Arrow-Debreu general equilibrium model*

Ivo Gjenero†

Abstract. *Here we formulate two general theorems of Arrow-Debreu on 1) the compactness of the attainable states of a general model of economy and 2) the existence of a competitive equilibrium, under fairly general assumptions.*

Key words: *economy, attainable state, competitive equilibrium*

Sažetak. **Arrow-Debreuov opći model ravnoteže.** *Formuliraju se dva vrlo općenita teorema Arrow-a i Debreu-a o 1) kompaktnosti dosiživih stanja jednog općenitog modela ekonomije i 2) postojanju konkurentske ravnoteže, uz vrlo općenite pretpostavke.*

Ključne riječi: *ekonomija, dostiživo stanje, konkurentska ravnoteža*

Let R^n denote the commodity space. For $i = 1, \ldots, m$, let $X_i \subset R^n$ denote the i - th consumer's consumption set, $\mathbf{e}_i \in R^n$ its private endowment and $\succeq_i$ its preference relation on X_i. For $h = 1, \ldots, k$ let Y_h denote the j-th producer's production set.

Let $X = \sum_{i=1}^m X_i$ be the total consumption set, $\mathbf{e} = \sum_{i=1}^m \mathbf{e}_i$ the total endowment and $Y = \sum_{h=1}^k Y_h$ the total production set.
Let θ_{ih} denote the share of the i-th consumer in the profits of h-th producer. Numbers θ_{ih} are nonnegative and for each h we have $\sum_{i=1}^m \theta_{ih} = 1$.

An economy is then described by tuple

$$\mathcal{E} = ((X_i, \mathbf{e}_i, \succeq_i), (\theta_{ih}), (Y_h)).$$

An attainable state of the economy is an $(m + k)$ tuple

$$((\mathbf{x}_i), (\mathbf{y}_h)) \in \prod_{i=1}^m X_i \times \prod_{h=1}^k Y_j,$$

*The lecture presented at the Mathematical Colloquium in Osijek organized by Croatian Mathematical Society - Division Osijek, November 17, 1995.

†Faculty of Economics, Department of Mathematics, Kennedyjev trg 6, HR-10 000 Zagreb, Croatia, e-mail: `gjenero@oliver.efzg.hr`

satisfying

$$\sum_{i=1}^{m} \mathbf{x}_i \leqq \sum_{h=1}^{k} \mathbf{y}_h - \mathbf{e}.$$

Let M denote the following set

$$M = \left\{ ((\mathbf{x}_i), (\mathbf{y}_h)) \in R^{n(m+k)} : \sum_{i=1}^{m} \mathbf{x}_i - \sum_{h=1}^{k} \mathbf{y}_j - \mathbf{e} \leqq \mathbf{0} \right\}$$

and let F denote the set of all attainable states. Then

$$F = (\prod X_i \times \prod Y_h) \bigcap M.$$

Let $\hat{X}_i \subseteq X_i$ be the projection of F on X_i and let $\hat{Y}_h \subseteq Y_h$ be the projection of F to Y_h.

Theorem 1 [Debreu]. *Let the economy* $\mathcal{E} = ((X_i, \mathbf{e}_i, \succeq_i), (Y_h), (\theta_{ih}))$ *satisfy:*

(D1) For each $i = 1, \ldots, m$,
X_i *is closed, convex and bounded from bellow; and* $\mathbf{e}_i \in X_i$.

(D2) For each $h = 1, \ldots, k$,
Y_h *is closed, convex and* $\mathbf{0} \in Y_h$.

(D3) $Y \bigcap (-Y) = \{\mathbf{0}\}$.

(D4) $-R^n_+ \subset Y$.

(D5) There exists some $\overline{\mathbf{x}}_i \in X_i$, *such that* $\overline{\mathbf{x}}_i < \mathbf{e}_i$.

Then the set F *of attainable states is compact and nonempty,* $\overline{\mathbf{x}}_i \in \hat{X}_i$ $(i = 1, \ldots, m)$ *and* $\mathbf{0} \in \hat{Y}_h$ $(h = 1, \ldots, k)$.

For the proof (Debreu, 1959, p.22) it suffices to show that $\mathbf{A}F = \{\mathbf{0}\}$, where $\mathbf{A}F$ is a recessive conus of F. (Rockafellar, 1970, §8.) For another proof see Smale (1982).

Definition 1. *A competitive equilibrium of economy*

$$\mathcal{E} = ((X_i, \mathbf{e}_i, \succeq_i), (Y_h), (\theta_{ih}))$$

is an $(m + k = 1)$ *tuple of vectors from* R^n

$$(\mathbf{p}^*, (\mathbf{x}^*_i), (\mathbf{y}^*_h),$$

where $\mathbf{p}^* \in P = \{\mathbf{p} \in R^n_+ : \sum_{j=1}^{k} p_j = 1\}$ *and* $((\mathbf{x}^*_i), (\mathbf{y}^*_h))$ *is an attainable state of economy* $\mathcal{E}$ *satisfying the following three conditions:*

1. *For each* $h = 1, \ldots, k$,
   $\langle \mathbf{p}^*, \mathbf{y}_h^* \rangle \geq \langle \mathbf{p}^*, \mathbf{y}_h \rangle$ *for all* $\mathbf{y}_h \in Y_h$.

2. *For each* $i = 1, \ldots, m$,
   $\mathbf{x}_i^* \succeq_i \mathbf{x}_i$ *for all* $\mathbf{x}_i \in \beta_i(\mathbf{p}^*)$,
   where
   $\beta_i(\mathbf{p}^*) = \{\mathbf{x}_i \in X_i : \langle \mathbf{p}^*, \mathbf{x}_i \rangle \leq \langle \mathbf{p}^*, \mathbf{e}_i \rangle + \sum_{h=1}^{k} \theta_{ih} \langle \mathbf{p}^*, \mathbf{y}_h^* \rangle \}$.

3. $\sum_{i=1}^{m} \mathbf{x}_i^* - \sum_{h=1}^{k} \mathbf{y}_h^* - \mathbf{e} = \mathbf{0}$.

The vector $\mathbf{p}^*$ *is called a vector of competitive prices.*

Theorem 2 [Debreu]. *Let economy* $\mathcal{E} = ((X_i, \mathbf{e}_i, \succeq_i), (Y_h), (\theta_{ih}))$ *satisfy:*

For each $i = 1, \ldots, m$,

(R1) X_i *is closed, convex and bounded from bellow;* $\mathbf{e}_i \in X_i$.

(R2) There exists some $\overline{\mathbf{x}}_i \in X_i$, *such that* $\overline{\mathbf{x}}_i < \mathbf{e}_i$.

(R3) For all $\mathbf{x}_i \in \hat{X}_i$ *exists* $\mathbf{x}_i' \in X_i$ *such that* $\mathbf{x}_i' \succ \mathbf{x}_i$.

(R4) The sets $\{(\mathbf{x} \in X_i : \mathbf{x} \succeq_i \mathbf{x}'\}$ *and* $\{\mathbf{x} : \mathbf{x}'' \succeq_i \mathbf{x}\}$ *are closed for every* $\mathbf{x}', \mathbf{x}'' \in X_i$.

(R5) If $\mathbf{x}$ *and* $\mathbf{x}'$ *are two points in* X_i *such that* $mx \succ_i \mathbf{x}'$ *and* $\lambda \in]0,1]$, *then* $\mathbf{x} \succ_i (1-\lambda)\mathbf{x} + \lambda \mathbf{x}'$.

For each $h = 1, \ldots, k$,

(R6) Y_h *is closed, convex and* $\mathbf{0} \in Y_h$.

(R7) $Y \bigcap (-Y) = \{\mathbf{0}\}$.

(R8) $-R_+^n \subset Y$.

Then there exists a competitive equilibrium of the economy $\mathcal{E}$. For the proof see Debreu (1959, 1982). The proof uses the conclusions of *Theorem 1* and Kakutani fixed point theorem for point to set mapping.

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