ${}_i\partial$ defined in 1.4. Then*
*a) $a_i, a_i^\dagger$ make $\mathbf{f}$ into a left $\mathcal{A}$ - module*
*b) $a_i^\dagger$ is adjoint to $a_i$ w.r.t. sesquilinear form $(\ ,\ ) = (\ ,\ )_{\mathbf{q}}$ defined in 1.3.*
*c) $a_i : \mathbf{f} \longrightarrow \mathbf{f}$ is locally nilpotent for every $i \in I$.*

## 1.6. The matrix $A(\mathbf{q})$ of the sesquilinear form $(\ ,\ )_{\mathbf{q}}$ on $\mathbf{f}$

Here we study the sesquilinear form $(\ ,\ )_{\mathbf{q}}$ on $\mathbf{f}$, defined in 1.2, via associated matrix w.r.t. the basis $B = \{\theta_{\mathbf{i}} = \theta_{i_1} \cdots \theta_{i_n} | i_j \in I, n \geq 0\}$ of the complex vector space
$\mathbf{f} = \bigoplus_\nu \mathbf{f}_\nu$. Let $B' = \{\theta_{\mathbf{i}} = \theta_{i_1} \cdots \theta_{i_n} | i_1, \ldots, i_n$ all distinct$\}$ and $B'' = B \setminus B' = \{\theta_{i_1} \cdots \theta_{i_n} |$ not all $i_1, \ldots, i_n$ distinct$\}$. Then we have the direct sum decomposition $\mathbf{f} = \mathbf{f}' \bigoplus \mathbf{f}''$, where $\mathbf{f}' = \operatorname{span} B'$, $\mathbf{f}'' = \operatorname{span} B''$. Note that for any weight $\nu = \sum \nu_i i \in \mathbf{N}[I]$ we have $\mathbf{f}_\nu \subset \mathbf{f}'$ (resp. $\mathbf{f}_\nu \subset \mathbf{f}''$) if all $\nu_i \leq 1$ (resp. some $\nu_i \geq 2$). Then we call such weight $\nu$ *generic* (resp. *degenerate* ) and we have further direct sum decompositions $\mathbf{f}' = \bigoplus_{\nu\ generic} \mathbf{f}_\nu$, $\mathbf{f}'' = \bigoplus_{\nu\ degenerate} \mathbf{f}_\nu$
.

**Proposition 1.6.1..** *i) Let $\mathbf{A} = \mathbf{A}(\mathbf{q}) : \mathbf{f} \longrightarrow \mathbf{f}$ be the linear operator, associated to the sesquilinear form $(\ ,\ ) = (\ ,\ )_{\mathbf{q}}$ on $\mathbf{f}$ defined by*

$$\mathbf{A}(\theta_{\mathbf{j}}) = \sum_{\mathbf{i}} (\theta_{\mathbf{j}}, \theta_{\mathbf{i}})_{\mathbf{q}} \theta_{\mathbf{i}}$$

*Then the $\mathbf{f}', \mathbf{f}'', \mathbf{f}_\nu$ ($\nu \in \mathbf{N}[I]$) are all invariant subspaces of $\mathbf{A}$, yielding the following block decompositions for the corresponding matrices $A = A' \bigoplus A''$, $A' = \bigoplus_{\nu \text{ generic}} A^{(\nu)}$, $A'' = \bigoplus_{\nu \text{ degenerate}} A^{(\nu)}$, with entries given by the following formulas:*

*ii) Let $\mathbf{i} = i_1 \ldots i_n$ and $\mathbf{j} = j_1 \ldots j_n$ be any two sequences with the same generic weight $\nu$ and let $\sigma = \sigma(\mathbf{i}, \mathbf{j}) \in S_n$ be the unique permutation such that*

$\sigma \cdot \mathbf{i} = \mathbf{j}$ *(i.e. $i_{\sigma^{-1}(p)} = j_p$, all $p$). Then*

$$A'_{\mathbf{i},\mathbf{j}} = A^{(\nu)}_{\mathbf{i},\mathbf{j}} = q_{\mathbf{i},\sigma} (= \bar{q}_{\mathbf{j},\sigma^{-1}})$$

*where (cf. 1.2) $q_{\mathbf{i},\sigma} := \prod_{(a,b)\in I(\sigma)} q_{i_a i_b}$ with $I(\sigma) = \{(a,b)|a < b, \sigma(a) > \sigma(b)\}$ denoting the set of inversions of $\sigma$.*

*iii) Let $\mathbf{i} = i_1 \dots i_n$ and $\mathbf{j} = j_1 \dots j_n$ be any two sequences of the same degenerate weight $\nu$ and let $\sigma(\mathbf{i},\mathbf{j}) = \{\sigma \in S_n | i_{\sigma^{-1}(p)} = j_p$, all $p\}$. Then*

$$A''_{\mathbf{i},\mathbf{j}} = A^{(\nu)}_{\mathbf{i},\mathbf{j}} = \sum_{\sigma\in\sigma(\mathbf{i},\mathbf{j})} q_{\mathbf{i},\sigma^{-1}} \ (= \sum_{\sigma\in\sigma(\mathbf{i},\mathbf{j})} \bar{q}_{\mathbf{j},\sigma^{-1}}).$$

**Proof.** i) follows from 1.3d). For ii) we have, by 1.4b)

$$A'_{\mathbf{i},\mathbf{j}} = A_{\mathbf{i},\mathbf{j}} = (\theta_{\mathbf{j}}, \theta_{\mathbf{i}})_{\mathbf{q}} = ({}_{i_1}\partial(\theta_{\mathbf{j}}), \theta_{i_2} \cdots \theta_{i_n})_{\mathbf{q}} = \cdots = {}_{i_n}\partial \cdots {}_{i_1}\partial(\theta_{j_1} \cdots \theta_{j_n})$$

By applying 1.4d) successively for $i = i_1, i_2, \dots$ and if $j_{\sigma(1)} = i_1, j_{\sigma(2)} = i_2, \cdots$ we obtain $(\prod_{1<b,\sigma(b)<\sigma(1)} q_{i_1 i_b})(\prod_{2<b,\sigma(b)<\sigma(2)} q_{i_2 i_b}) \cdots = \prod_{a<b,\sigma(b)<\sigma(a)} q_{i_a i_b} = q_{\mathbf{i},\sigma}$, so the claim follows. The proof of iii) is similar as for ii) except that $\sigma$ is not unique. □

**Remark 1.6.2..** *For any weight $\nu = \sum \nu_i i$ with $|\nu| = \sum \nu_i = n$, the size of the matrix $A^{(\nu)}$ is equal to $n!/\prod_i \nu_i! = \dim \mathbf{f}_\nu$. Hence for $\nu$ generic $A^{(\nu)}$ is an $n! \times n!$ matrix.*

**Example 1.6.3..** *Let $I = \{1,2,3\}$ and $\nu$ generic with $\nu_1 = \nu_2 = \nu_3 = 1$. Then w.r.t. basis $\{\theta_{123}, \theta_{132}, \theta_{312}, \theta_{321}, \theta_{231}, \theta_{213}\}$*

$$A^{123} = \begin{pmatrix} 1 & q_{23} & q_{23}q_{13} & q_{12}q_{13}q_{23} & q_{12}q_{13} & q_{12} \\ q_{32} & 1 & q_{13} & q_{13}q_{12} & q_{12}q_{13}q_{32} & q_{12}q_{32} \\ q_{32}q_{31} & q_{31} & 1 & q_{12} & q_{12}q_{32} & q_{12}q_{31}q_{32} \\ \cdot & \cdot & \cdot & 1 & q_{32} & q_{31}q_{32} \\ \cdot & \cdot & \cdot & q_{23} & 1 & q_{31} \\ \cdot & \cdot & \cdot & q_{13}q_{23} & q_{13} & 1 \end{pmatrix} = \begin{pmatrix} X & Y \\ \bar{Y} & \bar{X} \end{pmatrix}$$

*where $\bar{X}^T = X$, $Y^T = Y$.*

**Example 1.6.4..** *Let $I = \{1,2,3\}$ and $\nu$ degenerate with $\nu_1 = 2$, $\nu_2 = 0$, $\nu_3 = 1$. Then w.r.t. basis $\{\theta_{113}, \theta_{131}, \theta_{311}\}$*

$$A^{113} = \begin{pmatrix} 1 + q_{11} & q_{13} + q_{11}q_{13} & q_{13}^2 + q_{11}q_{13}^2 \\ q_{31} + q_{31}q_{11} & 1 + q_{11}q_{13}q_{31} & q_{13} + q_{11}q_{13} \\ q_{31}^2 + q_{31}^2 q_{11} & q_{31} + q_{31}q_{11} & 1 + q_{11} \end{pmatrix}.$$

## 1.7. A reduction to generic case

Some questions about the matrices $A^{(\nu)}$ for general $\nu$ (e.g. invertibility, positive definiteness) can be reduced to the generic situation by using the following observation. Let $\nu = \sum_i \nu_i i \in \mathbf{N}[I]$ be a degenerate weight. Let $\tilde{I}$ be any set of size equal to $n = |\nu| = \sum_i \nu_i$ and let $\phi : \tilde{I} \longrightarrow I$ be a function which maps exactly $\nu_i$ elements $\tilde{i}$ of $\tilde{I}$ to $i \in I$, and let $\tilde{\mathbf{q}}$ be the induced hermitian family of parameters $\tilde{q}_{\tilde{i},\tilde{j}} := q_{i,j}(\tilde{i},\tilde{j} \in \tilde{I})$ where $i = \phi(\tilde{i}), j = \phi(\tilde{j})$. Let $\tilde{\mathbf{f}}$ be the free associative algebra with generators $\tilde{\theta}_1, \ldots, \tilde{\theta}_n$ and let $(\ ,\ )_{\tilde{\mathbf{q}}}$ be the sesquilinear form on $\tilde{\mathbf{f}}$ associated to $\tilde{\mathbf{q}}$ (as in 1.3). Let $\tilde{\mathbf{f}}_{\tilde{\nu}}$ be the generic weight space corresponding to $\tilde{\nu} \in \mathbf{N}[\tilde{I}]$ where $\tilde{\nu}_{\tilde{i}} = 1$, for every $\tilde{i} \in \tilde{I}$. Let $H = H_\nu$ be the group of all bijections of $\tilde{I}$ which map $\phi^{-1}\{i\}$ to itself for every $i \in \phi(\tilde{I})$. This group is isomorphic to the Young subgroup $\prod_i S_{\nu_i} \subset S_n$. Let Y be the subspace of $\tilde{\mathbf{f}}_{\tilde{\nu}}$ spanned by *H-invariant vectors* $\tilde{\theta}_{H\tilde{\mathbf{i}}} = \sum_{h\in H} \tilde{\theta}_{h\cdot\tilde{\mathbf{i}}}$ where $\tilde{\theta}_{h\cdot\tilde{\mathbf{i}}} = \tilde{\theta}_{\tilde{i}_{h^{-1}(1)}} \cdots \tilde{\theta}_{\tilde{i}_{h^{-1}(n)}}$. Then for the operator $\tilde{\mathbf{A}}$ associated to the form $(\ ,\ )_{\tilde{\mathbf{q}}}$ we have

$$\tilde{\mathbf{A}}(\tilde{\theta}_{H\tilde{\mathbf{j}}}) = \sum_{h\in H} \tilde{\mathbf{A}}(\tilde{\theta}_{h\cdot\tilde{\mathbf{j}}}) = \sum_{h\in H} \sum_{\tilde{\mathbf{i}}} (\tilde{\theta}_{h\cdot\tilde{\mathbf{j}}}, \tilde{\theta}_{\tilde{\mathbf{i}}})_{\tilde{\mathbf{q}}} \tilde{\theta}_{\tilde{\mathbf{i}}}.$$

By *Prop.* 1.6.1. we can write $\tilde{\mathbf{A}}(\tilde{\theta}_{H\tilde{\mathbf{j}}}) = \sum_{\tilde{\mathbf{i}}} A^{(\nu)}_{\mathbf{i},\mathbf{j}} \tilde{\theta}_{\tilde{\mathbf{i}}} = \sum_{\mathbf{i}} A^{(\nu)}_{\mathbf{i},\mathbf{j}} \tilde{\theta}_{H\tilde{\mathbf{i}}}$. Thus we have proved that Y is an invariant subspace of the operator $\tilde{\mathbf{A}}$ associated to the form $(\ ,\ )_{\tilde{\mathbf{q}}}$ and moreover that the matrix of $\tilde{\mathbf{A}}|Y$ w.r.t basis of $H$-invariant vectors $\tilde{\theta}_{H\tilde{\mathbf{i}}}$ coincides with $A^{(\nu)}$. From this fact we conclude that

1) If $\tilde{\mathbf{A}}|_{\tilde{\mathbf{f}}_{\tilde{\nu}}}$ is invertible, then $\mathbf{A}^{(\nu)}$ is invertible, too. In particular $[A^{(\nu)}]^{-1}_{\mathbf{i},\mathbf{j}} = \sum_{h\in H} [\tilde{A}^{(\tilde{\nu})}]^{-1}_{\tilde{\mathbf{i}},h\tilde{\mathbf{j}}}$, where $\tilde{\mathbf{i}}, \tilde{\mathbf{j}}$ are chosen so that $\phi(\tilde{\mathbf{i}}) = \mathbf{i}, \phi(\tilde{\mathbf{j}}) = \mathbf{j}$. This means that the entries of $[A^{(\nu)}]^{-1}$($\nu$ degenerate)can be read off from the sums of $H$-equivalent columns of the matrix $[\tilde{A}^{(\tilde{\nu})}]^{-1}$($\tilde{\nu}$ generic).

2) The determinant of $A^{(\nu)}$ divides the determinant of $\tilde{A}^{(\tilde{\nu})}$.

3) If $\tilde{A}^{(\tilde{\nu})}$ is positive definite, then $A^{(\nu)}$ is positive definite, too.

## 1.8. Factorization of matrices $A^{(\nu)}$ for $\nu$ generic

First of all we point out that the rows of our multiparametric matrices $A^{(\nu)}$ are not equal up to reordering (what was true in [Zag], where all $q_{ij}$ are equal to q). Therefore, the factorization of the matrices $A^{(\nu)}$ can not be reduced to the factorization of the corresponding group algebra elements as was treated by Zagier. Instead, by a somewhat tricky extension of the Zagier's method we show how this can be done on the matrix level. This is achieved by studying a $q_{ij}$-deformation of the regular representation of the symmetric group which is only quasimultiplicative, i.e., multiplicative only up to factors which are diagonal

($q_{ij}$-dependent) matrices ("projective representation"). For $\nu = \sum \nu_i i \in \mathbf{N}[I]$ generic,$n = |\nu| = \sum \nu_i$ let $R_\nu$ denote the action of the symmetric group $S_n$ on the (generic) weight space $\mathbf{f}_\nu$, given on the basis $B_\nu = \{\theta_{\mathbf{i}} = \theta_{i_1} \cdots \theta_{i_n}, |\mathbf{i}| = \nu\}$ of $\mathbf{f}_\nu$ by place permutations:$R_\nu(g) : \theta_{\mathbf{j}} = \theta_{j_1} \cdots \theta_{j_n} \longrightarrow \theta_{g \cdot \mathbf{j}} = \theta_{j_{g^{-1}(1)}} \cdots \theta_{j_{g^{-1}(n)}}$. Then $R_\nu$ is equivalent to the *right regular representation* $R_n$ of $S_n$.The corresponding *matrix representation*, also denoted by $R_\nu(g)$, is given by $R_\nu(g)_{\mathbf{i},\mathbf{j}} := \delta_{\mathbf{i}, g\cdot\mathbf{j}}$.

Now, we need more notations. Let $Q^\nu_{a,b}$ for $1 \le a, b \le n$ and $Q^\nu(g)$, for $g \in S_n$ be the following diagonal matrices (multiplication operators on $\mathbf{f}_\nu$ ) defined by

$(Q^\nu_{a,b})_{\mathbf{i},\mathbf{i}} := q_{i_a i_b}$; e.g $(Q^{1234}_{2,4})_{4123,4123} = q_{13}$ if $I = \{1,2,3,4\}, \nu_1 = \nu_2 = \nu_3 = \nu_4 = 1$).

$$Q^\nu(g)_{\mathbf{i},\mathbf{i}} := q_{\mathbf{i},g^{-1}} = \prod_{a<b, g^{-1}(a)>g^{-1}(b)} q_{i_a i_b} \quad (\Longrightarrow Q^\nu(g) = \prod_{(a,b)\in I(g^{-1})} Q^\nu_{a,b}).$$

Note that $\bar{q}_{ij} = q_{ji}$ imply that $Q^\nu_{b,a} = [Q^\nu_{a,b}]^*$. We also denote by $|Q^\nu_{a,b}|$ the diagonal matrix defined by $|Q^\nu_{a,b}|_{\mathbf{i},\mathbf{i}} = |q_{i_a i_b}|$. The matrix $Q^\nu_{a,b} \cdot Q^\nu_{b,a} (= |Q^\nu_{a,b}|^2)$ we abbreviate as $Q^\nu_{\{a,b\}}$. More generally, for any subset $T \subseteq \{1, 2, \cdots, n\}$ we shall use the notations

$$Q^\nu_T := \prod_{a,b\in T, a\ne b} Q^\nu_{a,b}, \quad \Box^\nu_T := I - Q^\nu_T$$

(e.g. $Q^\nu_{\{3,5,6\}} = Q^\nu_{\{3,5\}} Q^\nu_{\{3,6\}} Q^\nu_{\{5,6\}} = Q^\nu_{3,5} Q^\nu_{5,3} Q^\nu_{3,6} Q^\nu_{5,6} Q^\nu_{6,5}$). The following $q_{ij}$-*deformation of the representation* $R_\nu$, defined by $\hat{R}_\nu(g) := Q^\nu(g) R_\nu(g),\ g \in S_n$ will be crucial in our method for factoring the matrices $A^{(\nu)}$ $\nu$-generic.

**Proposition 1.8.1..** *If $\nu$ is a generic weight with $|\nu| = n$, then for the matrix $A^{(\nu)}$ of $(\ ,\ )_{\mathbf{q}}$ on $\mathbf{f}_\nu$ we have*

$$A^{(\nu)} = \sum_{g\in S_n} \hat{R}_\nu(g)$$

**Proof.** The $(\mathbf{i},\mathbf{j})$-th entry of the r.h.s. is equal to $\sum_{g\in S_n} \hat{R}_\nu(g)_{\mathbf{i},\mathbf{j}} = \sum_{g\in S_n} Q(g)_{\mathbf{i},\mathbf{i}} \hat{R}_\nu(g)_{\mathbf{i},\mathbf{j}} = \sum_{g\in S_n} q_{\mathbf{i},g^{-1}} \delta_{\mathbf{i},g\cdot\mathbf{j}} = q_{\mathbf{i},\tau^{-1}}$, if $\mathbf{i} = \tau\mathbf{j}$ (such $\tau$ is unique, because $|\mathbf{i}| = |\mathbf{j}| = \nu$ is generic), what is just $A^{(\nu)}_{\mathbf{i},\mathbf{j}}$, according to *Prop.* 1.6.1 ii) and the proof follows. □

Before we proceed with factorization of matrices $A^{(\nu)}$ we need more detailed informations concerning our "projective" right regular representation $\hat{R}_\nu$:

PROPERTY 0. (quasimultiplicativity) $\hat{R}_\nu(g_1)\hat{R}_\nu(g_2) = \hat{R}_\nu(g_1 g_2)$ if $l(g_1 g_2) = l(g_1) + l(g_2)$, where $l(g) := Card\ I(g)$ is the length of $g \in S_n$. This property follows from the following general formula :

**Proposition 1.8.2..** *For $g_1, g_2 \in S_n$ we have $\hat{R}_\nu(g_1)\hat{R}_\nu(g_2) = M_\nu(g_1, g_2)\hat{R}_\nu(g_1 g_2)$ where the multiplication factor is the diagonal matrix*

$$M_\nu(g_1, g_2) = \prod_{(a,b)\in I(g_1^{-1})-I(g_2^{-1}g_1^{-1})} Q^\nu_{\{a,b\}} \quad (= \prod_{(a,b)\in I(g_1)\cap I(g_2^{-1})} Q^\nu_{\{g_1(a),g_1(b)\}}).$$

For $1 \leq a \leq b \leq n$ we denote by $t_{a,b}$ the following cyclic permutation in $S_n$

$$t_{a,b} := \begin{pmatrix} a & a+1 & \cdots & b \\ b & a & \cdots & b-1 \end{pmatrix}$$

which maps $b$ to $b-1$ to $b-2 \cdots$ to $a$ to $b$ and fixes all $1 \leq k < a$ and $b < k \leq n$. We also denote by $t_a := t_{a,a+1} (1 \leq a < n)$ the transposition of adjacent letters $a$ and $a+1$. Then, from *Proposition* 1.8.2, one gets the following more specific properties of $\hat{R}_\nu$ which we shall need later on:

PROPERTY 1. (braid relations)

$$\hat{R}_\nu(t_a)\hat{R}_\nu(t_{a+1})\hat{R}_\nu(t_a) = \hat{R}_\nu(t_{a+1})\hat{R}_\nu(t_a)\hat{R}_\nu(t_{a+1}), \text{ for } a = 1, \ldots, n-2.$$

$$\hat{R}_\nu(t_a)\hat{R}_\nu(t_b) = \hat{R}_\nu(t_b)\hat{R}_\nu(t_a), \text{ for } a, b \in \{1, \ldots, n-1\} \text{ with } |a-b| \geq 2.$$

PROPERTY 2. $\hat{R}_\nu(g)\hat{R}_\nu(t_{k,m}) = \hat{R}_\nu(g t_{k,m})$, for $g \in S_{m-1} \times S_{n-m+1}, 1 \leq k \leq m \leq n$.

PROPERTY 3. (commutation rules) i) For $1 \leq a \leq a' < m \leq n$

$$\hat{R}_\nu(t_{a',m})\hat{R}_\nu(t_{a,m}) = Q^\nu_{\{m-1,m\}}\hat{R}_\nu(t_{a,m-1})\hat{R}_\nu(t_{a'+1,m}).$$

ii) Let $w_n = nn-1 \cdots 21$ be the longest permutation in $S_n$. Then for any $g \in S_n$

$$\hat{R}_\nu(g w_n)\hat{R}_\nu(w_n) = \hat{R}_\nu(w_n)\hat{R}_\nu(w_n g) = \Big(\prod_{a<b, g^{-1}(a)<g^{-1}(b)} Q^\nu_{\{a,b\}}\Big)\hat{R}(g)$$

**Proposition 1.8.3..** *For $m \leq n$, let $A^{(\nu),m} := \hat{R}_\nu(t_{1,m}) + \hat{R}_\nu(t_{2,m}) + \cdots + \hat{R}_\nu(t_{m,m})$ $(A^{(\nu),1} = I)$. Then we have the following factorization*

$$A^{(\nu)} = A^{(\nu),1}A^{(\nu),2} \cdots A^{(\nu),n}.$$

We now make a second reduction by expressing the matrices $A^{(\nu),m}$ in turn as a product of yet simpler matrices.

**Proposition 1.8.4..** *Let $C^{(\nu),m} (m \leq n)$ and $D^{(\nu),m} (m < n)$ be the following matrices* $C^{(\nu),m} := [I - \hat{R}_\nu(t_{1,m})][I - \hat{R}_\nu(t_{2,m})] \cdots [I - \hat{R}_\nu(t_{m-1,m})]$,
$D^{(\nu),m} := [I - Q^\nu_{\{m,m+1\}}\hat{R}_\nu(t_{1,m})][I - Q^\nu_{\{m,m+1\}}\hat{R}_\nu(t_{2,m})] \cdots [I - Q^\nu_{\{m,m+1\}}\hat{R}_\nu(t_{m,m})]$.
*Then*

$$A^{(\nu),m} = D^{(\nu),m-1}[C^{(\nu),m}]^{-1}$$

## 1.9. Formula for the determinant of $A^{(\nu)}$, $\nu$ generic.

So far we have expressed the matrix $A^{(\nu)}$ as a product of matrices like $[I - \hat{R}_\nu(t_{k,m})]^{-1}$ or $I - Q^{\nu}_{\{m,m+1\}}\hat{R}_\nu(t_{k,m})$. Thus, in order to evaluate $\det A^{(\nu)}$, we first compute the determinant of such matrices.

**Lemma 1.9.1..** *For $\nu$ generic with $|\nu| = n$, we have*
*a)* $\det(I - \hat{R}_\nu(t_{a,b})) = \prod_{\mu\subseteq\nu, |\mu|=b-a+1} (\Box_\mu)^{(b-a)!(n+a-b-1)!}, (a < b \leq n)$
*b)* $\det(I - Q^{\nu}_{\{b,b+1\}}\hat{R}_\nu(t_{a,b})) = \prod_{\mu\subseteq\nu, |\mu|=b-a+2} (\Box_\mu)^{(b-a)!(b-a+2)!(n+a-b-2)!}, (a \leq b < n)$
*where for any subset $T \subset I$ we denote by $\Box_T$ the quantity*

$$\Box_T := 1 - q_T; \qquad q_T = \prod_{i\neq j\in T} q_{ij} \qquad (= \prod_{\{i\neq j\}\subset T} |q_{ij}|^2)$$

*in which the last product is over all two-element subsets of $T$ (We view $\nu$ as a subset of $I$, hence $\mu \subseteq \nu$ means that $\mu$ is a subset of $\nu$).*

**Proof.** a) Let $H :=< t_{a,b} >\subset S_n$ be the cyclic subgroup of $S_n$ generated by the cycle $t_{a,b}$. Then, each $H$-orbit on $\mathbf{f}_\nu$, $\mathbf{f}_\nu^{[\mathbf{i}]_a^b} = span\{\theta_{t^k_{a,b}\cdot\mathbf{i}} | 0 \leq k \leq b-a\}$, (which clearly corresponds to a cyclic $t_{a,b}$-equivalence class $[\mathbf{i}]_a^b = i_1 \cdots (i_a i_{a+1} \cdots i_b) \cdots i_n$ of the sequence $\mathbf{i} = i_1 \ldots i_n$ of weight $\nu$) is an invariant subspace of $R_\nu(t_{a,b})$ (and hence of $\hat{R}_\nu(t_{a,b})$). Note that $\hat{R}_\nu(t_{a,b})(\theta_{t^k_{a,b}\cdot\mathbf{i}}) = c_k\theta_{t^{k+1}_{a,b}\cdot\mathbf{i}}$ where $c_k = q_{t^k_{a,b}\cdot\mathbf{i}, t^{-1}_{a,b}} (0 \leq k \leq b-a)$ i.e. $\begin{cases} c_0 &= q_{i_a i_b} q_{i_{a+1} i_b} \cdots q_{i_{b-1} i_b}, \\ c_1 &= q_{i_{a+1} i_a} q_{i_{a+2} i_a} \cdots q_{i_b i_a}, \\ &\vdots \\ c_{b-a} &= q_{i_b i_{b-1}} q_{i_a i_{b-1}} \cdots q_{i_{b-2} i_{b-1}}. \end{cases}$

Thus $\hat{R}_\nu(t_{a,b})|f_\nu^{[\mathbf{i}]_a^b}$ is a cyclic operator, and

$$\det(I - \hat{R}_\nu(t_{a,b})|f_\nu^{[\mathbf{i}]_a^b}) = 1 - c_0c_1\cdots c_{b-a} = 1 - \prod_{i\neq j\in\{i_a,\ldots,i_b\}} q_{ij} = \Box_{\{i_a,\ldots,i_b\}}.$$

Note that this determinant depends only on the set $\{i_a, i_{a+1}, \ldots, i_b\}$ and that there are $(b-a)!(n-(b-a+1))!$ cyclic $t_{a,b}$ -equivalence classes corresponding to any given $(b-a+1)$-set $\mu = \{i_a, \ldots, i_b\} \subset \nu$. (Here we identify a generic weight $\nu = \sum \nu_i \cdot i$, $\nu_i \leq 1$ with the set $\{i \in I | \nu_i = 1\}$). b) Quite analogous to).
$\Box$

**Theorem 1.9.2. [DETERMINANT FORMULA].** *For $\nu$ generic, we have*

$$\det A^{(\nu)} = \prod_{\mu\subseteq\nu, |\mu|\geq 2} (\Box_\mu)^{(|\mu|-2)!(|\nu|-|\mu|+1)!}.$$

In particular, in *Example* 1.6.3 we have

$$\det A^{123} = (1 - |q_{12}|^2)^2(1 - |q_{13}|^2)^2(1 - |q_{23}|^2)^2(1 - |q_{12}|^2|q_{13}|^2|q_{23}|^2)$$

**Remark 1.9.3..** Theorem *1.9.2 is a multiparametric extension of* Theorem *2 in [Zag] which (case* $(q_{ij} = q)$*) reads as:* $\det A_n(q) = \prod_{k=2}^{n} (1-q^{k(k-1)})^{\frac{n!(n-k+1)}{k(k-1)}}$ *(e.g.* $\det A_3(q) = (1-q^2)^6(1-q^6)$*).*

**Theorem 1.9.4..** *The matrix* $A = A(\mathbf{q})$ *associated to the sesquilinear form* $(\ ,\ )_{\mathbf{q}}$ *on* $\mathbf{f}$*,(see 1.3 and 1.6) is positive definite if* $|q_{ij}| < 1$*, for all* $i, j \in I$*,so that the* $q_{ij}$*-canonical commutation relations 1.1(1) have a Hilbert space realization (cf.1.5).*

## 2. Formulas for the inverse of $A^{(\nu)}$, $\nu$ generic.

The problem of computing the inverse of matrices $A^{(\nu)}$ appears in the expansions of the number operators and transition operators (c.f [MSP]). It is also related to a random walk problem on symmetric groups and in several other situations (hyperplane arrangements, contravariant forms on certain quantum groups). We shall give here two types of formulas for $[A^{(\nu)}]^{-1}$: Zagier type formula and Božejko-Speicher type formulas.

### 2.1. Zagier type formula

First we give a formula for the inverse of $A^{(\nu)}$, $\nu$ generic, which follows from *Prop.* 1.8.3 and *Prop.* 1.8.4 :

$$\begin{aligned} [A^{(\nu)}]^{-1} &= [A^{(\nu),n}]^{-1} \cdots [A^{(\nu),1}]^{-1} \\ &= C^{(\nu),n} \cdot [D^{(\nu),n-1}]^{-1} \cdot C^{(\nu),n-1} \cdot [D^{(\nu),n-2}]^{-1} \cdots C^{(\nu),2} \cdot [D^{(\nu),1}]^{-1} \end{aligned}$$

To invert $A^{(\nu)}$, therefore, the first step is to invert $D^{(\nu),m}$ for each $m < n$. Then one can use multiparametric extensions of *Propositions* 3. and 4. of [Zag] which are too long to state them here.

### 2.2. Božejko-Speicher type formulas

In addition to the, multiplicative in spirit, Zagier type formula for the inverse of $A^{(\nu)}$ ($\nu$ generic), given in 2.1., one also has another, additive in spirit, Božejko-Speicher type formula (c.f. [BSp1], Lemma 2.6.) which, in the case of the symmetric group $S_n$, we shall present here, in a slightly different notation, together with several improvements. For $J = \{j_1 < j_2 < \cdots < j_{l-1}\} \subseteq \{1, 2, \ldots, n-1\}$ let $S_J$ be the following Young subgroup of $S_n$ defined by $S_J := S_{j_1} \times S_{j_2 - j_1} \times \cdots \times S_{n - j_{l-1}}$, $S_\phi = S_n$. Then the following is the left coset decomposition: $S_n = \gamma_J S_J$, where $\gamma_J = \{g \in S_n | g(1) < g(2) < \cdots < g(j_1), g(j_1+1) < \cdots < g(j_2), \cdots, g(j_{l-1}+1) < \cdots < g(n)\}$. The definition of $\gamma_J$ can also be put in the way.

**Fact 2.2.1..** $g \in \gamma_J \Leftrightarrow g(1)g(2)\cdots g(n)$ *is the shuffle of the sets* $[1..j_1], [j_1+1..j_2], \ldots [j_{l-1}+1, n] \Leftrightarrow$ *the descent set* $Des(g) = \{1 \leq i \leq n -$

*$1|g(i) > g(i+1)\}$ of $g$ is contained in the set $J$ (c.f. [Sta, pp. 69-70]). (Here $[a..b]$ denotes the set $\{a, a+1, \ldots, b\}$.) Moreover, each $g \in S_n$ has the unique factorization $g = a_J g_J$ with $g_J \in S_J$ and $a_J \in \gamma_J$ and with $l(g) = l(a_J) + l(g_J)$. For an arbitrary subset $X \subseteq S_n$ we define the matrix $\hat{R}_\nu(X)$ by*

$$\hat{R}_\nu(X) := \sum_{g \in X} \hat{R}_\nu(g)$$

**Proposition 2.2.2..** *Let $\nu$ be a generic weight, $|\nu| = n$. For any subset $J = \{j_1 < j_2 < \cdots j_{l-1}\}$ of $\{1, 2, \ldots, n-1\}$ let $A_J^{(\nu)}, \Gamma_J^{(\nu)}$ be the following matrices*

$$A_J^{(\nu)} := \hat{R}_\nu(S_J)(= \sum_{g \in S_J} \hat{R}_\nu(g)), \quad \Gamma_J^{(\nu)} := \hat{R}_\nu(\gamma_J)(= \sum_{g \in \gamma_J} \hat{R}_\nu(g)).$$

*Then the matrix $A^{(\nu)}(= A_\phi^{(\nu)})$ of the sesquilinear form $(\ ,\ )_{\mathbf{q}}$ (see* Prop. *1.8.1) has the following factorizations*

$$A^{(\nu)} = \Gamma_J^{(\nu)} A_J^{(\nu)} \quad (\Rightarrow \Gamma_J^{(\nu)} = A^{(\nu)}[A_J^{(\nu)}]^{-1})$$

**Proof.** By quasimultiplicativity of $\hat{R}_\nu$ and Fact 2.2.1. □

The following formula is the Božejko-Speicher adaptation of an Euler-type character formula of Solomon. In the case $W = S_n$ it reads as follows :

**Lemma 2.2.3..** *(c.f. [BSp2] Lemma 2.6) Let $w_n = nn-1\ldots1$ be the longest permutation in $S_n$. Then we have*

$$\sum_{J \subseteq \{1,2,\ldots,n-1\}} (-1)^{n-1-|J|} \Gamma_J^{(\nu)} = \hat{R}_\nu(w_n)$$

For reader's convenience we include here a variant of the proof (our notation is slightly different). For any subset $M \subseteq \{1, 2, \ldots, n-1\}$ we denote by $\delta_M$ the subset of $S_n$ consisting of all permutations $g \in S_n$ whose descent set $Des(g)$ is equal to M. Then by Fact 2.2.1 it is clear that $\gamma_J = \bigcup_{M \subseteq J} \delta_M$ (disjoint union), implying that

$$\hat{R}_\nu(\gamma_J) = \sum_{M \subseteq J} \hat{R}_\nu(\delta_M)$$

By the inclusion-exclusion principle we obtain

$$\hat{R}_\nu(\delta_M) = \sum_{J \subseteq M} (-1)^{|M-J|} \hat{R}_\nu(\gamma_J)$$

By letting $M = \{1, 2, \ldots, n-1\}(\Rightarrow \delta_M = \{w_n\})$ we obtain the desired identity. By combining *Prop.* 2.2.2. and *Lemma* 2.2.3 we obtain the following relation among the inverses of matrices $A_J^{(\nu)}$'s.

**Proposition 2.2.4.. (Long recursion for the inverse of $A^{(\nu)}$):** *We have*

$$[A^{(\nu)}]^{-1} = (\sum_{\emptyset\neq J\subseteq\{1,2,\dots,n-1\}} (-1)^{|J|+1}[A_J^{(\nu)}]^{-1})(I+(-1)^n\hat{R}_\nu(w_n))^{-1}$$

**Remark 2.2.5..** *Let us associate to each subset $\phi \neq J = \{j_1 < j_2 < \cdots < j_{l-1}\} \subseteq \{1,2,\dots,n-1\}$* a subdivision *$\sigma(J)$ of the set $\{1,2,\dots,n\}$ into intervals by $\sigma(J) := J_1J_2\cdots J_l$, where $J_k := [j_{k-1}+1..j_k](j_0 = 1, j_l = n)$. (Here $[a..b]$ denotes the interval $\{a, a+1,\dots,b-1,b\}$ and abbreviate $[a..a](=\{a\})$ to $[a]$). The Young subgroup $S_J$ can be written as a direct product of commuting subgroups*

$$S_J = S_{[1..j_1]}S_{[j_1+1..j_2]}\cdots S_{[j_{l-1}+1..n]} = S_{J_1}S_{J_2}\cdots S_{J_l}$$

*where for each interval $I = [a..b]$, $1 \le a \le b \le n$ we denote by $S_I = S_{[a..b]}$ the subgroup of $S_n$ consisting of permutations which are identity on the complement of $[a..b]$ (i.e. $S_{[a..b]} = S_1^{a-1}\times S_{b-a+1}\times S_1^{n-b}$). By denoting accordingly $A_I^{(\nu)} = A_{[a..b]}^{(\nu)} := \hat{R}_\nu(S_{[a..b]})$, we can rewrite the formula for $[A^{(\nu)}]^{-1} = [A_{[1..n]}^{(\nu)}]^{-1}$ in* Prop. *2.2.4. as follows:*

$$[A_{[1..n]}^{(\nu)}]^{-1} = (\sum_{\sigma=J_1\cdots J_l, l\geq 2} (-1)^l[A_{J_1}^{(\nu)}]^{-1}\cdots[A_{J_l}^{(\nu)}]^{-1})(I+(-1)^n\hat{R}_\nu(w_n))^{-1} \quad (*)$$

*where the sum is over all subdivisions of the set $\{1,2,\dots,n\}$. Similar formula we can write for $[A_{[a..b]}^{(\nu)}]^{-1}$ for any nondegenerate interval $[a..b]$, $1 \le a < b \le n$. Of course, if $a = b, [A_{[a..b]}^{(\nu)}]^{-1}$ is the identity matrix. Now we shall use an ordering denoted by $\prec$ on the set $\Sigma_n$ of all subdivisions of the set $\{1,2,\dots,n\}$, called* reverse refinement order, *defined by $\sigma \prec \sigma'$ if $\sigma'$ is finer than $\sigma$, i.e. $\sigma'$ is obtained by subdividing each nontrivial interval in $\sigma$. The minimal and maximal elements in $\Sigma_n$ are denoted by $\hat{0}_n(=[1..n])$ and $\hat{1}_n = [1][2]\cdots[n]$. We shall call $(\Sigma_n, \prec)$ the* lattice of subdivisions *of $\{1,2,\dots,n\}$. For example, we have $\Sigma_1 = \{[1]\}$, $\Sigma_2 = \{[12],[1][2]\}$, $\Sigma_3 = \{[123],[1][23],[12][3],[1][2][3]\}$, $\Sigma_4 = \{[1234],[123][4],[12][34],[1][234],[12][3][4], [1][23][4],[1][2][34],[1][2][3][4]\}$(see Figure 1).*
*(Here $[1234]$ denotes the interval $[1..4] = \{1,2,3,4\}$ etc.)*

*Now for each interval $I = [a..b], 1 \le a < b \le n$ we denote by $w_I = w_{[a..b]} := 1\ 2\cdots a-1\ b\ b-1\cdots a\ b+1\cdots n$ the longest permutation in $S_{[a..b]}(= S_1^{a-1}\times S_{b-a+1}\times S_1^{n-b})$ and by $\Psi_I^\nu = \Psi_{[a..b]}^\nu, a < b$ the following matrix*

$$[I+(-1)^{b-a+1}\hat{R}_\nu(w_{[a..b]})]^{-1} = \frac{1}{\square_{[a..b]}^\nu}[I-(-1)^{b-a+1}\hat{R}_\nu(w_{[a..b]})]$$
$$= \frac{1}{\square_I^\nu}\Phi_I^\nu, \quad \text{where} \qquad \Phi_I^\nu := I-(-1)^{|I|}\hat{R}_\nu(w_I)$$

Figure 1: $\Sigma_4 =$ The lattice of subdivisions of $\{1,2,3,4\}$.

*and where* $\Box^{\nu}_{[a..b]}$ *is the diagonal matrix (cf. the definition of* $\Box^{\nu}_{T}$ *given in 1.8):*

$$\Box^{\nu}_{[a..b]} = \Box^{\nu}_{\{a,a+1,\cdots,b\}} = I - Q^{\nu}_{\{a,a+1,\ldots,b\}} = I - \prod_{a\le k<l\le b} |Q^{\nu}_{k,l}|^2,\ [Q^{\nu}_{k,l}]_{i_1\cdots i_n, i_1\cdots i_n} = q_{i_k i_l}.$$

*Accordingly, for any subdivision* $\sigma = I_1 I_2 \cdots I_l \in \Sigma_n$ *we define* $\Psi^{\nu}_{\sigma} := \prod_{j:|I_j|\ge 2} \Psi^{\nu}_{I_j}$ *(factors commute here, because* ${I_j}'s$ *are disjoint!), and similarly, for any chain* $\mathcal{C} : \sigma^{(1)} \prec \cdots \prec \sigma^{(m)}$ *in* $\Sigma_n$ *we define*

$$\Psi^{\nu}_{\mathcal{C}} = \overleftarrow{\prod}_{1\le j\le m} \Psi^{\nu}_{\sigma^{(j)}} = \Psi^{\nu}_{\sigma^{(m)}} \cdots \Psi^{\nu}_{\sigma^{(1)}}$$

*In the same way we introduce notations* $\Box^{\nu}_{\mathcal{C}}$ *and* $\Phi^{\nu}_{\mathcal{C}}$ *and observe that then* $\Psi^{\nu}_{\mathcal{C}} = \frac{1}{\Box^{\nu}_{\mathcal{C}}} \Phi^{\nu}_{\mathcal{C}}$*. For example, if* $\mathcal{C} : \hat{0}_5 = [12345] \prec [12][345] \prec [1][2][34][5] \prec \hat{1}_5$*, then for any* generic weight $\nu, |\nu| = 5$ *we have*

$$\begin{aligned}
\Psi^{\nu}_{\mathcal{C}} &= \Psi^{\nu}_{\{3,4\}} (\Psi^{\nu}_{\{1,2\}} \Psi^{\nu}_{\{3,4,5\}}) \Psi^{\nu}_{\{1,2,3,4,5\}} \\
&= \frac{1}{\Box^{\nu}_{\{3,4\}} \Box^{\nu}_{\{1,2\}} \Box^{\nu}_{\{3,4,5\}} \Box^{\nu}_{\{1,2,3,4,5\}}} \Phi^{\nu}_{\{3,4\}} \Phi^{\nu}_{\{1,2\}} \Phi^{\nu}_{\{3,4,5\}} \Phi^{\nu}_{\{1,2,3,4,5\}}.
\end{aligned}$$

*Now we can state our first explicit formula for the inverse of* $A^{(\nu)}$ *in terms of the involutions* $w_I = w_{[a..b]}, 1 \le a < b \le n$*.*

**Theorem 2.2.6. [INVERSION FORMULA – CHAIN VERSION].** *Let $\nu$ be a generic weight, $|\nu| = n$. Then*

$$[A^{(\nu)}]^{-1} = \sum_{\mathcal{C}} (-1)^{b_+(\mathcal{C})+n-1} \Psi^\nu_{\mathcal{C}} = \sum_{\mathcal{C}} \frac{(-1)^{b_+(\mathcal{C})+n-1}}{\Box^\nu_{\mathcal{C}}} \Phi^\nu_{\mathcal{C}}$$

*where the summation is over all chains $\mathcal{C} : \hat{0}_n = \sigma^{(0)} \prec \sigma^{(1)} \cdots \prec \sigma^{(m)} \prec \hat{1}_n$ in the subdivision lattice $\Sigma_n$ and where $b_+(\mathcal{C})$ denotes the total number of nondegenerate intervals appearing in members of $\mathcal{C}$.*

**Proof.** The formula follows by iterating the formula (*) in *Remark* 2.2.5. □

**Remark 2.2.7..** *If we represent chains $\mathcal{C} : \hat{0}_n = \sigma^{(0)} \prec \sigma^{(1)} \prec \cdots \prec \sigma^{(m-1)} \prec \hat{1}_n$ of length $m \geq 1$ as* generalized bracketing *(of* depth $m$*) of the word $12 \cdots n$ with one pair of brackets for each nondegenerate interval appearing in the members of $\mathcal{C}$ (e.g. $\hat{0}_5 = [12345] \prec [12][345] \prec [1][2][34][5] \prec \hat{1}_5$ is represented as $[[12][[34]5]]$), then we can write the bracketing version of the Inversion formula of Thm. 2.2.6 as*

$$[A^{(\nu)}]^{-1} = \sum_{\beta} (-1)^{b(\beta)+n-1} \Psi^\nu_{\beta} = \sum_{\beta} \frac{(-1)^{b(\beta)+n-1}}{\Box^\nu_{\beta}} \Phi^\nu_{\beta}$$

*where the sum is over all generalized bracketings of the word $12 \cdots n$ and where $b(\beta)$ denotes the number of pairs of brackets in $\beta$ and where $\Psi^\nu_\beta := \Psi^\nu_{\mathcal{C}}$, $\Phi^\nu_\beta := \Phi^\nu_{\mathcal{C}}$, $\Box^\nu_\beta := \Box^\nu_{\mathcal{C}}$ if $\beta$ is associated to the (unique!) chain $\mathcal{C}$ in $\Sigma_n$, e.g.*
$\Psi^\nu_{[[12][[34]5]]} = \Psi^\nu_{[3..4]}(\Psi^\nu_{[1..2]}\Psi^\nu_{[3..5]})\Psi^\nu_{[1..5]} = \Psi^\nu_{[1..2]}\Psi^\nu_{[3..4]}\Psi^\nu_{[3..5]}\Psi^\nu_{[1..5]}$
$= \frac{1}{\Box^\nu_{\{1,2\}}\Box^\nu_{\{3,4\}}\Box^\nu_{\{3,4,5\}}\Box^\nu_{\{1,\ldots,5\}}}(I - \hat{R}_\nu(w_{[1..2]}))(I - \hat{R}_\nu(w_{[3..4]}))(I + \hat{R}_\nu(w_{[3..5]}))(I + \hat{R}_\nu(w_{[1..5]})$.

*In particular for* Example *1.6.3 ($I = \{1,2,3\}, \nu_1 = \nu_2 = \nu_3 = 1$) we have*

$[A^{123}]^{-1} = -\Psi_{[123]} + \Psi_{[[12]3]} + \Psi_{[1[23]]} = \frac{-1}{\Box_{\{1,2,3\}}}(I - \hat{R}_{123}(321)) + \frac{1}{\Box_{\{1,2\}}\Box_{\{1,2,3\}}}$
$(I + \hat{R}_{123}(213))(I - \hat{R}_{123}(321)) + \frac{1}{\Box_{\{2,3\}}\Box_{\{1,2,3\}}}(I + \hat{R}_{123}(132))(I - \hat{R}_{123}(321))$.

*Similarly, for $I = \{1,2,3,4\}, \nu_1 = \nu_2 = \nu_3 = \nu_4 = 1$ we have*

$$\begin{aligned}[A^{1234}]^{-1} &= \Psi_{[1234]} - \Psi_{[1[234]]} - \Psi_{[12[34]]} - \Psi_{[1[23]4]} - \Psi_{[[12]34]} - \Psi_{[[123]4]} + \\ &+ \Psi_{[[12][34]]} + \Psi_{[[[12]3]4]} + \Psi_{[[1[23]]4]} + \Psi_{[1[[23]4]]} + \Psi_{[1[2[34]]]}\end{aligned}$$

*(Here we suppressed the upper indices in $\Psi^{123}_\beta$ and $\Psi^{1234}_\beta$).*

**Corollary 2.2.8. [EXTENDED ZAGIER'S CONJECTURE].** *For $\nu$ generic, $|\nu| = n$, for the inverse of the matrix $A^{(\nu)} = A^{(\nu)}(\mathbf{q})$ we have*

*i)* $$[A^{(\nu)}]^{-1} \in \frac{1}{\Box^\nu}\mathrm{Mat}_{n!}(Z[q_{ij}])$$

*where $\Box^\nu$ denotes the diagonal matrix* $\prod_{1\leq a<b\leq n} \Box^\nu_{[a..b]} = \prod_{1\leq a<b\leq n}(I - \prod_{a\leq k\neq l\leq b} Q^\nu_{k,l})$

*i′)* $$[A^{(\nu)}]^{-1} \in \frac{1}{d_\nu}\mathrm{Mat}_{n!}(Z[q_{ij}])$$

*where $d_\nu$ is the following quantity $\prod_{\mu\subseteq\nu,|\mu|\geq 2}\Box_\mu = \prod_{\mu\subseteq\nu,|\mu|\geq 2}(1-\prod_{i\neq j\in\mu} q_{ij})$ ($\Box_\mu$ is the same as in* Lemma 1.9.1*). In particular when all $q_{ij}=q$ (Zagier's case) we have from i):*

$$ii) \qquad [A^\nu(q)]^{-1} \in \frac{1}{\delta_n(q)}\mathrm{Mat}_{n!}(Z[q])$$

*where $\delta_n(q)=\prod_{1\leq a<b\leq n}(1-q^{(b-a+1)(b-a)})=\prod_{k=2}^{n}(1-q^{k(k-1)})^{n-k+1}$.*

**Proof.** *i*) follows from *Theorem* 2.2.6 by taking the common denominator which turns out to be $\Box^\nu = \prod_{1\leq a<b\leq n}\Box^\nu_{[a..b]}$ because any $\Box^\nu_{[a..b]}$ appears at most once in each of the denominators $\Box^\nu_C$ (and actually appears in at least one of them).
*i'*) The entries of $\Box^\nu$ are zero or $\Box^\nu_{\mathbf{i},\mathbf{i}}$ where $\mathbf{i}=i_1\cdots i_n$ is any permutation of $\nu$ ($|\mathbf{i}|=\nu$) considered as a subset of $I$ (because $\nu$ is generic!). Since

$$\Box^\nu_{\mathbf{i},\mathbf{i}} \quad = \prod_{1\leq a<b\leq n}(1-\prod_{a\leq k\neq l\leq b} q_{i_k i_l}) = \prod_{1\leq a<b\leq n}\Box_{\{i_a,i_{a+1},\ldots,i_b\}}$$

we see that $\Box^\nu_{\mathbf{i},\mathbf{i}}$ divides $d_\nu$.
*ii*) Note that in case all $q_{ij}=q$:

$$\Box^\nu_{\mathbf{i},\mathbf{i}} \quad = \prod_{1\leq a<b\leq n}(1-\prod_{a\leq k\neq l\leq b} q) = \prod_{k=2}^{n}(1-q^{k(k-1)})^{n-k+1} = \delta_n(q).$$

This completes the proof of Extended Zagier's conjecture. □

**Remark 2.2.9..** *In [Zag] p.201 Zagier conjectured that $A_n(q)^{-1}\in\frac{1}{\triangle_n}\mathrm{Mat}_{n!}(Z[q])$, where $\triangle_n := \prod_{k=2}^{n}(1-q^{k(k-1)})$ and checked this conjecture for $n\leq 5$. But we found that this conjecture failed for $n=8$ (see Examples to* Proposition *2.2.15). It seems that our statement in* Corollary *2.2.8 ii) is the right form of a conjecture valid for all $n$ when all $q_{ij}$ are equal.*

**Proposition 2.2.10..** *Let $c_n$ be the number of $\hat{0}_n-\hat{1}_n$ chains in the subdivision lattice $\Sigma_n$ (i.e. the number of $\Psi$-terms in the formula for $[A^{(\nu)}]^{-1}$ $\nu$ generic, $|\nu|=n$ in Thm. 2.2.6 ), $c_0:=0, c_1:=1$. Then*

$$C(t)=\sum_{n\geq 0}c_n t^n = \frac{1}{4}(1+t-\sqrt{1-6t+t^2}) = t+t^2+3t^3+11t^4+45t^5+197t^6+\cdots$$

**Proof.** By *Remark* 2.2.7 this counting is equivalent to the Generalized bracketing problem of Schröder (1870) (see [Com], p. 56). In fact, the numbers $c_n$ can be computed faster via linear recurrence relation (following from the fact that $C(t)$ is algebraic): $(n+1)c_{n+1}=3(2n-1)c_n-(n-2)c_{n-1},\ \ n\geq 2,\ \ c_1=c_2=1.$
□

More generally, by a formal language method we found in [MS] that the number $c_{n,k}$ of chains, as above, having altogether $k$ nondegenerate intervals is

equal to $c_{n,k} = \binom{n+k+1}{k}\binom{n-2}{k-1}/n$ (e.g.$c_{3,1} = 1, c_{3,2} = 2, c_{4,1} = 1, c_{4,2} = 5, c_{4,3} = 5$ ) yielding a simple formula for $c_n = c_{n,1} + \cdots + c_{n,n-1}$. (This formula is relevant to non-intersecting diagonals structures in one proof of Four Color Theorem and it is much simpler than the one given in [CM1,CM2]).
Now we turn our attention to computation of entries in the inverse of $A^{(\nu)}$, $\nu$ generic. First we note that any $n! \times n!$ matrix $A$ can be written as $A = \sum_{g \in S_n} A(g) R_n(g)$, where $A(g)$ are diagonal matrices defined by $A(g)_{\mathbf{i},\mathbf{i}} = A_{\mathbf{i},g^{-1}\cdot\mathbf{i}}$ (all $\mathbf{i}$) ($R_n(g)$ is the right regular representation matrix $R_n(g)_{\mathbf{i},\mathbf{j}} = \delta_{\mathbf{i},g\cdot\mathbf{j}}$, c.f. 1.8).We call $A(g)$ the *g-th diagonal* of $A$. Hence, if we write

$$A^{(\nu)} = \sum_{g \in S_n} A^{(\nu)}(g) R_\nu(g), \quad [A^{(\nu)}]^{-1} = \sum_{g \in S_n} [A^{(\nu)}]^{-1}(g) R_\nu(g)$$

then by *Prop.* 1.8.1 ($\nu$ generic ) we have

$$A^{(\nu)}(g) = Q^\nu(g) = \prod_{(a,b) \in I(g^{-1})} Q^\nu_{a,b} \ ; \ (Q^\nu_{a,b})_{\mathbf{i},\mathbf{i}} = q_{i_a i_b}$$

To compute $[A^{(\nu)}]^{-1}(g)$ we first write $[A^{(\nu)}]^{-1}(g) = \Lambda^\nu(g) A^{(\nu)}(g)$ where $\Lambda^\nu(g)$ are yet unknown diagonal matrices.

Similarly, for each $\emptyset \neq J \subseteq \{1, 2, \ldots, n-1\}$ we write $[A_J^{(\nu)}]^{-1}(g) = \Lambda_J^\nu(g) A_J^{(\nu)}(g)$ and for any $I = [a..b] \subseteq \{1, 2, \ldots, n\}$ we write $[A_I^{(\nu)}]^{-1}(g) = \Lambda_I^\nu(g) A_I^{(\nu)}(g)$ where $\Lambda_J^\nu(g)$ and $\Lambda_I^\nu(g)$ are unknown diagonal matrices.

If $\sigma(J) = J_1 J_2 \cdots J_l$ is the subdivision of $\{1, 2, \ldots, n\}$ (cf. *Remark* 2.2.5) associated to $J$, and if $g = g_1 g_2 \cdots g_l \in S_J = S_{J_1} S_{J_2} \cdots S_{J_l}$, then $\Lambda_J^\nu(g) = \Lambda_{J_1}^\nu(g_1) \cdots \Lambda_{J_l}^\nu(g_l)$.

Let us denote by $S_n^>$ (resp. $S_n^<$) the subset of $S_n$ of all elements $g$ such that $g(1) > g(n)$ (resp. $g(1) < g(n)$). It is evident that $S_n^< = S_n^> w_n$, $S_n^> = S_n^< w_n$, where $w_n = nn-1\cdots 21$.

**Proposition 2.2.11..** *The diagonal matrices $\Lambda^\nu(g)$ are real and satisfy the following recurrences:*
*i)* $\Lambda^\nu(g) = (-1)^{n-1} |Q^\nu(g w_n)|^2 \Lambda^\nu(g w_n)$, *if* $g \in S_n^>$
*ii)* $\Lambda^\nu(g) = \Lambda^\nu_{[1..n]}(g) = \frac{1}{\Box^\nu_{[1..n]}} \sum_{\emptyset \neq J \subseteq \{1,2,\ldots,n-1\}, g \in S_J} (-1)^{|J|+1} \Lambda_J^\nu(g)$, *if* $g \in S_n^<$
*ii')* $\Lambda^\nu(g) = \frac{1}{\Box^\nu_{[1..n]}} \sum_{g = g'g'' \in S_k \times S_{n-k}, 1 \leq k \leq n-1} (Q^\nu_{[1..k]})^{[g(1)<g(k)]} \Lambda^\nu_{[1..k]}(g') \Lambda^\nu_{[k+1..n]}(g'')$,
*if* $g \in S_n^<$. *In particular,* $[A^{(\nu)}]^{-1}(g) = [A^{(\nu)}]^{-1}(g w_n) = 0$ *if both $g$ and $g w_n$ are not splittable, i.e. if the minimal Young subgroup containing $g$ (resp. $g w_n$) is equal to $S_n$.*

**Proof.** Substituting the formula $(I + (-1)^n \hat{R}_\nu(w_n))^{-1} = \frac{1}{\Box^\nu_{[1..n]}} (I - (-1)^n \hat{R}_\nu(w_n))$ into formula for $[A^{(\nu)}]^{-1}$ in *Prop.* 2.2.4 we see immediately that for $g \in S_n^<$

$$[A^{(\nu)}]^{-1}(g) = \sum_{\emptyset \neq J \subseteq \{1,2,\ldots,n-1\}} (-1)^{|J|+1} [A_J^{(\nu)}]^{-1}(g) \frac{1}{\Box^\nu_{[1..n]}} \qquad (*)$$

Then for $g \in S_n^>$, we again use *Prop.* 2.2.4 and *Property* 3.ii) in 1.8 The property ii) is immediate from (*) because $[A_J^\nu]^{-1}(g) \neq 0 \Rightarrow g \in S_J$. To prove ii') one can use the following lemma which we are going to state without proof. $\square$

**Lemma 2.2.12. [Short recursion for the inverse of $A^{(\nu)}$].** *We have*

$$[A^{(\nu)}]^{-1} = (\sum_{k=1}^{n-1} (-1)^{k-1} [A^{(\nu)}_{\{k\}}]^{-1} \hat{R}_\nu(w_{[1..k]})(I + (-1)^n \hat{R}_\nu(w_n))^{-1}$$

*where* $A^{(\nu)}_{\{k\}} = \hat{R}_\nu(S_k \times S_{n-k})(= A^{(\nu)}_{[1..k]} A^{(\nu)}_{[k+1..n]})$ *is just* $A_J^{(\nu)}$ *when* $J = \{k\}$.

**Corollary 2.2.13..** *With notations of* Remark *2.2.7 and* Proposition *2.2.11 we have the following formulas for the diagonal entries of the inverse of* $A^\nu$, $\nu$ *generic,* $|\nu| = n$.

*i)*
$$[A^{(\nu)}]^{-1}(id) = \sum_\beta \frac{(-1)^{b(\beta)+n-1}}{\square^\nu_\beta}$$

*where the sum is over all generalized bracketings* $\beta$ *of the word* $12 \cdots n$, *which have outer brackets.*

*i')*
$$[A^{(\nu)}]^{-1}(id) = \frac{1}{\square^\nu_{[1..n]}} \sum_\beta \frac{Q^\nu_\beta}{\square^\nu_\beta}$$

*where the sum is over all generalized bracketings* $\beta$, *without outer brackets, of the word* $12 \cdots n$, *and where* $Q^\nu_\beta$ *is defined, analogously as* $\square^\nu_\beta$, *to be the product of* $Q^\nu_{[a..b]}$ *over all bracket pairs in* $\beta$.

**Proof.** *i)* follows from *Remark* 2.2.7 because $\hat{R}_\nu$-terms contribute only to nondiagonal entries. *i')* follows by iterating *Proposition* 2.2.11 *ii')* in case $g = id$ and using that $[A^{(\nu)}]^{-1}(id) = \Lambda^\nu(id) A^{(\nu)}(id) = \Lambda^\nu(id) Q^\nu(id) = \Lambda^\nu(id)$. $\square$

In particular if $I = \{1,2\}, \nu_1 = \nu_2 = 1$, we have $\Lambda^{12}(id) = [A^{12}]^{-1}(id) = \frac{1}{\square_{\{1,2\}}}$.

In Example 1.6.3 ($I = \{1,2,3\}, \nu_1 = \nu_2 = \nu_3 = 1$)we have

$$\begin{aligned} \Lambda^{123}(id) = [A^{123}]^{-1}(id) &= \frac{-1}{\square_{123}} + \frac{1}{\square_{12}\square_{123}} + \frac{1}{\square_{23}\square_{123}} \\ &= \frac{1}{\square_{123}}(1 + \frac{Q_{12}}{\square_{12}} + \frac{Q_{23}}{\square_{23}}) \end{aligned}$$

Similarly, for $I = \{1,2,3,4\}, \nu_1 = \nu_2 = \nu_3 = \nu_4 = 1$ we have

$$\begin{aligned} \Lambda^{1234}(id) = & \ [A^{1234}]^{-1}(id) \\ &= \tfrac{1}{\square_{1234}} \Big\{ 1 + \tfrac{Q_{12}}{\square_{12}} + \tfrac{Q_{23}}{\square_{23}} + \tfrac{Q_{34}}{\square_{34}} + \tfrac{Q_{12}Q_{34}}{\square_{12}\square_{34}} \\ &\qquad + (1 + \tfrac{Q_{12}}{\square_{12}} + \tfrac{Q_{23}}{\square_{23}}) \tfrac{Q_{123}}{\square_{123}} + (1 + \tfrac{Q_{23}}{\square_{23}} + \tfrac{Q_{34}}{\square_{34}}) \tfrac{Q_{234}}{\square_{234}} \Big\} \end{aligned}$$

(Here we abbreviated $Q_{\{1,2\}}, Q_{\{2,3,4\}}$ to $Q_{12}, Q_{234}$ etc.). If we take all $q_{ij} = q$ (Zagier's case), then we obtain easily that

$$[A_3(q)]^{-1}(id) = \frac{1+q^2}{(1-q^2)(1-q^6)}I, \quad [A_4(q)]^{-1}(id) = \frac{1+2q^2+q^4+2q^6+q^8}{(1-q^2)(1-q^6)(1-q^{12})}I$$

what agrees with Zagier's computations.

**Remark 2.2.14..** *The formula i') in* Corollary *2.2.13 can be interpreted also as a regular language expression for closed walks in the weighted digraph (a Markov chain) $\mathcal{D}^\nu$ on the symmetric group $S_n$ where the adjacency matrix $A(\mathcal{D}^\nu)$ is given by nondiagonal entries of $A^{(\nu)}$ multiplied by -1, i.e. $A(\mathcal{D}^\nu) = -(A^{(\nu)} - I)$. Then the walk generating matrix function of $\mathcal{D}^\nu$ is nothing but the inverse of $A^{(\nu)}$ because $W(\mathcal{D}^\nu) = (I - A(\mathcal{D}^\nu))^{-1} = [A^{(\nu)}]^{-1}$. For example, we have*

$$\begin{aligned} W(\mathcal{D}^{123})_{closed} &= [A^{123}]^{-1}(id) = Q_{\{1,2,3\}}{}^{*}(I + Q_{\{1,2\}}{}^{+} + Q_{\{2,3\}}{}^{+}) \\ W(\mathcal{D}^{1234})_{closed} &= [A^{1234}]^{-1}(id) = Q_{[1..4]}{}^{*}\Big\{1 + Q_{[1..2]}{}^{+} + Q_{[2..3]}{}^{+} + Q_{[3..4]}{}^{+} + \end{aligned}$$
$$Q_{[1..2]}{}^{+}Q_{[3..4]}{}^{+} + (1 + Q_{[1..2]}{}^{+} + Q_{[2..3]}{}^{+})Q_{[1..3]}{}^{+} + (1 + Q_{[2..3]}{}^{+} + Q_{[3..4]}{}^{+})Q_{[2..4]}{}^{+}\Big\}$$

*in the familiar formal language notation ($x^* = \frac{1}{1-x}, x^+ = \frac{x}{1-x}$).*

Now we turn our attention to computing a general entry of the inverse of $A^\nu$, $\nu$ generic, $|\nu| = n$.

Let $g \in S_n^<$(i.e. $g(1) < g(n)$) be given. Let $J(g) = \{j_1 < j_2 < \cdots < j_{n(g)-1}\} \subset \{1, 2, \ldots, n-1\}$ be the label of the minimal Young subgroup of $S_n$ containing g. It is clear that $J(g)$ can be given explicitly as $J(g) = \{1 \le j \le n-1 | g(1)+g(2)+\cdots+g(j) = 1+2+\cdots+j\}$ Then by $\sigma(g) = J_1 J_2 \cdots J_{n(g)} \in \Sigma_n$ we denote the subdivision associated to $J(g)$ i.e

$$J_1 = J_1(g) := [1..j_1], J_2 = J_2(g) := [j_1{+}1..j_2], \cdots, J_{n(g)} := J_{n(g)}(g) = [j_{n(g)-1}{+}1..n]$$

and by $g = g_1 g_2 \cdots g_{n(g)}$ we denote the corresponding factorization of $g$ with $g_k \in S_{J_k(g)}, 1 \le k \le n(g)$. By noting that $g \in S_J \Leftrightarrow J \subseteq J(g)$, we can rewrite the formula *Proposition* 2.2.11 ii) as follows

$$\begin{aligned} \Lambda^\nu(g) &= \Lambda^\nu_{[1..n]}(g) = \frac{1}{\square^\nu_{[1..n]}} \sum_{\emptyset \ne J \subseteq J(g)} (-1)^{|J|+1} \Lambda^\nu_J(g) \\ &= \frac{1}{\square^\nu_{[1..n]}} \sum_{\emptyset \ne K \subseteq \{1,2,\ldots,n(g)-1\}} (-1)^{|K|+1} \Lambda^\nu_{J(K)}(g) \end{aligned}$$

where $J(K) := \{j_k | k \in K\} \subseteq \{1, 2, \ldots, n-1\}$. (Note that if $J(g) = \emptyset$ ($\Rightarrow g$ and $gw_n$ are not splittable), then $\Lambda^\nu(g) = 0$ by this formula, too.) In terms of subdivisions this can be viewed as a recursion formula:

$$\Lambda^\nu_{[1..n]}(g) = \frac{1}{\square^\nu_{[1..n]}} \sum_{\tau = K_1 K_2 \cdots K_l \in \Sigma_{n(g)}, l \ge 2} (-1)^l \Lambda^\nu_{I(K_1)}(g_{K_1}) \cdots \Lambda^\nu_{I(K_l)}(g_{K_l}) \quad (*)$$

where $I(K_s) := \bigcup_{k\in K_s} J_k(g)$, $g_{K_s} := \prod_{k\in K_s} g_k, s = 1, ..., l$.

By iterating this recursion formula (*) (as in *Theorem* 2.2.6, *Remark* 2.2.7, *Corollary* 2.2.13) we obtain

$$\Lambda^\nu_{[1..n]}(g) = (\sum_\beta (-1)^{b(\beta)+n(g)-1}\tilde\Psi_\beta)\Lambda^\nu_{J_1(g)}(g_1)\cdots\Lambda^\nu_{J_{n(g)}(g)}(g_{n(g)}) \qquad (**)$$

where $\beta$ runs over all generalized bracketings of the word $12\cdots n(g)$ which have outer brackets and where for each bracket pair $[a..b]$, $1 \le a < b \le n(g)$, we set

$$\tilde\Psi_{[a..b]} := \frac{1}{\Box_{J_a \bigcup J_{a+1}\bigcup\cdots\bigcup J_b}}$$

($b(\beta)$ :=number of bracket pairs in $\beta$). Thus the expression in the parentheses can be viewed as a *"thickened" identity coefficient*

$$\Lambda^{12\cdots n(g)}(id)|^\nu_{1\to J_1, 2\to J_2,\cdots, n(g)\to J_{n(g)}}$$

which we shall denote by

$$\Lambda^\nu_{\sigma(g)} = \Lambda^\nu_{J_1(g)J_2(g)\cdots J_{n(g)}(g)} := \Lambda^{12\cdots n(g)}(id)|_{1\to J_1, 2\to J_2,\ldots, n(g)\to J_{n(g)}}.$$

(In particular we can now write $\Lambda^\nu_{[1..n]}(id)$ also as $\Lambda^\nu_{[1][2]\cdots[n]}$).

As an example for this notation we take $g = 41325786$. Then $\sigma(g) = [1..4][5][6..8]$ i.e $J_1(g) = [1..4], J_2(g) = [5], J_3(g) = [6..8]$. So

$$\Lambda^\nu_{[1..4][5][6..8]} = \Lambda^{123}(id)|^\nu_{1\to[1..4],2\to[5],3\to[6..8]} = \frac{1}{\Box^\nu_{[1..8]}}(-1 + \frac{1}{\Box^\nu_{[1..5]}} + \frac{1}{\Box^\nu_{[5..8]}})$$

(c.f. *Corollary* 2.2.13).

Now we have one more observation concerning the formula $(**)$. To each nonzero factor $\Lambda^\nu_{J_k(g)}(g_k), 1 \le k \le n(g)$ in $(**)$ we can apply *Proposition* 2.2.11 i) because $g_k$, being a minimal Young factor of $g$, is not splittable and hence $g_k(j_{k-1}+1) > g_k(j_k)$ (otherwise $g_k w_{J_k}$ would also be nonsplittable $\Rightarrow \Lambda^\nu_{J_k(g)}(g_k) = 0$)

$$\Lambda^\nu_{J_k(g)}(g_k) = (-1)^{|J_k(g)|-1}|Q^\nu(g_k w_{J_k(g)})|^2\Lambda^\nu_{J_k(g)}(g_k w_{J_k(g)})$$

Substituting this into $(**)$ we obtain the following algorithm for computing the diagonal matrices $\Lambda^\nu(g)$ describing the inverse of $A^{(\nu)}$(recall $[A^{(\nu)}]^{-1} = \sum_{g\in S_n}\Lambda^\nu(g)\hat R(g)$).

**Proposition 2.2.15. [An algorithm for $\Lambda^\nu(g)$, $\nu$ generic, $|\nu| = n$)].** *For $g \in S_n$ we have*

$$\Lambda^\nu_{[1..n]}(g) = (-1)^{n-n(g)}\Lambda^\nu_{\sigma(g)}|Q^\nu(g')|^2\Lambda^\nu_{J(g)}(g')$$

*where $g' := gw_{J(g)}$ ($w_{J(g)}$ = the maximal element in the minimal Young subgroup $S_{J(g)}$ containing $g$). Similar statement holds true if we replace $[1..n]$ by any interval $[a..b], 1 \le a \le b \le n$.*

**Proof.** If $g(1) < g(n)$, this is what we get from $(**)$. If $g(1) > g(n)$, then $J(g) = \emptyset, S_{J(g)} = S_n, w_{J(g)} = nn{-}1\ldots 21 = w_n, n(g) = 1, \sigma(g) = [1..n], \Lambda^\nu_{\sigma(g)} = \Lambda^1(id)|_{1\to[1..n]} = I, g' = gw_{J(g)} = gw_n$, so what we needed to prove is just the claim in *Proposition* 2.2.11 i). $\square$

To illustrate this algorithm we take (again!) $g = 41325786$ ($\nu$ generic weight, $|\nu| = 8$) for which $J(g) = \{4,5\}, J_1(g) = [1..4], J_2(g) = [5], J_3(g) = [6..8], n(g) = 3, n = 8, w_{J(g)} = 43215876, g' = gw_{J(g)} = 23145687, Q^\nu(g') = Q^\nu_{1,2}Q^\nu_{1,3}Q^\nu_{7,8}$, $|Q^\nu(g')|^2 = Q^\nu(g')Q^\nu(g')^* = Q^\nu_{\{1,2\}}Q^\nu_{\{1,3\}}Q^\nu_{\{7,8\}}$. Then the first step of our algorithm gives

$$\begin{aligned}&\Lambda^\nu_{[1..8]}(g) = \Lambda^\nu_{[1..8]}(41325786) =\\ &= (-1)^{8-3}\Lambda^\nu_{[1..4][5][6..8]}Q^\nu_{\{1,2\}}Q^\nu_{\{1,3\}}Q^\nu_{\{7,8\}}\Lambda^\nu_{[1..4]}(2314)\Lambda^\nu_{[5]}(5)\Lambda^\nu_{[6..8]}(687).\end{aligned}$$

In the second step of our algorithm we compute

$$\Lambda^\nu_{[1..4]}(2314) = (-1)^{4-2}\Lambda^\nu_{[1..3][4]}Q^\nu_{\{2,3\}}\Lambda^\nu_{[1..3]}(132)\Lambda^\nu_{[4]}(4)$$

$$\Lambda^\nu_{[6..8]}(687) = (-1)^{3-2}\Lambda^\nu_{[6][7..8]}\Lambda^\nu_{[6]}(6)\Lambda^\nu_{[7..8]}(78)$$

In the third (last) step we need only to compute

$$\Lambda^\nu_{[1..3]}(132) = (-1)^{3-2}\Lambda^\nu_{[1][2..3]}\Lambda^\nu_{[1]}(1)\Lambda^\nu_{[2..3]}(23).$$

Since $\Lambda^\nu_{[7..8]}(78) = \Lambda^\nu_{[7][8]}, \Lambda^\nu_{[2..3]}(23) = \Lambda^\nu_{[2][3]}, (Q^\nu_{\{1,2\}}Q^\nu_{\{1,3\}})Q^\nu_{\{2,3\}} = Q^\nu_{[1..3]}, \Lambda^\nu_{[1]}(1) = \cdots = \Lambda^\nu_{[8]}(8) = I$, we finally obtain

$$\Lambda^\nu_{[1..8]}(41325786) = -\Lambda^\nu_{[1..4][5][6..8]}\Lambda^\nu_{[1..3][4]}\Lambda^\nu_{[1][2..3]}\Lambda^\nu_{[2][3]}\Lambda^\nu_{[6][7..8]}\Lambda^\nu_{[7][8]}Q^\nu_{[1..3]}Q^\nu_{[7..8]}.$$

As a general example we take $g = w_J$ where $J = \{j_1 < \cdots < j_{l-1}\}$ is an arbitrary subset of $\{1,2,\ldots,n-1\}$. Here $n(g) = l$ and $g' = id$, so by one application of our algorithm we obtain

$$\Lambda^\nu_{[1..n]}(w_J) = (-1)^{n-l}\Lambda^\nu_{J_1J_2\cdots J_l}\Lambda^\nu_{J_1}(id)\Lambda^\nu_{J_2}(id)\cdots\Lambda^\nu_{J_l}(id)$$

where $J_1 = [1..j_1], J_2 = [j_1+1..j_2], \ldots, J_l = [j_{l-1}+1..n]$.In particular for $n = 8$, $J = \{4\}$ we obtain

$$\begin{aligned}\Lambda^\nu_{[1..8]}(43218765) &= (-1)^{8-2}\Lambda^\nu_{[1..4][5..8]}\Lambda^\nu_{[1..4]}(1234)\Lambda^\nu_{[5..8]}(5678)\\ &= \frac{1}{\square^\nu_{[1..8]}}\Lambda^\nu_{[1][2][3][4]}\Lambda^\nu_{[5][6][7][8]}\end{aligned}$$

In Zagier's case, when all $q_{ij} = q$, we would then have (c.f. Examples to *Cor.* 2.2.13)

$$\Lambda^\nu_{[1..8]}(43218765) = \frac{1}{1-q^{7\cdot 8}}\frac{(1+2q^2+q^4+2q^6+q^8)^2}{(1-q^{1\cdot 2})^2(1-q^{2\cdot 3})^2(1-q^{3\cdot 4})^2}I$$

But, the denominator $D_8$ of this expression does not divide Zagier's $\triangle_8 = (1-q^{2\cdot 1})(1-q^{3\cdot 2})(1-q^{4\cdot 3})(1-q^{5\cdot 4})(1-q^{6\cdot 5})(1-q^{7\cdot 6})(1-q^{8\cdot 7})$. Namely, $\triangle_8/D_8 = (1-q^{4\cdot 5})(1-q^{5\cdot 6})(1-q^{6\cdot 7})/(1-q^{1\cdot 2})(1-q^{2\cdot 3})(1-q^{3\cdot 4})$ is not a polynomial due to the factor $1-q^2+q^4$ in the denominator. This computation shows that the original Zagier's conjecture (c.f. *Remark* 2.2.9) fails for $n=8$.

Now we return back to our algorithm. We shall show now that it is somewhat better to combine two steps of our algorithm into one step. This can be observed already in our illustrative example ($g = 41325786$) where after the second step the "unrelated factors" $Q^\nu_{\{1,2\}}$ and $Q^\nu_{\{1,3\}}$ from the first step were completed, with the factor $Q^\nu_{\{2,3\}}$, into a "nicer" term $Q^\nu_{[1..3]}$ having a contiguous indexing set. Fortunately this holds in general, but first we need more notations to state the results. To each permutation $g \in S_n$ we can associate a sequence of permutations $g, g', g'', \ldots$, where $g^{(k+1)}$ is obtained from $g^{(k)}$ by reversing all minimal Young factors in $g^{(k)}$ i.e $g' = gw_{J(g)}, g'' = gw_{J(g')}, \ldots, g^{(k+1)} = (g^{(k)})' = g^{(k)}w_{J(g^{(k)})}, \ldots$. We shall call this sequence a *Young sequence* of $g$. Furthermore, we call $g$ *tree-like* if $g^{(k)} = id$ for some $k$, and by *depth of $g$* we call the minimal such $k$. Besides the notation $\Lambda^\nu_{\sigma(g)} = \Lambda^\nu_{J_1(g)J_2(g)\cdots J_{n(g)}(g)}$, where $\sigma(g) = J_1(g)\cdots J_{n(g)}(g)$ is the subdivision of $\{1,2,\ldots,n\}$ associated to the minimal Young subgroup $S_{J(g)}$ containing $g$ we need a relative version $\Lambda^\nu_{\sigma(g'):\sigma(g)}$ which we define by

$$\Lambda^\nu_{\sigma(g'):\sigma(g)} := \Lambda^\nu_{\sigma(g'|J_1(g))}\Lambda^\nu_{\sigma(g'|J_2(g))}\cdots\Lambda^\nu_{\sigma(g'|J_{n(g)}(g))}$$

For example when $g = 41325786 (\Rightarrow g' = 23145687)$, $J_1(g) = [1..4], J_2(g) = [5], J_3(g) = [6..8]$, we have $\Lambda^\nu_{\sigma(g'):\sigma(g)} = \Lambda^\nu_{[123][4]}\Lambda^\nu_{[5]}\Lambda^\nu_{[6][7..8]}$. Also, besides the notation, for $T \subseteq \{1,2,\ldots,n\}, Q^\nu_T = \prod_{a,b\in T, a\neq b} Q^\nu_{a,b}$ (introduced in 1.8), we define for any subdivision $\sigma = J_1J_2\cdots J_l$ of $\{1,2,\ldots,n\}$:

$$Q^\nu_\sigma := Q^\nu_{J_1}Q^\nu_{J_2}\cdots Q^\nu_{J_l}$$

For example: $Q^\nu_{[1..3][4][5][6][7..8]} = Q^\nu_{[1..3]}Q^\nu_{[4]}Q^\nu_{[5]}Q^\nu_{[6]}Q^\nu_{[7..8]} = Q^\nu_{[1..3]}Q^\nu_{[7..8]}$.

**Proposition 2.2.16. [Fast algorithm for $\Lambda^\nu(g)$, $\nu$ generic, $|\nu| = n$].** *With the notations above we have*

$$\Lambda^\nu_{[1..n]}(g) = (-1)^{n(g)+n(g')}\Lambda^\nu_{\sigma(g)}\Lambda^\nu_{\sigma(g'):\sigma(g)}Q^\nu_{\sigma(g')}\Lambda^\nu_{J(g')}(g'')$$

*($n(g)$ = the number of minimal Young factors of $g$)*

**Proof.** By applying twice the algorithm in *Proposition* 2.2.15. □

Now we shall state our principal result concerning the inversion of matrices $A^{(\nu)}$ of the sesquilinear form $(\ ,\ )_{\mathbf{q}}$, defined in 1.3, on the generic weight space $\mathbf{f}_\nu, |\nu| = n$.

**Theorem 2.2.17. [INVERSE MATRIX ENTRIES].** *Let $\nu$ be a generic weight, $|\nu| = n$. For the coefficients $\Lambda^\nu(g)$ in the expansion*

$$[A^{(\nu)}]^{-1} = \sum_{g\in S_n}\Lambda^\nu(g)\hat{R}_\nu(g)$$

*we have, with the notations above, the following formulas:*
*i) If $g \in S_n$ is a tree-like permutation of depth d, then*

$$\Lambda^\nu(g) = (-1)^N \Lambda^\nu_{\sigma(g)} \Lambda^\nu_{\sigma(g'):\sigma(g)} \Lambda^\nu_{\sigma(g''):\sigma(g')} \cdots \Lambda^\nu_{\sigma(g^{(d)}):\sigma(g^{(d-1)})} Q^\nu_{\sigma(g')} Q^\nu_{\sigma(g''')} \cdots Q^\nu_{\sigma(g^{(d')})}$$

*where* $N = N(g) := \sum_{k=0}^{d} \sum_{I \in \sigma(g^{(k)})} (Card\ I - 1),\ d' = 2\lfloor (d-1)/2 \rfloor + 1$
*ii) If $g \in S_n$ is not tree-like, then $\Lambda^\nu(g) = 0$.*

**Proof.** *i*) follows by iterating our fast algorithm (of *Proposition* 2.2.16). *ii*) If $g$ is not tree-like then in the Young sequence of $g$ we encounter some Young factor which together with its reverse is not splittable, but then the corresponding $\Lambda^\nu_{[..]}$(the factor) $= 0$ (c.f. *Proposition* 2.2.11), hence $\Lambda^\nu(g) = 0$. □

Now we give explicit formulas for the inverses of $A^{123}$ and $A^{1234}$: We have

$$\begin{aligned}
[A^{123}]^{-1} &= \frac{1}{\square_{[1..3]}}\{\frac{I - Q_{[1..2]}Q_{[2..3]}}{\square_{[1..2]}\square_{[2..3]}}(\hat{R}(123) + \hat{R}(321)) - \\
&- \frac{1}{\square_{[1..2]}}(\hat{R}(213) + Q_{[1..2]}\hat{R}(312)) - \frac{1}{\square_{[2..3]}}(\hat{R}(132) + Q_{2..3]}\hat{R}(231))\}. \\
[A^{1234}]^{-1} &= \Lambda^{1234}(id)\hat{R}(1234) + \frac{1}{\square_{1234}}\{-\frac{I - Q_{123}Q_{34}}{\square_{12}\square_{123}\square_{34}}\hat{R}(2134) \\
&- \frac{I - Q_{123}Q_{234}}{\square_{23}\square_{123}\square_{234}}\hat{R}(1324) - \frac{I - Q_{12}Q_{234}}{\square_{12}\square_{34}\square_{234}}\hat{R}(1243) + \frac{1}{\square_{12}\square_{34}}\hat{R}(2143) + \\
&+ \frac{I - Q_{12}Q_{23}}{\square_{12}\square_{23}\square_{123}}\hat{R}(3214) - \frac{Q_{12}}{\square_{12}\square_{123}}\hat{R}(3124) - \frac{Q_{23}}{\square_{23}\square_{123}}\hat{R}(2314) \\
&+ \frac{I - Q_{23}Q_{34}}{\square_{23}\square_{34}\square_{234}}\hat{R}(1432) - \frac{Q_{23}}{\square_{23}\square_{234}}\hat{R}(1423) - \frac{Q_{34}}{\square_{34}\square_{234}}\hat{R}(1342)\} \\
&+ \text{(eleven terms obtained by multiplying with } -\hat{R}(4321)).
\end{aligned}$$

where $\Lambda^{123}(id)$ and $\Lambda^{1234}(id)$ are given as examples illustrating *Corollary* 2.2.13.(Here we abbreviated $Q_{[1..2]}, Q_{[2..4]}$ to $Q_{12}$ (not to be confused with $Q_{1,2}$), $Q_{234}$ etc.). Note that $A^{1234}$ is a $24 \times 24$ symbolic matrix so the inversion of such a matrix by standard methods on a computer is almost impossible (the output may contain huge number of pages of messy expressions!).

**Remark 2.2.18..** *By using our reduction to the generic case formula 1.7.1* $[A^{(\nu)}]^{-1}_{\mathbf{ij}} = \sum_{h \in H} [\tilde{A}^{(\tilde{\nu})}]^{-1}_{\tilde{\mathbf{i}}, h\tilde{\mathbf{j}}}$ *we can also write formulas for the inverse matrix entries in the case of degenerate weights $\nu$. E.g. for the inverse of $A^{113}$ (see* Example *1.6.4) one gets*

$$[A^{113}]^{-1} = \frac{1}{\Delta}\begin{pmatrix} 1 & -(1+q_{11})q_{13} & q_{11}q_{13}^2 \\ -q_{31}(1+q_{11}) & (1+q_{11})(1+q_{13}q_{31}) & -(1+q_{11})q_{13} \\ q_{13}^2 q_{11} & -q_{31}(1+q_{11}) & 1 \end{pmatrix}$$

*where* $\Delta = (1+q_{11})(1-q_{13}q_{31})(1-q_{11}q_{13}q_{31})$.

# 3. Applications

## 3.1. Quantum bilinear form of the discriminant arrangement of hyperplanes

Here we briefly recall the definition of the quantum bilinear form in case of the configuration $\mathcal{A}_n$ of diagonal hyperplanes $H_{ij} = H_{ij}^n : x_i = x_j, 1 \le i < j \le n$ in $\mathbf{R}^n$ (for general case see [Var]). This arrangement $\mathcal{A}_n$ is also called the *discriminant arrangement* of hyperplanes in $\mathbf{R}^n$. The *domains* of $\mathcal{A}_n$ (i.e connected components of the complement of the union of hyperplanes in $\mathcal{A}_n$) are clearly of the form

$$P_\pi = \{x \in \mathbf{R}^n | x_{\pi(1)} < x_{\pi(2)} < \cdots < x_{\pi(n)}\}, \pi \in S_n$$

Let $a(H_{ij}^n) = q_{ij}$ be the *weight* of the hyperplane $H_{ij} \in \mathcal{A}_n$, where $q_{ij}$ are given real numbers, $1 \le i < j \le n$. Then the *quantum bilinear form* $B_n$ of $\mathcal{A}_n$ is defined on the free vector space $M_n = M_{\mathcal{A}_n}$, generated by the domains of $\mathcal{A}_n$, by

$$B_n(P_\pi, P_\tau) = \prod a(H)$$

where the product is taken over all hyperplanes $H \in \mathcal{A}_n$ which separate $P_\pi$ from $P_\tau$.

**Proposition 3.1.1..** *We have*

$$B_n(P_\pi, P_\tau) = \prod_{(a,b)\in I(\pi^{-1})\triangle I(\tau^{-1})} q_{ab}$$

*where $I(\sigma) = \{(a,b) | a < b, \sigma(a) > \sigma(b)\}$ denotes the set of inversions of $\sigma \in S_n$ and $X \triangle Y = (X \setminus Y) \bigcup (Y \setminus X)$ denotes the symmetric difference of sets $X$ and $Y$.*

**Corollary 3.1.2..** *The matrix of the quantum bilinear form $B_n$ of the discriminant arrangements $\mathcal{A}_n = \{H_{ij}\}$ of hyperplanes in $\mathbf{R}^n$ coincides with the matrix $A^{12\cdots n} = A^{12\cdots n}(\mathbf{q})$ of the form $(\ ,\ )_{\mathbf{q}}$ (defined in 1.3), restricted to the generic weight space $\mathbf{f}_\nu$, where $I = \{1, 2, \ldots, n\}, \nu_1 = \nu_2 = \cdots = \nu_n = 1$ and where $\mathbf{q} = \{q_{ij} \in \mathbf{R}, 1 \le i, j \le n, q_{ij} = q_{ji}\}, q_{ij} =$ the weight of $H_{ij}$ for $1 \le i < j \le n$.* This Corollary enables us to translate all our results concerning matrices $A^\nu, \nu =$ generic, $|\nu| = n$ into results about the quantum bilinear form $B_n$. As an example we shall reinterpret our determinantal formula given in *Theorem* 1.9.2.

**Theorem 3.1.3..** *The determinant of the quantum bilinear form $B_n$ of the discriminant arrangement $\mathcal{A}_n$ is given by the formula*

$$\det B_n = \prod_{L \in \mathcal{E}'(\mathcal{A}_n)} (1 - a(L)^2)^{l(L)}$$

*where for* $L = \{x_{i_1} = x_{i_2} = \cdots = x_{i_k}\} \in \mathcal{A}_{n,k} \subset \mathcal{E}'(\mathcal{A}_n)$ *we have*

$$a(L) = \prod_{1 \le a < b \le k} q_{i_a i_b},\ l(L) = (k-2)!(n-k+1)!$$

Note that our formula for $\det B_n$ is more explicit then Varchenko's formula, and in particular we conclude that the multiplicity $l(L) = 0$ for all $L \in \mathcal{E}(\mathcal{A}_n) \setminus \mathcal{E}'(\mathcal{A}_n)$. Note added in proof. After receiving a new book by Varchenko [Multidimensional hypergeometric functions and representation theory of Lie algebras and quantum groups, World Scientific (1995)] we found in it a result ( *Theorem* 3.11.2.—proved by different techniques) equivalent to our *Theorem* 3.1.3, but there seems to be no results equivalent to our inversion formulas applied to discriminant arrangements.

## 3.2. Quantum groups

We shall use the notations from [SVa]. Our *Theorem* 1.9.2 implies the following

**Theorem 3.2.1..** *The determinant of the contravariant form* $S$ *on the weight space* $(U_q\mathbf{n}_-)_{(1,1,\ldots,1)}$ *is given by the following formula*

$$\begin{aligned}
\det S|_{(U_q\mathbf{n}_-)_{(1,1\ldots,1)}} &= \\
&= q^{-\frac{n!}{4}\sum_{1\le k<l\le n} b_{kl}} \prod_{m=2}^{n} \prod_{1\le i_1<\cdots<i_m\le n} (1 - q^{\sum_{1\le k<l\le m} b_{i_k i_l}})^{(m-2)!(n-m+1)!} \\
&= \prod_{m=2}^{n} \prod_{1\le i_1<\cdots<i_m\le n} (q^{-\frac{1}{2}\sum_{1\le k<l\le m} b_{i_k i_l}} - q^{\frac{1}{2}\sum_{1\le k<l\le m} b_{i_k i_l}})^{(m-2)!(n-m+1)!}
\end{aligned}$$

**Proof.** By factoring out from the matrix $S(f_I, f_J)$ the factor $q^{-\frac{1}{4}\sum_{1\le k<l\le n} b_{kl}}$ we get a matrix which (up to permutation of rows and columns) coincides with the matrix $A^{12\cdots n}(\mathbf{q})$, where $\mathbf{q} = \{q_{ij}\}$, $q_{ij} := q^{-\frac{1}{2}b_{ij}}$. Then we apply *Theorem* 1.9.2 and the result follows. □

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