Math. Commun. **23**(2018), 161–180

Approximation of functions by bivariate q-Stancu-Durrmeyer type operators

Trapti Neer[1], Ana Maria Acu[2,*] and Purshottam Agrawal[1]

[1] *Department of Mathematics, Indian Institute of Technology Roorkee, Roorkee 247 667, India*
[2] *Department of Mathematics and Informatics, Lucian Blaga University of Sibiu, Str. Dr. I. Ratiu, No. 5-7, RO-550 012 Sibiu, Romania*

Received May 25, 2017; accepted July 18, 2017

Abstract. This paper is a continuation of our work in [24], wherein we studied some approximation properties of the Stancu-Durrmeyer operators based on q-integers. Here, we construct a bivariate generalization of these operators and study the rate of convergence by means of the complete modulus of continuity and the partial moduli of continuity and the degree of approximation with the aid of Peetre's K functional. Subsequently, we define the GBS (Generalized Boolean Sum) operators of Stancu-Durrmeyer type and give the rate of approximation by means of the mixed modulus of smoothness and the Lipschitz class of Bögel-continuous functions.

AMS subject classifications: 41A25, 41A36

Key words: Complete modulus of continuity, partial moduli of continuity, B-continuous functions and B-differentiable functions

1. Introduction

In the last decade, the application of q-calculus in the field of approximation theory has been an active research area. In 1987, Lupas [23] initiated the study of q-analogue of classical Bernstein polynomials. Later, Phillips [28] proposed another q-generalization of Bernstein polynomials and established the rate of convergence and Voronovskaja type asymptotic formula for these operators. Subsequently, researchers introduced q-analogues of several linear positive operators. For some significant contributions in this direction we refer to (cf. [1, 3, 16, 20, 21, 22, 25, 32, 36]).

In 2009, for any function $f \in C[0,1], q > 0, \alpha = \alpha(n) \geq 0$ $(\alpha(n) \to 0,\ as\, n \to \infty)$ and each $n \in \mathbb{N}$ Nowak [26] defined the q-analogue for the operators defined by Stancu [34] as

$$B_n^{q,\alpha}(f;x) = \sum_{k=0}^{n} v_{n,k}^{q,\alpha}(x) f\left(\frac{[k]_q}{[n]_q}\right), \quad x \in [0,1], \tag{1}$$

*Corresponding author. *Email addresses:* triptineeriitr@gmail.com (T. Neer), acuana77@yahoo.com (A. M. Acu), pna_iitr@yahoo.co.in (P. Agrawal)

where,

$$v_{n,k}^{q,\alpha}(x)=\begin{bmatrix}n\\k\end{bmatrix}_q\frac{\prod_{\nu=0}^{k-1}(x+\alpha[\nu]_q)\prod_{\mu=0}^{n-k-1}(1-q^\mu x+\alpha[\mu]_q)}{\prod_{\lambda=0}^{n-1}(1+\alpha[\lambda]_q)}.$$

Recently, Neer and Agrawal [24] introduced the Durrmeyer type integral modification for the operators (1) as

$$D_n^\alpha(f;q;x)=[n+1]_q\sum_{k=0}^{n}v_{n,k}^{q,\alpha}(x)\int_0^1 p_{n,k}^q(t)f(t)d_qt, \tag{2}$$

where

$$p_{n,k}^q(t)=\begin{bmatrix}n\\k\end{bmatrix}_q t^k(1-qt)_q^{n-k},$$

and investigated some approximation properties.

Now, we propose a two-dimensional case of the operators (2). Let $I=[0,1]$, $I^2=I\times I$ and $C(I^2)$ denote the class of all real valued continuous functions on I^2 endowed with the norm $||f||=\sup_{(x,y)\in I^2}|f(x,y)|$. Then, for $f\in C(I^2)$, the bivariate q-Stancu-Durrmeyer type operators of (2) is defined as

$$\begin{aligned}&D_{n_1,n_2}^{\alpha_{n_1},\alpha_{n_2}}(f;q_{n_1},q_{n_2},x,y)\\&=[n_1+1]_{q_{n_1}}[n_2+1]_{q_{n_2}}\sum_{k_1=0}^{n_1}\sum_{k_2=0}^{n_2}v_{n_1,k_1}^{q_{n_1},\alpha_{n_1}}(x)v_{n_2,k_2}^{q_{n_2},\alpha_{n_2}}(y)\\&\int_0^1\int_0^1 p_{n_1,k_1}^{q_{n_1}}(t)p_{n_2,k_2}^{q_{n_2}}(s)f(t,s)d_{q_{n_1}}t\,d_{q_{n_2}}s,\quad\text{for all } (x,y)\in I^2.\end{aligned}\tag{3}$$

We study the degree of approximation of these operators and the associated GBS operators.

2. Auxiliary results

Lemma 1 (see [24]). *For $D_n^\alpha(t^m;q;x)$, $m=0,1,2$, one has*

(i) $D_n^\alpha(1;q;x)=1$;

(ii) $D_n^\alpha(t;q;x)=\frac{1}{[n+2]_q}(1+q[n]_qx)$;

(iii) $D_n^\alpha(t^2;q;x)=\frac{1}{[n+2]_q[n+3]_q}\left\{[2]_q+q(1+2q)[n]_qx+\frac{q^3[n]_q^2}{1+\alpha}\left(x(x+\alpha)+\frac{x(1-x)}{[n]_q}\right)\right\}$.

Consequently,

(a) $D_n^\alpha(t-x;q;x)=\frac{1}{[n+2]_q}+\frac{1}{[n+2]_q}(q[n]_q-[n+2]_q)x$;

(b) $D_n^\alpha((t-x)^2;q;x)=\frac{[2]_q}{[n+2]_q[n+3]_q}+\left\{\frac{1}{[n+2]_q[n+3]_q}\left(q(1+2q)[n]_q+\frac{q^3[n]_q([n]_q\alpha+1)}{(1+\alpha)}\right)-\frac{2}{[n+2]_q}\right\}x+\left\{\frac{q^3[n]_q([n]_q-1)}{[n+2]_q[n+3]_q(1+\alpha)}-\frac{2q[n]_q}{[n+2]_q}+1\right\}x^2$.

Lemma 2. *For* $D_{n_1,n_2}^{\alpha_{n_1},\alpha_{n_2}}(e_{ij};q_1,q_2,x,y),\ e_{ij}=x^iy^j,\ i,j\in\mathbb{N}\cup\{0\},\ x,y\in[0,1],$ *we have*

(i) $D_{n_1,n_2}^{\alpha_{n_1},\alpha_{n_2}}(e_{00};q_{n_1},q_{n_2},x,y)=1;$

(ii) $D_{n_1,n_2}^{\alpha_{n_1},\alpha_{n_2}}(e_{10};q_{n_1},q_{n_2},x,y)=\frac{1}{[n_1+2]_{q_{n_1}}}(1+q_{n_1}[n_1]_{q_{n_1}}x);$

(iii) $D_{n_1,n_2}^{\alpha_{n_1},\alpha_{n_2}}(e_{01};q_{n_1},q_{n_2},x,y)=\frac{1}{[n_2+2]_{q_{n_2}}}(1+q_{n_2}[n_2]_{q_{n_2}}y);$

(iv) $D_{n_1,n_2}^{\alpha_{n_1},\alpha_{n_2}}(e_{11};q_{n_1},q_{n_2},x,y)=\frac{1}{[n_1+2]_{q_{n_1}}}(1+q_{n_1}[n_1]_{q_{n_1}}x)\frac{1}{[n_2+2]_{q_{n_2}}}(1+q_{n_2}[n_2]_{q_{n_2}}y);$

(v) $D_{n_1,n_2}^{\alpha_{n_1},\alpha_{n_2}}(e_{20};q_{n_1},q_{n_2},x,y)=\frac{1}{[n_1+2]_{q_{n_1}}[n_1+3]_{q_{n_1}}}\Bigg\{[2]_{q_{n_1}}+q_{n_1}(1+2q_{n_1})[n_1]_{q_{n_1}}x$
$+\frac{q_{n_1}^3[n_1]_{q_{n_1}}^2}{1+\alpha_{n_1}}\left(x(x+\alpha_{n_1})+\frac{x(1-x)}{[n_1]_{q_{n_1}}}\right)\Bigg\};$

(vi) $D_{n_1,n_2}^{\alpha_{n_1},\alpha_{n_2}}(e_{02};q_{n_1},q_{n_2},x,y)=\frac{1}{[n_2+2]_{q_{n_2}}[n_2+3]_{q_{n_2}}}\Bigg\{[2]_{q_{n_2}}+q_{n_2}(1+2q_{n_2})[n_2]_{q_{n_2}}y$
$+\frac{q_{n_2}^3[n_2]_{q_{n_2}}^2}{1+\alpha_{n_2}}\left(y(y+\alpha_{n_2})+\frac{y(1-y)}{[n_2]_{q_{n_2}}}\right)\Bigg\};$

(vii) $D_{n_1,n_2}^{\alpha_{n_1},\alpha_{n_2}}(e_{20}+e_{02};q_1,q_2,x,y)=\frac{1}{[n_1+2]_{q_{n_1}}[n_1+3]_{q_{n_1}}}\Bigg\{[2]_{q_{n_1}}+q_{n_1}(1+2q_{n_1})[n_1]_{q_{n_1}}x$
$+\frac{q_{n_1}^3[n_1]_{q_{n_1}}^2}{1+\alpha_{n_1}}\left(x(x+\alpha_{n_1})+\frac{x(1-x)}{[n_1]_{q_{n_1}}}\right)\Bigg\}\frac{1}{[n_2+2]_{q_{n_2}}[n_2+3]_{q_{n_2}}}\Bigg\{[2]_{q_{n_2}}+q_{n_2}(1+2q_{n_2})[n_2]_{q_{n_2}}y$
$+\frac{q_{n_2}^3[n_2]_{q_{n_2}}^2}{1+\alpha_{n_2}}\left(y(y+\alpha_{n_2})+\frac{y(1-y)}{[n_2]_{q_{n_2}}}\right)\Bigg\}.$

Lemma 3. *(i)* $D_{n_1,n_2}^{\alpha_{n_1},\alpha_{n_2}}(t-x;q_{n_1},q_{n_2},x,y)=\frac{1}{[n_1+2]_{q_{n_1}}}+\frac{1}{[n_1+2]_{q_{n_1}}}(q_{n_1}[n_1]_{q_{n_1}}$
$-[n+2]_{q_{n_1}})x;$

(ii) $D_{n_1,n_2}^{\alpha_{n_1},\alpha_{n_2}}(s-y;q_{n_1},q_{n_2},x,y)=\frac{1}{[n_2+2]_{q_{n_2}}}+\frac{1}{[n_2+2]_{q_{n_2}}}(q_{n_2}[n_2]_{q_{n_2}}-[n+2]_{q_{n_2}})y;$

(iii) $D_{n_1,n_2}^{\alpha_{n_1},\alpha_{n_2}}((t-x)^2;q_{n_1},q_{n_2},x,y)=\frac{[2]_{q_{n_1}}}{[n_1+2]_{q_{n_1}}[n_1+3]_{q_{n_1}}}+\Bigg\{\frac{1}{[n_1+2]_{q_{n_1}}[n_1+3]_{q_{n_1}}}\Bigg(q_{n_1}(1$
$+2q_{n_1})[n_1]_{q_{n_1}}+\frac{q_{n_1}^3[n_1]_{q_{n_1}}([n_1]_{q_{n_1}}\alpha_{n_1}+1)}{(1+\alpha_{n_1})}\Bigg)-\frac{2}{[n_1+2]_{q_{n_1}}}\Bigg\}x+\Bigg\{\frac{q_{n_1}^3[n_1]_{q_{n_1}}([n_1]_{q_{n_1}}-1)}{[n_1+2]_{q_{n_1}}[n_1+3]_{q_{n_1}}(1+\alpha_{n_1})}$
$-\frac{2q_{n_1}[n_1]_{q_{n_1}}}{[n_1+2]_{q_{n_1}}}+1\Bigg\}x^2;$

(iv) $D_{n_1,n_2}^{\alpha_{n_1},\alpha_{n_2}}((s-y)^2;q_{n_1},q_{n_2},x,y)=\Bigg\{\frac{1}{[n_2+2]_{q_{n_2}}[n_2+3]_{q_{n_2}}}\Bigg(q_{n_2}(1+2q_{n_2})[n_2]_{q_{n_2}}$
$+\frac{q_{n_2}^3[n_2]_{q_{n_2}}([n_2]_{q_{n_2}}\alpha_{n_2}+1)}{(1+\alpha_{n_2})}\Bigg)-\frac{2}{[n_2+2]_{q_{n_2}}}\Bigg\}y+\Bigg\{\frac{q_{n_2}^3[n_2]_{q_{n_2}}([n_2]_{q_{n_2}}-1)}{[n_2+2]_{q_{n_2}}[n_2+3]_{q_{n_2}}(1+\alpha_{n_2})}$
$-\frac{2q_{n_2}[n_2]_{q_{n_2}}}{[n_2+2]_{q_{n_2}}}+1\Bigg\}y^2+\frac{[2]_{q_{n_2}}}{[n_2+2]_{q_{n_2}}[n_2+3]_{q_{n_2}}}.$

3. Main results

In what follows, let $0 < q_{n_i} < 1$ and $\alpha_{n_i} \geq 0$ be sequences such that $\lim\limits_{n_i \to \infty} q_{n_i} = 1$, $\lim\limits_{n_i \to \infty} q_{n_i}^{n_i} = a_i (0 \leq a_i < 1)$ and $\lim\limits_{n_i \to \infty} \alpha_{n_i} = 0,\ i = 1, 2$. Also, assume that

$$\begin{aligned} \delta_{n_1,q_{n_1}}^{\alpha_{n_1}}(x) &= \sqrt{D_{n_1}^{\alpha_{n_1}}((t-x)^2; q_{n_1}, x)}, \\ \delta_{n_2,q_{n_2}}^{\alpha_{n_2}}(y) &= \sqrt{D_{n_2}^{\alpha_{n_2}}((s-y)^2; q_{n_2}, y)}. \end{aligned} \tag{4}$$

Let $e_{i,j} = t^i s^j,\ i, j \in \mathbb{N} \cup \{0\}$.

Theorem 1 (see [35])**.** *Let I_1 and I_2 be two compact intervals of the real line. Let T_{n_1,n_2} with $(n_1, n_2) \in \mathbb{N} \times \mathbb{N}$ be linear positive operators on $C(I_1 \times I_2)$ such that*

$$\lim_{n_1,n_2 \to \infty} T_{n_1,n_2}(e_{ij}) = e_{ij}, \quad 0 \leq i + j \leq 1$$

and

$$\lim_{n_1,n_2 \to \infty} T_{n_1,n_2}(e_{20} + e_{02}) = e_{20} + e_{02}$$

uniformly on $I_1 \times I_2$ Then $(T_{n_1,n_2} f)$ converges uniformly to f on $I_1 \times I_2$, for any $f \in C(I_1 \times I_2)$.

As a consequence of the above theorem and Lemma 2, we have the following theorem.

Theorem 2. *The sequence of bivariate q-Stancu-Durrmeyer operators $D_{n_1,n_2}^{\alpha_{n_1},\alpha_{n_2}}(f; q_{n_1}, q_{n_2}, x, y)$ converges uniformly to $f(x, y)$, for any $f \in C(I^2)$.*

Now we give some numerical results which show the rate of convergence of the operator $D_{n_1,n_2}^{\alpha_{n_1},\alpha_{n_2}}$ to certain functions using Maple algorithms.

Example 1. *Let $f : \mathbb{R}^2 \to \mathbb{R}$, $f(x, y) = x^2y^2 + x^3y - 2x^4$. The convergence of the operator $D_{n_1,n_2}^{\alpha_{n_1},\alpha_{n_2}}$ to the function f is illustrated in Figure 1 for $n_1 = n_2 = 10$, $q_{n_1} = q_{n_2} = 0.3, \alpha_{n_1} = \alpha_{n_2} = 0.2$ and $n_1 = n_2 = 100, q_{n_1} = q_{n_2} = 0.9$, $\alpha_{n_1} = \alpha_{n_2} = 0.2$, respectively. We note that as the values of n_1 and n_2 increase and the values of q_{n_1} and q_{n_2} are close to 1, the error in the approximation of the function by the operator becomes smaller.*

Also, for $n_1 = n_2 = 100, q_{n_1} = q_{n_2} = 0.9$, the convergence of the operator $D_{n_1,n_2}^{\alpha_{n_1},\alpha_{n_2}}$ to the function f is illustrated in Figure 2 with respect to parameters α_{n_1} and α_{n_2}. It should be noted that as the values of α_{n_1} and α_{n_2} are close to 0, the error in the approximation of the function by the operator becomes smaller.

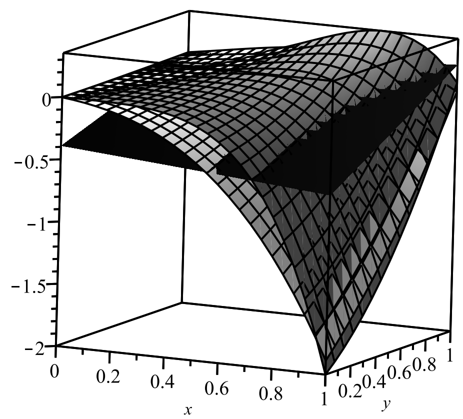

Figure 1: *The convergence of $D^{\alpha_{n_1},\alpha_{n_2}}_{n_1,n_2}(f;q_{n_1},q_{n_2},x,y)$ to $f(x,y)$ (grey f, white $D^{0.2,0.2}_{100,100}(f;0.9,0.9;x,y)$, black $D^{0.2,0.2}_{10,10}(f;0.3,0.3;x,y)$)*

Theorem 3. *Let $f \in C^1(I^2)$ and $(x,y) \in I^2$. Then we have*

$$|D^{\alpha_{n_1},\alpha_{n_2}}_{n_1,n_2}(f;q_{n_1},q_{n_2},x,y) - f(x,y)| \leq ||f'_x||\delta^{\alpha_{n_1}}_{n_1,q_{n_1}}(x) + ||f'_y||\delta^{\alpha_{n_2}}_{n_2,q_{n_2}}(y), \quad (5)$$

where $\delta^{\alpha_{n_1}}_{n_1,q_{n_1}}(x)$ and $\delta^{\alpha_{n_2}}_{n_2,q_{n_2}}(x)$ are defined in (4).

Proof. For a fixed point $(x,y) \in I^2$

$$f(t,s) - f(x,y) = \int_x^t f'_u(u,s)du + \int_y^s f'_v(x,v)dv. \quad (6)$$

Applying $D^{\alpha_{n_1},\alpha_{n_2}}_{n_1,n_2}$ to the above equation (6), we get

$$|D^{\alpha_{n_1},\alpha_{n_2}}_{n_1,n_2}(f;q_{n_1},q_{n_2},x,y) - f(x,y)| \leq D^{\alpha_{n_1},\alpha_{n_2}}_{n_1,n_2}\left(\left|\int_x^t |f'_u(u,s)|du\right|;q_{n_1},q_{n_2},x,y\right) + D^{\alpha_{n_1},\alpha_{n_2}}_{n_1,n_2}\left(\left|\int_y^s |f'_v(x,v)|dv\right|;q_{n_1},q_{n_2},x,y\right).$$

Since $\left|\int_x^t |f'_u(u,s)|du\right| \leq ||f'_x|||t-x|$ and $\left|\int_y^s |f'_v(x,v)|dv\right| \leq ||f'_y|||s-y|$, we have

$$|D^{\alpha_{n_1},\alpha_{n_2}}_{n_1,n_2}(f;q_{n_1},q_{n_2},x,y) - f(x,y)| \leq ||f'_x||D^{\alpha_{n_1}}_{n_1}(|t-x|;q_{n_1},x) + ||f'_y||D^{\alpha_{n_2}}_{n_2}(|s-y|;q_{n_2},y).$$

Now applying the Cauchy-Schwarz inequality and Lemma 1,

$$\begin{aligned}|D^{\alpha_{n_1},\alpha_{n_2}}_{n_1,n_2}(f;q_{n_1},q_{n_2},x,y) - f(x,y)| &\leq ||f'_x||\sqrt{D^{\alpha_{n_1}}_{n_1}((t-x)^2;q_{n_1},x)}\sqrt{D^{\alpha_{n_1}}_{n_1}(1;q_{n_1},x)} \\ &\quad + ||f'_y||\sqrt{D^{\alpha_{n_2}}_{n_2}((s-y)^2;q_{n_2},y)}\sqrt{D^{\alpha_{n_2}}_{n_2}(1;q_{n_2},y)} \\ &= ||f'_x||\delta^{\alpha_{n_1}}_{n_1,q_{n_1}}(x) + ||f'_y||\delta^{\alpha_{n_2}}_{n_2,q_{n_2}}(y).\end{aligned}$$

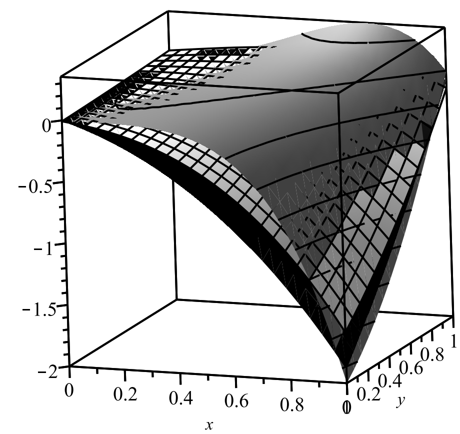

Figure 2: *The convergence of* $D_{n_1,n_2}^{\alpha_{n_1},\alpha_{n_2}}(f;q_{n_1},q_{n_2},x,y)$ *to* $f(x,y)$*, for* $q_{n_1}=q_{n_2}=0.9$*,* $n_1=n_2=100$ *(grey* f*, white* $D_{100,100}^{0.1,0.1}$*, black* $D_{100,100}^{0.9,0.9}$*)*

$q_{n_1}=q_{n_2}$	Error of approximation
0.4	2.683089189
0.5	2.314780978
0.6	1.915542164
0.7	1.503058754
0.8	1.241435619
0.9	1.061760430

Table 1: *Error of approximation for* $D_{n_1,n_2}^{\alpha_{n_1},\alpha_{n_2}}$

This completes the proof. □

Example 2. *Let* $f\in C^1(I^2)$*. Considering* $n_1=n_2=10$ *and* $\alpha_{n_1}=\alpha_{n_2}=0.2$*, in Table 1 we compute the error of approximation of* $f(x,y)=x^2y^2+x^3y-2x^4$ *by using relation (5).*

For $f\in C(I^2)$, the total modulus of continuity [2] is given by

$$\bar{\omega}(f;\delta_1,\delta_2)=\sup\left\{|f(t,s)-f(x,y)|:(t,s),(x,y)\in I^2 \text{ and } |t-x|\le\delta_1,\ |s-y|\le\delta_2\right\}.$$

Alternately,

$$\bar{\omega}(f;\delta_1,\delta_2)=\sup\left\{|f(t,s)-f(x,y)|:(t,s),(x,y)\in I^2 \text{ and } \sqrt{(t-x)^2+(s-y)^2}\le\delta\right\},$$

where,

(i) $\bar{\omega}(f;\delta_1,\delta_2)\to 0$, if $\delta_1\to 0$ and $\delta_2\to 0$;

(ii) $|f(t,s)-f(x,y)|\leq \bar{\omega}(f;\delta_1,\delta_2)\bigg(1+\dfrac{|t-x|}{\delta_1}\bigg)\bigg(1+\dfrac{|s-y|}{\delta_2}\bigg).$

Further, the partial moduli of continuity with respect to x and y are defined as

$$\omega^1(f;\delta)=\sup\bigg\{|f(x_1,y)-f(x_2,y)|: y\in I \ \textit{and}\ |x_1-x_2|\leq\delta\bigg\},$$

and

$$\omega^2(f;\delta)=\sup\bigg\{|f(x,y_1)-f(x,y_2)|: x\in I \ \textit{and}\ |y_1-y_2|\leq\delta\bigg\}.$$

Let

$$C^2(I^2):=\bigg\{f\in C(I^2): f_{xx},f_{xy},f_{yx},f_{yy}\in C(I^2)\bigg\},$$

equipped with the norm

$$||f||_{C^2(I^2)}=||f||+\sum_{i=1}^{2}\left(\left\|\frac{\partial^i f}{\partial x^i}\right\|+\left\|\frac{\partial^i f}{\partial y^i}\right\|\right).$$

For $f\in C(I^2)$, the appropriate Peetre's $K-$functional is defined by

$$\mathcal{K}(f;\delta)=\inf_{g\in C^2(I^2)}\{||f-g||+\delta||g||\},\delta>0.$$

Also,

$$\mathcal{K}(f;\delta)\leq M\bigg\{\bar{\omega}_2(f;\sqrt{\delta})+\min(1,\delta)||f||\bigg\},$$

where M is a constant independent of δ and f and $\bar{\omega}_2(f;\sqrt{\delta})$ is given by

$$\bar{\omega}_2(f;\sqrt{\delta})=\sup\ \bigg\{\bigg|\sum_{\nu=0}^{2}(-1)^{2-\nu}f(x+\nu h,y+\nu k)\bigg|:$$
$$(x,y),(x+2h,y+2k)\in J^2,|h|\leq\delta,|k|\leq\delta\bigg\}.$$

Theorem 4. *For $f\in C(I^2)$ and each $(x,y)\in I^2$, there follows:*

$$|D_{n_1,n_2}^{\alpha_{n_1},\alpha_{n_2}}(f;q_{n_1},q_{n_2},x,y)-f(x,y)|\leq 4\bar{\omega}\bigg(f,\delta_{n_1,q_{n_1}}^{\alpha_{n_1}}(x),\delta_{n_2,q_{n_2}}^{\alpha_{n_2}}(y)\bigg).$$

Proof. In view of property (ii) of the total modulus of continuity and Lemma 2, we have

$$\begin{aligned}&|D_{n_1,n_2}^{\alpha_{n_1},\alpha_{n_2}}(f;q_{n_1},q_{n_2},x,y)-f(x,y)|\\ &\quad\leq D_{n_1,n_2}^{\alpha_{n_1},\alpha_{n_2}}(|f(t,s)-f(x,y)|;q_{n_1},q_{n_2},x,y)\\ &\quad\leq\bar{\omega}\bigg(f,\delta_{n_1,q_{n_1}}^{\alpha_{n_1}}(x),\delta_{n_2,q_{n_2}}^{\alpha_{n_2}}(y)\bigg)\bigg(1+\frac{1}{\delta_{n_1,q_{n_1}}^{\alpha_{n_1}}(x)}D_{n_1}^{\alpha_{n_1}}(|t-x|;q_{n_1},x)\bigg)\\ &\qquad\times\bigg(1+\frac{1}{\delta_{n_2,q_{n_2}}^{\alpha_{n_2}}(y)}D_{n_2}^{\alpha_{n_2}}(|s-y|;q_{n_2},y)\bigg).\end{aligned}$$

Now applying the Cauchy-Schwarz inequality, we get

$$\begin{aligned}
&|D_{n_1,n_2}^{\alpha_{n_1},\alpha_{n_2}}(f;q_{n_1},q_{n_2},x,y)-f(x,y)|\\
&\quad\le D_{n_1,n_2}^{\alpha_{n_1},\alpha_{n_2}}(|f(t,s)-f(x,y)|;q_{n_1},q_{n_2},x,y)\\
&\quad\le \bar{\omega}\left(f,\delta_{n_1,q_{n_1}}^{\alpha_{n_1}}(x),\delta_{n_2,q_{n_2}}^{\alpha_{n_2}}(y)\right)\left(1+\frac{1}{\delta_{n_1,q_{n_1}}^{\alpha_{n_1}}(x)}\sqrt{D_{n_1}^{\alpha_{n_1}}((t-x)^2;q_{n_1},x)}\right)\\
&\qquad\times\left(1+\frac{1}{\delta_{n_2,q_{n_2}}^{\alpha_{n_2}}(y)}\sqrt{D_{n_2}^{\alpha_{n_2}}((s-y)^2;q_{n_2},y)}\right). \qquad (7)
\end{aligned}$$

Considering (4) from the above inequality, the desired result follows easily. □

Theorem 5. *Let $f\in C(I^2)$ and $(x,y)\in I^2$. Then,*

$$|D_{n_1,n_2}^{\alpha_{n_1},\alpha_{n_2}}(f;q_{n_1},q_{n_2},x,y)-f(x,y)|\le 2\omega^1\left(f;\delta_{n_1,q_{n_1}}^{\alpha_{n_1}}(x)\right)+2\omega^2\left(f;\delta_{n_2,q_{n_2}}^{\alpha_{n_2}}(y)\right).$$

Proof. Using the definition of the partial moduli of continuity and the Cauchy-Schwarz inequality,

$$\begin{aligned}
&|D_{n_1,n_2}^{\alpha_{n_1},\alpha_{n_2}}(f;q_{n_1},q_{n_2},x,y)-f(x,y)|\\
&\quad\le D_{n_1,n_2}^{\alpha_{n_1},\alpha_{n_2}}(|f(t,s)-f(x,y)|;q_{n_1},q_{n_2},x,y)\\
&\quad\le D_{n_1,n_2}^{\alpha_{n_1},\alpha_{n_2}}(|f(t,s)-f(t,y)|;q_{n_1},q_{n_2},x,y)\\
&\qquad+D_{n_1,n_2}^{\alpha_{n_1},\alpha_{n_2}}(|f(t,y)-f(x,y)|;q_{n_1},q_{n_2},x,y)\\
&\quad\le D_{n_1,n_2}^{\alpha_{n_1},\alpha_{n_2}}\left(\omega^1(f;|t-x|);q_{n_1},q_{n_2},x,y\right)+D_{n_1,n_2}^{\alpha_{n_1},\alpha_{n_2}}\left(\omega^2(f;|s-y|);q_{n_1},q_{n_2},x,y\right)\\
&\quad\le \omega^1(f,\delta_{n_1,q_{n_1}}^{\alpha_{n_1}}(x))\left\{1+\frac{1}{\delta_{n_1,q_{n_1}}^{\alpha_{n_1}}(x)}D_{n_1}^{\alpha_{n_1}}(|t-x|;q_{n_1},x)\right\}\\
&\qquad+\omega^2(f,\delta_{n_2,q_{n_2}}^{\alpha_{n_2}}(y))\left\{1+\frac{1}{\delta_{n_2,q_{n_2}}^{\alpha_{n_2}}(y)}D_{n_2}^{\alpha_{n_2}}(|s-y|;q_{n_2},y)\right\}\\
&\quad\le \omega^1(f,\delta_{n_1,q_{n_1}}^{\alpha_{n_1}}(x))\left\{1+\frac{1}{\delta_{n_1,q_{n_1}}^{\alpha_{n_1}}(x)}\left(D_{n_1}^{\alpha_{n_1}}((t-x)^2;q_{n_1},x)\right)^{1/2}\right\}\\
&\qquad+\omega^2(f,\delta_{n_2,q_{n_2}}^{\alpha_{n_2}}(y))\left\{1+\frac{1}{\delta_{n_2,q_{n_2}}^{\alpha_{n_2}}(y)}\left(D_{n_2}^{\alpha_{n_2}}((s-y)^2;q_{n_2},y)\right)^{1/2}\right\}.
\end{aligned}$$

Thus, we get the desired result. □

In our next result we establish the rate of approximation of the Stancu-Durrmeyer type operators to the function $f\in C(I^2)$ by means of Peetre's K-functional.

Theorem 6. *For the function $f \in C(I^2)$, we have*

$$|D_{n_1,n_2}^{\alpha_{n_1},\alpha_{n_2}}(f;q_{n_1},q_{n_2},x,y)-f(x,y)|$$
$$\leq M\left\{\bar{\omega}_2\left(f;\sqrt{A_{n_1,n_2}^{\alpha_{n_1},\alpha_{n_2}}(q_{n_1},q_{n_2},x,y)}\right)+\min\{1,A_{n_1,n_2}^{\alpha_{n_1},\alpha_{n_2}}(q_{n_1},q_{n_2},x,y)\}||f||\right\}$$
$$+\omega\left(f;\sqrt{\left(\frac{1}{[n_1+2]_{q_{n_1}}}(1+q_{n_1}[n_1]_{q_{n_1}}x)-x\right)^2+\left(\frac{1}{[n_2+2]_{q_{n_2}}}(1+q_{n_2}[n_2]_{q_{n_2}}y)-y\right)^2}\right),$$

where

$$A_{n_1,n_2,p}^{\alpha_{n_1},\alpha_{n_2}}(q_{n_1},q_{n_2},x,y)$$
$$=\left\{(\delta_{n_1,q_{n_1}}^{\alpha_{n_1}}(x))^2+(\delta_{n_2,q_{n_2}}^{\alpha_{n_2}}(y))^2+\left(\frac{1}{[n_1+2]_{q_{n_1}}}(1+q_{n_1}[n_1]_{q_{n_1}}x)-x\right)^2\right.$$
$$\left.+\left(\frac{1}{[n_2+2]_{q_{n_2}}}(1+q_{n_2}[n_2]_{q_{n_2}}y)-y\right)^2\right\},$$

and the constant $M>0$ does not depend onf and $A_{n_1,n_2,p}^{\alpha_{n_1},\alpha_{n_2}}(q_{n_1},q_{n_2},x,y)$.

Proof. Let us define an auxiliary operator

$$D_{n_1,n_2}^{*\alpha_{n_1},\alpha_{n_2}}(f;q_{n_1},q_{n_2},x,y)$$
$$=D_{n_1,n_2}^{\alpha_{n_1},\alpha_{n_2}}(f;q_{n_1},q_{n_2},x,y)+f(x,y)$$
$$-f\left(\frac{1}{[n_1+2]_{q_{n_1}}}(1+q_{n_1}[n_1]_{q_{n_1}}x),\frac{1}{[n_2+2]_{q_{n_2}}}(1+q_{n_2}[n_2]_{q_{n_2}}y)\right).$$

Then using Lemma 3, we have

$$D_{n_1,n_2}^{*\alpha_{n_1},\alpha_{n_2}}(1;q_{n_1},q_{n_2},x,y)=1,$$
$$D_{n_1,n_2}^{*\alpha_{n_1},\alpha_{n_2}}(t-x;q_{n_1},q_{n_2},x,y)=0,$$
$$D_{n_1,n_2}^{*\alpha_{n_1},\alpha_{n_2}}(s-y;q_{n_1},q_{n_2},x,y)=0.$$

For any $g\in C^2(I^2)$ and $t,s\in I$, by Taylor's theorem

$$g(t,s)-g(x,y)=\frac{\partial g(x,y)}{\partial x}(t-x)+\int_x^t(t-u)\frac{\partial^2 g(u,y)}{\partial u^2}du$$
$$+\frac{\partial g(x,y)}{\partial y}(s-y)+\int_y^s(s-v)\frac{\partial^2 g(t,v)}{\partial v^2}dv.$$

Hence,

$$D_{n_1,n_2}^{*\alpha_{n_1},\alpha_{n_2}}(f;q_{n_1},q_{n_2},x,y)$$
$$=D_{n_1,n_2}^{*\alpha_{n_1},\alpha_{n_2}}\left(\int_x^t(t-u)\frac{\partial^2 g(u,y)}{\partial u^2}du;q_{n_1},q_{n_2},x,y\right)$$
$$+D_{n_1,n_2}^{*\alpha_{n_1},\alpha_{n_2}}\left(\int_y^s(s-v)\frac{\partial^2 g(x,v)}{\partial v^2}dv;q_{n_1},q_{n_2},x,y\right)$$

$$
\begin{aligned}
&= D_{n_1,n_2}^{\alpha_{n_1},\alpha_{n_2}}\left(\int_x^t (t-u)\frac{\partial^2 g(u,y)}{\partial u^2}du; q_{n_1},q_{n_2},x,y\right)\\
&\quad - \int_x^{\frac{1}{[n_1+2]_{q_{n_1}}}(1+q_{n_1}[n_1]_{q_{n_1}}x)} \left(\frac{1}{[n_1+2]_{q_{n_1}}}(1+q_{n_1}[n_1]_{q_{n_1}}x)-u\right)\frac{\partial^2 g(u,y)}{\partial u^2}du\\
&\quad + D_{n_1,n_2}^{\alpha_{n_1},\alpha_{n_2}}\left(\int_y^s (s-v)\frac{\partial^2 g(x,v)}{\partial v^2}dv; q_{n_1},q_{n_2},x,y\right)\\
&\quad - \int_y^{\frac{1}{[n_2+2]_{q_{n_2}}}(1+q_{n_2}[n_2]_{q_{n_2}}y)} \left(\frac{1}{[n_2+2]_{q_{n_2}}}(1+q_{n_2}[n_2]_{q_{n_2}}y)-v\right)\frac{\partial^2 g(x,v)}{\partial v^2}dv.
\end{aligned}
$$

Thus,

$$
\begin{aligned}
&|D_{n_1,n_2}^{*\alpha_{n_1},\alpha_{n_2}}(g;q_{n_1},q_{n_2},x,y)-g(x,y)|\\
&\quad\le D_{n_1,n_2}^{\alpha_{n_1},\alpha_{n_2}}\left(\left|\int_x^t |t-u|\left|\frac{\partial^2 g(u,y)}{\partial u^2}\right|du\right|;x,y\right)\\
&\qquad + \left|\int_x^{\frac{1}{[n_1+2]_{q_{n_1}}}(1+q_{n_1}[n_1]_{q_{n_1}}x)}\left|\frac{1}{[n_1+2]_{q_{n_1}}}(1+q_{n_1}[n_1]_{q_{n_1}}x)-u\right|\left|\frac{\partial^2 g(u,y)}{\partial u^2}\right|du\right|\\
&\qquad + D_{n_1,n_2}^{\alpha_{n_1},\alpha_{n_2}}\left(\left|\int_y^s |s-v|\left|\frac{\partial^2 g(x,v)}{\partial v^2}\right|dv\right|;x,y\right)\\
&\qquad + \left|\int_y^{\frac{1}{[n_2+2]_{q_{n_2}}}(1+q_{n_2}[n_2]_{q_{n_2}}y)}\left|\frac{1}{[n_2+2]_{q_{n_2}}}(1+q_{n_2}[n_2]_{q_{n_2}}y)-v\right|\left|\frac{\partial^2 g(x,v)}{\partial v^2}\right|dv\right|\\
&\quad\le \left\{D_{n_1,n_2}^{\alpha_{n_1},\alpha_{n_2}}((t-x)^2;q_{n_1},q_{n_2},x,y)+\left(\frac{1}{[n_1+2]_{q_{n_1}}}(1+q_{n_1}[n_1]_{q_{n_1}}x)-x\right)^2\right\}||g||_{C^2(I^2)}\\
&\qquad + \left\{D_{n_1,n_2}^{\alpha_{n_1},\alpha_{n_2}}((s-y)^2;q_{n_1},q_{n_2},x,y)+\left(\frac{1}{[n_2+2]_{q_{n_2}}}(1+q_{n_2}[n_2]_{q_{n_2}}y)-y\right)^2\right\}\\
&\qquad \times ||g||_{C^2(I^2)}\\
&\quad\le \left\{(\delta_{n_1,q_{n_1}}^{\alpha_{n_1}}(x))^2+\left(\frac{1}{[n_1+2]_{q_{n_1}}}(1+q_{n_1}[n_1]_{q_{n_1}}x)-x\right)^2\right\}||g||_{C^2(I^2)}\\
&\qquad + \left\{(\delta_{n_2,q_{n_2}}^{\alpha_{n_2}}(y))^2+\left(\frac{1}{[n_2+2]_{q_{n_2}}}(1+q_{n_2}[n_2]_{q_{n_2}}y)-y\right)^2\right\}||g||_{C^2(I^2)}\\
&\quad= \left\{(\delta_{n_1,q_{n_1}}^{\alpha_{n_1}}(x))^2+(\delta_{n_2,q_{n_2}}^{\alpha_{n_2}}(y))^2+\left(\frac{1}{[n_1+2]_{q_{n_1}}}(1+q_{n_1}[n_1]_{q_{n_1}}x)-x\right)^2\right.\\
&\qquad \left.+\left(\frac{1}{[n_2+2]_{q_{n_2}}}(1+q_{n_2}[n_2]_{q_{n_2}}y)-y\right)^2\right\}||g||_{C^2(I^2)}\\
&\quad= A_{n_1,n_2}^{\alpha_{n_1},\alpha_{n_2}}(q_{n_1},q_{n_2},x,y)||g||_{C^2(I^2)}. \qquad (8)
\end{aligned}
$$

Also,

$$\begin{aligned}
&|D^{*\alpha_{n_1},\alpha_{n_2}}_{n_1,n_2}(f;q_{n_1},q_{n_2},x,y)|\\
&\quad\le |D^{\alpha_{n_1},\alpha_{n_2}}_{n_1,n_2}(f;q_{n_1},q_{n_2},x,y)|\\
&\qquad+\left|f\left(\frac{1}{[n_1+2]_{q_{n_1}}}(1+q_{n_1}[n_1]_{q_{n_1}}x),\frac{1}{[n_2+2]_{q_{n_2}}}(1+q_{n_2}[n_2]_{q_{n_2}}y)\right)\right|+|f(x,y)|\\
&\quad\le 3||f||. \qquad (9)
\end{aligned}$$

Hence for any $g\in C^2(I^2)$, in view of (8) and (9), we get

$$\begin{aligned}
&|D^{\alpha_{n_1},\alpha_{n_2}}_{n_1,n_2}(f;q_{n_1},q_{n_2},x,y)-f(x,y)|\\
&\quad=\Bigg|D^{*\alpha_{n_1},\alpha_{n_2}}_{n_1,n_2}(f;q_{n_1},q_{n_2},x,y)-f(x,y)\\
&\qquad+f\left(\frac{1}{[n_1+2]_{q_{n_1}}}(1+q_{n_1}[n_1]_{q_{n_1}}x),\frac{1}{[n_2+2]_{q_{n_2}}}(1+q_{n_2}[n_2]_{q_{n_2}}y)\right)-f(x,y)\Bigg|\\
&\quad\le |D^{*\alpha_{n_1},\alpha_{n_2}}_{n_1,n_2}(f-g;q_{n_1},q_{n_2},x,y)|+|D^{*\alpha_{n_1},\alpha_{n_2}}_{n_1,n_2}(g;q_{n_1},q_{n_2},x,y)-g(x,y)|\\
&\qquad+|g(x,y)-f(x,y)|\\
&\qquad+\left|f\left(\frac{1}{[n_1+2]_{q_{n_1}}}(1+q_{n_1}[n_1]_{q_{n_1}}x),\frac{1}{[n_2+2]_{q_{n_2}}}(1+q_{n_2}[n_2]_{q_{n_2}}y)\right)-f(x,y)\right|\\
&\quad\le 4||f-g||+A^{\alpha_{n_1},\alpha_{n_2}}_{n_1,n_2}(q_{n_1},q_{n_2},x,y)||g||_{C^2(I^2)}\\
&\qquad+\omega\left(f;\sqrt{\left(\frac{1}{[n_1+2]_{q_{n_1}}}(1+q_{n_1}[n_1]_{q_{n_1}}x)-x\right)^2+\left(\frac{1}{[n_2+2]_{q_{n_2}}}(1+q_{n_2}[n_2]_{q_{n_2}}y)-y\right)^2}\right).
\end{aligned}$$

Thus,

$$\begin{aligned}
&|D^{\alpha_{n_1},\alpha_{n_2}}_{n_1,n_2}(f;q_{n_1},q_{n_2},x,y)-f(x,y)|\\
&\quad\le 4\inf_{g\in C^2(I^2)}\left\{||f-g||+A^{\alpha_{n_1},\alpha_{n_2}}_{n_1,n_2}(q_{n_1},q_{n_2},x,y)||g||_{C^2(I^2)}\right\}\\
&\qquad+\omega\left(f;\sqrt{\left(\frac{1}{[n_1+2]_{q_{n_1}}}(1+q_{n_1}[n_1]_{q_{n_1}}x)-x\right)^2+\left(\frac{1}{[n_2+2]_{q_{n_2}}}(1+q_{n_2}[n_2]_{q_{n_2}}y)-y\right)^2}\right)\\
&\quad\le 4\mathcal{K}(f;A^{\alpha_{n_1},\alpha_{n_2}}_{n_1,n_2}(q_{n_1},q_{n_2},x,y))\\
&\qquad+\omega\left(f;\sqrt{\left(\frac{1}{[n_1+2]_{q_{n_1}}}(1+q_{n_1}[n_1]_{q_{n_1}}x)-x\right)^2+\left(\frac{1}{[n_2+2]_{q_{n_2}}}(1+q_{n_2}[n_2]_{q_{n_2}}y)-y\right)^2}\right)\\
&\quad\le M\left\{\bar{\omega}_2\left(f;\sqrt{A^{\alpha_{n_1},\alpha_{n_2}}_{n_1,n_2}(q_{n_1},q_{n_2},x,y)}\right)+\min\{1,A^{\alpha_{n_1},\alpha_{n_2}}_{n_1,n_2}(q_{n_1},q_{n_2},x,y)\}||f||_{C(I^2)}\right\}\\
&\qquad+\omega\left(f;\sqrt{\left(\frac{1}{[n_1+2]_{q_{n_1}}}(1+q_{n_1}[n_1]_{q_{n_1}}x)-x\right)^2+\left(\frac{1}{[n_2+2]_{q_{n_2}}}(1+q_{n_2}[n_2]_{q_{n_2}}y)-y\right)^2}\right),
\end{aligned}$$

which is the desired conclusion. □

4. A Voronovskaya-type theorm

Lemma 4. *Assume that* $0 < q_n < 1$, $q_n \to 1$ *and* $q_n^n \to a$, $a \in [0,1)$ *as* $n \to \infty$. *If* $\lim\limits_{n\to\infty} \alpha_n = 0$ *and* $\lim\limits_{n\to\infty} n\alpha_n = l \in \mathbb{R}$, *then*

$$\lim_{n\to\infty} [n]_{q_n} D_n^{\alpha_n}(t-x;q_n;x) = 1-(1+a)x,$$

$$\lim_{n\to\infty} [n]_{q_n} D_n^{\alpha_n}\left((t-x)^2;q_n;x\right) = (l+2)x(1-x), \tag{10}$$

$$\lim_{n\to\infty} [n]_{q_n}^2 D_n^{\alpha_n}\left((t-x)^4;q_n;x\right) = 3x^2(1-x)^2 l(l+4)+x^2(1-x)(7x^2-7x+5). \tag{11}$$

Proof. Using Lemma 1, we get

$$\lim_{n\to\infty} [n]_{q_n} D_n^{\alpha_n}(t-x;q_n;x) = \frac{[n]_{q_n}}{[n+2]_{q_n}} - \frac{[n]_{q_n}}{[n+2]_{q_n}}(1+q_n^{n+1}) = 1-(1+a)x,$$

$$\begin{aligned}\lim_{n\to\infty} [n]_{q_n} D_n^{\alpha_n}\left((t-x)^2;q_n;x\right) &= (2+l)x-x^2+\lim_{n\to\infty}\frac{[n]_{q_n}}{[n+2]_{q_n}[n+3]_{q_n}(1+\alpha_n)}\\ &\quad\times\left\{q^3[n]_{q_n}^2-2q_n[n]_{q_n}[n+2]_{q_n}[n+3]_{q_n}+[n+2]_{q_n}[n+3]_{q_n}\right.\\ &\quad\left.+[n+3]_{q_n}\alpha_n([n+2]_{q_n}-2q_n[n]_{q_n})\right\}x^2\\ &= (l+2)x-(l+1)x^2+\lim_{n\to\infty}\frac{[n]_{q_n}}{[n+2]_{q_n}[n+3]_{q_n}(1+\alpha_n)}\\ &\quad\times\left\{q_n^3(q_n^n)^2+2q_n(1+q_n)q_n^n-q_n^2[n]_{q_n}+1\right\}x^2\\ &= (l+2)x(1-x).\end{aligned}$$

Relation (11) is obtained in a similar way using Lemma 1 and Lemma 3.2 ([27]). □

Theorem 7. *Let* $f \in C^2\left(I^2\right)$ *and* $(q_n)_n$ *be a sequence in the interval* $(0,1)$ *such that* $q_n \to 1$ *and* $q_n^n \to a$, $a \in [0,1)$ *as* $n \to \infty$. *If* $\lim\limits_{n\to\infty}\alpha_n = 0$ *and* $\lim\limits_{n\to\infty} n\alpha_n = l \in \mathbb{R}$, *then for every* $(x,y) \in I^2$, *one has*

$$\begin{aligned}&\lim_{n\to\infty}[n]_{q_n}\left\{D_{n,n}^{\alpha_n,\alpha_n}(f;q_n,q_n,x,y)-f(x,y)\right\}\\ &\quad= [1-(a+1)x]f_x'(x,y)+[1-(a+1)y]f_y'(x,y)+\frac{l+2}{2}\left\{x(1-x)f_{x^2}''(x,y)\right.\\ &\qquad\left.+y(1-y)f_{y^2}''(x,y)\right\}.\end{aligned}$$

Proof. For $(x_0,y_0) \in I^2$, by Taylor's formula it follows

$$\begin{aligned}f(t,s) =& f(x_0,y_0)+f_x'(x_0,y_0)(t-x_0)+f_y'(x_0,y_0)(s-y_0)\\ &+\frac{1}{2}\left\{f_{x^2}''(x_0,y_0)(t-x_0)^2+2f_{xy}''(x_0,y_0)(t-x_0)(s-y_0)+f_{y^2}''(x_0,y_0)(s-y_0)^2\right\}\\ &+\varphi(t,s)\left((t-x_0)^2+(s-y_0)^2\right),\end{aligned}$$

where $(t,s) \in I^2$ and $\lim\limits_{(t,s)\to(x_0,y_0)}\varphi(s,t) = 0$.

From the linearity of $D_{n,n}^{\alpha_n,\alpha_n}$, we have

$$\begin{aligned}
&D_{n,n}^{\alpha_n,\alpha_n}(f(t,s);q_n,q_n,x_0,y_0)\\
&\quad= f(x_0,y_0)+f'_x(x_0,y_0)D_{n,n}^{\alpha_n,\alpha_n}(t-x_0;q_n,q_n,x_0,y_0)\\
&\qquad+f'_y(x_0,y_0)D_{n,n}^{\alpha_n,\alpha_n}(s-y_0;q_n,q_n,x_0,y_0)\\
&\qquad+\frac{1}{2}\left\{f''_{x^2}(x_0,y_0)D_{n,n}^{\alpha_n,\alpha_n}((t-x_0)^2;q_n,q_n,x_0,y_0)\right.\\
&\qquad+2f''_{xy}(x_0,y_0)D_{n,n}^{\alpha_n,\alpha_n}((t-x_0)(s-y_0);q_n,q_n,x_0,y_0)\\
&\qquad+\left.f''_{y^2}(x_0,y_0)D_{n,n}^{\alpha_n,\alpha_n}((s-y_0)^2;q_n,q_n,x_0,y_0)\right\}\\
&\qquad+D_{n,n}^{\alpha_n,\alpha_n}\left(\varphi(t,s)\left((t-x_0)^2+(s-y_0)^2\right);q_n,q_n,x_0,y_0\right)\\
&\quad= f(x_0,y_0)+f'_x(x_0,y_0)D_n^{\alpha_n}(t-x_0;q_n,x_0)+f'_y(x_0,y_0)D_n^{\alpha_n}(s-y_0;q_n,y_0)\\
&\qquad+\frac{1}{2}\left\{f''_{x^2}(x_0,y_0)D_n^{\alpha_n}((t-x_0)^2;q_n,x_0)+f''_{y^2}(x_0,y_0)D_n^{\alpha_n}((s-y_0)^2;q_n,y_0)\right.\\
&\qquad\left.+2f''_{xy}(x_0,y_0)D_n^{\alpha_n}((t-x_0);q_n,x_0)D_n^{\alpha_n}((s-y_0);q_n,y_0)\right\}\\
&\qquad+D_{n,n}^{\alpha_n,\alpha_n}\left(\varphi(s,t)\left((t-x_0)^2+(s-y_0)^2\right);q_n,q_n,x_0,y_0\right).
\end{aligned}$$

By the Cauchy-Schwarz inequality, we have

$$\begin{aligned}
&\left|D_{n,n}^{\alpha_n,\alpha_n}\left(\varphi(t,s)\left((t-x_0)^2+(s-y_0)^2\right);q_n,q_n,x_0,y_0\right)\right|\\
&\quad\le\left\{D_{n,n}^{\alpha_n,\alpha_n}\left(\varphi^2(t,s);q_n,q_n,x_0,y_0\right)\right\}^{1/2}\\
&\qquad\times\left\{D_{n,n}^{\alpha_n,\alpha_n}\left(\left((t-x_0)^2+(s-y_0)^2\right)^2;q_n,q_n,x_0,y_0\right)\right\}^{1/2}\\
&\quad\le\sqrt{2}\left\{D_{n,n}^{\alpha_n,\alpha_n}\left(\varphi^2(t,s);q_n,q_n,x_0,y_0\right)\right\}^{1/2}\\
&\qquad\times\left\{D_{n,n}^{\alpha_n,\alpha_n}\left((t-x_0)^4;q_n,q_n,x_0,y_0\right)+D_{n,n}^{\alpha_n,\alpha_n}\left((s-y_0)^4;q_n,q_n,x_0,y_0\right)\right\}^{1/2}.
\end{aligned}$$

By Theorem 2, we get

$$\lim_{n\to\infty}D_{n,n}^{\alpha_n,\alpha_n}\left(\varphi^2(t,s);q_n,q_n,x_0,y_0\right)=\varphi^2(x_0,y_0)=0,$$

and hence using Lemma 4 we have

$$\lim_{n\to\infty}[n]_{q_n}D_{n,n}^{\alpha_n,\alpha_n}\left(\varphi(t,s)\left((t-x_0)^2+(s-y_0)^2\right);q_n,q_n,x_0,y_0\right)=0.$$

The theorem is proved by applying Lemma 4. □

5. Construction of q-GBS operators of Stancu-Durrmeyer type

In [11] and [12], Bögel introduced a new concept of Bögel-continuous (B-continuous) functions and established some significant results. Dobrescu and Matei [15] proved the convergence of the Boolean sum of bivariate generalization of Bernstein polynomials to the B-continuous function on a bounded interval. Badea and Cottin [5] obtained Korovkin theorems for GBS operators. After that, Pop [29] introduced

Voronovoskaja-type theorems for certain GBS operators. For further contribution by researchers in this direction, we refer to (cf.[8, 9, 10, 17, 18, 19, 30, 31, 33], etc.).

Following [13], a function $f : E \times F \to R$ is said to be B-continuous at $(x_0, y_0) \in E \times F$, where E and F are compact subsets of $\mathbb{R}$ if

$$\lim_{(x,y)\to(x_0,y_0)} \Delta f[(x, y); (x_0, y_0)] = 0,$$

where

$$\Delta f[(x, y); (x_0, y_0)] = f(x, y) - f(x_0, y) - f(x, y_0) + f(x_0, y_0)$$

denotes the mixed difference of f.

A function $f : E \times F \to \mathbb{R}$ is called B-differentiable at $(x_0, y_0) \in E \times F$ if the limit

$$\lim_{(x,y)\to(x_0,y_0)} \frac{\Delta f[(x, y); (x_0, y_0)]}{(x - x_0)(y - y_0)}$$

exists and is finite. This limit is called B-differential of f at the point (x_0, y_0) and is denoted by $D_b f(x_0, y_0)$.

The function $f : A \subset E \times F \to \mathbb{R}$ is called B-bounded on A if we can find a constant $M > 0$ such that $|\Delta f\,[t, s; x, y]| \le M,\ \forall (x, y), (t, s) \in A$. In case $A \subset \mathbb{R}^2$ is compact, it follows that each B-continuous function on A is a B-bounded function. In what follows, let $B_b\,(A)$, be the class of all B-bounded functions on A. We denote by $C_b\,(A)$ and $D_b\,(A)$ the spaces of all B-continuous functions and B differentiable functions on A, respectively. Further, let $B\,(A)$ and $C\,(A)$ be the spaces of all bounded functions and continuous functions on A, respectively, endowed with the sup-norm $\|.\|_\infty$. Clearly, $C\,(A) \subset C_b\,(A)$.

Theorem 8 (see [6]). *Let (L_{n_1,n_2}), $L_{n_1,n_2} : C_b(A) \to B(A)$, $n_1, n_2 \in \mathbb{N}$ be a sequence of bivariate linear positive operators, G_{n_1,n_2} GBS-operators associated to L_{n_1,n_2} and let the following conditions be satisfied*

(i) $L_{n_1,n_2}(e_{00}; x, y) = 1$

(ii) $L_{n_1,n_2}(e_{10}; x, y) = x + u_{n_1,n_2}(x, y)$

(iii) $L_{n_1,n_2}(e_{01}; x, y) = y + v_{n_1,n_2}(x, y)$

(iv) $L_{n_1,n_2}(e_{20} + e_{02}; x, y) = x^2 + y^2 + w_{n_1,n_2}(x, y)$

for all $(x, y) \in A$. If the sequences (u_{n_1,n_2}), (v_{n_1,n_2}) and (w_{n_1,n_2}) converges to zero uniformly on A, then the sequence $(G_{n_1,n_2} f)$ converge to f uniformly on A for all $f \in C_b(A)$.

The mixed modulus of smoothness for $f \in C_b\left(I^2\right)$ is defined as

$$\omega_{mixed}\,(f; \delta_1, \delta_2) := \sup\left\{|\Delta f\,[t, s; x, y]| : |x - t| < \delta_1, |y - s| < \delta_2\right\},$$

for all $(x, y), (t, s) \in I^2$ and for any $(\delta_1, \delta_2) \in (0, \infty) \times (0, \infty)$ with $\omega_{mixed} : [0, \infty) \times [0, \infty) \to \mathbb{R}$.

For the properties of ω_{mixed} we refer to papers [5] and [7]. For any $f \in C_b(I^2)$ and $n_1, n_2 \in \mathbb{N}$, the GBS operator associated with $D_{n_1,n_2}^{\alpha_{n_1},\alpha_{n_2}}$ is given by

$$G_{n_1,n_2}^{\alpha_{n_1},\alpha_{n_2}}(f; q_{n_1}, q_{n_2}, x, y) = D_{n_1,n_2}^{\alpha_{n_1},\alpha_{n_2}}(f(t, y) + f(x, s) - f(t, s); q_{n_1}, q_{n_2}, x, y),$$

for all $(x, y) \in I^2$. Alternately,

$$\begin{aligned} G_{n_1,n_2}^{\alpha_{n_1},\alpha_{n_2}}&(f; q_{n_1}, q_{n_2}, x, y) \\ &= [n_1 + 1]_{q_{n_1}} [n_2 + 1]_{q_{n_2}} \sum_{k_1=0}^{n_1} \sum_{k_2=0}^{n_2} p_{n_1,k_1}^{q_{n_1},\alpha_{n_1}}(x) p_{n_2,k_2}^{q_{n_2},\alpha_{n_2}}(y) \\ &\times \int_0^1 \int_0^1 p_{n_1,k_1}^{q_{n_1}}(t) p_{n_2,k_2}^{q_{n_2}}(s)[f(t, y) + f(x, s) - f(t, s)] d_{q_{n_1}} t d_{q_{n_2}} s. \end{aligned} \tag{12}$$

Evidently, the operator $G_{n_1,n_2}^{\alpha_{n_1},\alpha_{n_2}}$ is a linear operator.

Example 3. *Let $f(x) = x^3y^3 + 3x^2y - 2y^4$, $n_1 = n_2 = 50$, $\alpha_{n_1} = \alpha_{n_2} = 0.1$, $q_{n_1} = q_{n_2} = 0.9$. In Figure 3, we compare the q-Stancu-Durrmeyer operator and its GBS-type operator. We note that $G_{n_1,n_2}^{\alpha_{n_1},\alpha_{n_2}}$ gives a better approximation than the operator $D_{n_1,n_2}^{\alpha_{n_1},\alpha_{n_2}}$. In Table 2, we computed the error of approximation for $D_{n_1,n_2}^{\alpha_{n_1},\alpha_{n_2}}$ and $G_{n_1,n_2}^{\alpha_{n_1},\alpha_{n_2}}$ at certain points.*

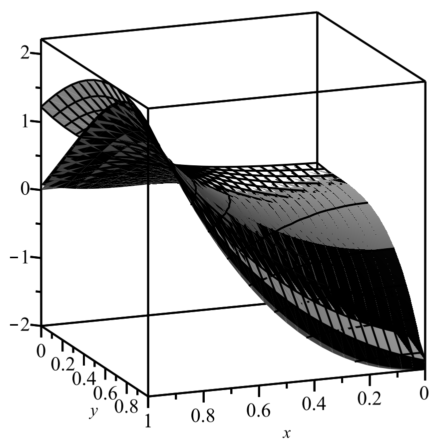

Figure 3: *The convergence of $D_{n_1,n_2}^{\alpha_{n_1},\alpha_{n_2}}$ and $G_{n_1,n_2}^{\alpha_{n_1},\alpha_{n_2}}$ to f (grey f, white $D_{n_1,n_2}^{\alpha_{n_1},\alpha_{n_2}}$, black $G_{n_1,n_2}^{\alpha_{n_1},\alpha_{n_2}}$)*

Theorem 9 (see [7])**.** *Let $L : C_b(A) \to C_b(A)$ be a bivariate linear positive operator and $G : C_b(A) \to C_b(A)$ be associated GBS-operator. Then*

$$\begin{aligned} |G(f; x, y) - f(x, y)| \le |f(x, y)||L(1; x, y) - 1| + \{L(1; x, y) \\ +\delta_1^{-1}\sqrt{L((t - x)^2; x, y)} + \delta_2^{-1}\sqrt{L((s - y)^2; x, y)} \\ +\delta_1^{-1}\sqrt{L((t-x)^2; x, y)}\delta_2^{-1}\sqrt{L((s-y)^2; x, y)}\}\omega_{mixed}(f; \delta_1, \delta_2), \end{aligned}$$

x	y	$\lvert(D^{\alpha_{n_1},\alpha_{n_2}}_{n_1,n_2}(f;q_{n_1},q_{n_2};x,y) - f(x,y)\rvert$	$\lvert(G^{\alpha_{n_1},\alpha_{n_2}}_{n_1,n_2}(f;q_{n_1},q_{n_2};x,y) - f(x,y)\rvert$
0.1	0.1	0.0008661426	0.0150800080
0.1	0.2	0.0356938734	0.0133166036
0.1	0.4	0.1607531616	0.0089474496
0.1	0.5	0.2300166971	0.0061010298
0.1	0.6	0.2819979805	0.0026529348
0.2	0.5	0.1803563457	0.0094698811
0.2	0.6	0.2267702167	0.0033050660
0.6	0.1	0.2396496598	0.0656001107
0.6	0.2	0.2046883968	0.0617182500
0.7	0.1	0.3621377633	0.0689935569
0.7	0.2	0.3185327583	0.0656191341
0.7	0.3	0.2511961928	0.0611420405
0.8	0.1	0.5408351349	0.0571732421
0.8	0.2	0.4855005567	0.0546858009
0.8	0.3	0.4044064178	0.0514608256
0.9	0.1	0.7979254620	0.0162967181
0.9	0.2	0.7277334820	0.0151054460

Table 2: *Error of approximation for* $D^{\alpha_{n_1},\alpha_{n_2}}_{n_1,n_2}$ *and* $G^{\alpha_{n_1},\alpha_{n_2}}_{n_1,n_2}$

for all $f \in C_b(A)$, $(x,y) \in A$ *and* $\delta_1, \delta_2 > 0$.

Theorem 10. *Let* $f \in C_b(I^2)$ *and* $(x,y) \in I^2$. *Then*

$$|G^{\alpha_{n_1},\alpha_{n_2}}_{n_1,n_2}(f;q_{n_1},q_{n_2},x,y) - f(x,y)| \leq 4\omega_{mixed}(f,\delta^{\alpha_{n_1}}_{n_1,q_{n_1}}(x),\delta^{\alpha_{n_2}}_{n_2,q_{n_2}}(y)).$$

Proof. Applying Theorem 9, we have

$$\begin{aligned}
&|G^{\alpha_{n_1},\alpha_{n_2}}_{n_1,n_2}(f;q_{n_1},q_{n_2},x,y) - f(x,y)| \\
&\quad \leq |f(x,y)||G^{\alpha_{n_1},\alpha_{n_2}}_{n_1,n_2}(1;q_{n_1},q_{n_2},x,y) - 1| + \{G^{\alpha_{n_1},\alpha_{n_2}}_{n_1,n_2}(1;q_{n_1},q_{n_2},x,y) \\
&\quad + \delta_1^{-1}\sqrt{G^{\alpha_{n_1},\alpha_{n_2}}_{n_1,n_2}((t-x)^2;q_{n_1},q_{n_2},x,y)} + \delta_2^{-1}\sqrt{G^{\alpha_{n_1},\alpha_{n_2}}_{n_1,n_2}((s-y)^2;q_{n_1},q_{n_2},x,y)} \\
&\quad + \delta_1^{-1}\delta_2^{-1}\sqrt{G^{\alpha_{n_1},\alpha_{n_2}}_{n_1,n_2}((t-x)^2;q_{n_1},q_{n_2},x,y)} \\
&\quad \times \sqrt{G^{\alpha_{n_1},\alpha_{n_2}}_{n_1,n_2}((s-y)^2;q_{n_1},q_{n_2},x,y)}\}\omega_{mixed}(f;\delta_1,\delta_2).
\end{aligned}$$

Now choosing $\delta_1 = \delta^{\alpha_{n_1}}_{n_1,q_{n_1}}(x)$ and $\delta_2 = \delta^{\alpha_{n_2}}_{n_2,q_{n_2}}(y)$, we obtain the desired result. □

Let

$$Lip_M(\beta,\gamma) = \left\{f \in C_b\left(I^2\right) : |\Delta f[t,s;x,y]| \leq M|t-x|^\beta |s-y|^\gamma, \text{for } (t,s), (x,y) \in I^2\right\},$$

$\beta, \gamma \in (0,1]$, be the Lipschitz class for B-continuous functions.

Theorem 11. *Let* $f \in Lip_M(\beta,\gamma)$ *and* $(x,y) \in I^2$. *Then for* $M > 0$, $\beta,\gamma \in (0,1]$, *we have*

$$|G^{\alpha_{n_1},\alpha_{n_2}}_{n_1,n_2}(f;q_{n_1},q_{n_2},x,y) - f(x,y)| \leq M\left(\delta^{\alpha_{n_1}}_{n_1,q_{n_1}}(x)\right)^\beta \left(\delta^{\alpha_{n_2}}_{n_2,q_{n_2}}(y)\right)^\gamma.$$

Proof. By the definition of $G_{n_1,n_2}^{\alpha_{n_1},\alpha_{n_2}}$, we may write

$$\begin{aligned}
&G_{n_1,n_2}^{\alpha_{n_1},\alpha_{n_2}}(f;q_{n_1},q_{n_2},x,y)\\
&\quad= D_{n_1,n_2}^{\alpha_{n_1},\alpha_{n_2}}(f(x,s)+f(t,y)-f(t,s);q_{n_1},q_{n_2},x,y)\\
&\quad= D_{n_1,n_2}^{\alpha_{n_1},\alpha_{n_2}}(f(x,y)-\Delta f[t,s;x,y];q_{n_1},q_{n_2},x,y)\\
&\quad= f(x,y)D_{n_1,n_2}^{\alpha_{n_1},\alpha_{n_2}}(e_{00};q_{n_1},q_{n_2},x,y)-D_{n_1,n_2}^{\alpha_{n_1},\alpha_{n_2}}(\Delta f[t,s;x,y];q_{n_1},q_{n_2},x,y).
\end{aligned}$$

Hence,

$$\begin{aligned}
&|G_{n_1,n_2}^{\alpha_{n_1},\alpha_{n_2}}(f;q_{n_1},q_{n_2},x,y)-f(x,y)|\\
&\quad\leq D_{n_1,n_2}^{\alpha_{n_1},\alpha_{n_2}}(|\Delta f[t,s;x,y]|;q_{n_1},q_{n_2},x,y)\\
&\quad\leq M D_{n_1,n_2}^{\alpha_{n_1},\alpha_{n_2}}(|t-x|^{\beta}|s-y|^{\gamma};q_{n_1},q_{n_2},x,y)\\
&\quad= M D_{n_1}^{\alpha_{n_1}}(|t-x|^{\beta};q_{n_1},x,)D_{n_2}^{\alpha_{n_2}}(|s-y|^{\gamma};q_{n_2},x).
\end{aligned}$$

Applying Hölder's inequality with $p_1=2/\beta, q_1=2/\left(2-\beta\right)$ and $p_2=2/\gamma$, $q_2=2/\left(2-\gamma\right)$, we have

$$\begin{aligned}
&|G_{n_1,n_2}^{\alpha_{n_1},\alpha_{n_2}}(f;q_{n_1},q_{n_2},x,y)-f(x,y)|\\
&\quad\leq M\left(D_{n_1}^{\alpha_{n_1}}((t-x)^2;q_{n_1},x,)\right)^{\beta/2}\times\left(D_{n_2}^{\alpha_{n_2}}((s-y)^2;q_{n_2},y)\right)^{\gamma/2}\\
&\quad\leq M\left(\delta_{n_1,q_{n_1}}^{\alpha_{n_1}}(x)\right)^{\beta}\left(\delta_{n_2,q_{n_2}}^{\alpha_{n_2}}(y)\right)^{\gamma}.
\end{aligned}$$

□

Theorem 12. *Let $f\in D_b(I^2)$ with $D_B f\in B(I^2)$. Then, for each $(x,y)\in I$, we have*

$$\begin{aligned}
&|G_{n_1,n_2}^{\alpha_{n_1},\alpha_{n_2}}f;q_{n_1},q_{n_2},x,y-f(x,y)|\\
&\quad\leq\frac{M}{[n_1]_{q_{n_1}}^{1/2}[n_2]_{q_{n_2}}^{1/2}}\left(||D_B f||_\infty+\omega_{mixed}(D_B f;[n_1]_{q_{n_1}}^{-1/2},[n_2]_{q_{n_2}}^{-1/2})\right).
\end{aligned}$$

Proof. Since $f\in D_b(I^2)$ and $D_B f\in B(I^2)$, then from

$$D_B f(x,y)=\lim_{(x,y)\to(x_0,y_0)}\frac{\Delta f[(t,s);(x,y)]}{(t-x)(s-y)}$$

it follows that

$$\Delta f[t,s;x,y]=(t-x)(s-y)D_B f(\xi,\eta),$$

where $\xi,\ \eta$ lie between t and x and s and y, respectively.

Since $D_B f\in B(I^2)$ and using the following relation

$$D_B f(\xi,\eta)=\Delta D_B f(\xi,\eta)+D_B f(\xi,y)+D_B f(x,\eta)-D_B f(x,y),$$

we obtain

$$\begin{aligned}
&|D_{n_1,n_2}^{\alpha_{n_1},\alpha_{n_2}}(\Delta f[t,s;x,y];q_{n_1},q_{n_2},x,y)|\\
&\quad= |D_{n_1,n_2}^{\alpha_{n_1},\alpha_{n_2}}((t-x)(s-y)D_Bf(\xi,\eta);q_{n_1},q_{n_2},x,y)|\\
&\quad\le D_{n_1,n_2}^{\alpha_{n_1},\alpha_{n_2}}(|t-x||s-y||\Delta D_Bf(\xi,\eta)|;q_{n_1},q_{n_2},x,y)\\
&\qquad+D_{n_1,n_2}^{\alpha_{n_1},\alpha_{n_2}}(|t-x||s-y|(|D_Bf(\xi,y)|\\
&\qquad+|D_Bf(x,\eta)|+|D_Bf(x,y)|);q_{n_1},q_{n_2},x,y)\\
&\quad\le D_{n_1,n_2}^{\alpha_{n_1},\alpha_{n_2}}(|t-x||s-y|\omega_{mixed}(D_Bf;|\xi-x|,|\eta-y|);q_{n_1},q_{n_2},x,y)\\
&\qquad+3\;||D_Bf||_\infty\; D_{n_1,n_2}^{\alpha_{n_1},\alpha_{n_2}}(|t-x||s-y|;q_{n_1},q_{n_2},x,y).
\end{aligned}$$

Using the basic properties of ω_{mixed}, we have

$$\begin{aligned}
\omega_{mixed}(D_Bf;|\xi-x|,|\eta-y|)&\le \omega_{mixed}(D_Bf;|t-x|,|s-y|)\\
&\le (1+\delta_{n_1}^{-1}|t-x|)(1+\delta_{n_2}^{-1}|s-y|)\;\omega_{mixed}(D_Bf;\delta_{n_1},\delta_{n_2}).
\end{aligned}$$

Hence applying the Cauchy-Schwarz inequality, we get

$$\begin{aligned}
&|G_{n_1,n_2}^{\alpha_{n_1},\alpha_{n_2}}f;q_{n_1},q_{n_2},x,y-f(x,y)|\\
&\quad= |D_{n_1,n_2}^{\alpha_{n_1},\alpha_{n_2}}\Delta f[t,s;x,y];q_{n_1},q_{n_2},x,y|\\
&\quad\le 3||D_Bf||_\infty\sqrt{D_{n_1,n_2}^{\alpha_{n_1},\alpha_{n_2}}((t-x)^2(s-y)^2;q_{n_1},q_{n_2},x,y)}\\
&\qquad+\Big(D_{n_1,n_2}^{\alpha_{n_1},\alpha_{n_2}}(|t-x||s-y|;q_{n_1},q_{n_2},x,y)\\
&\qquad+\delta_{n_1}^{-1}D_{n_1,n_2}^{\alpha_{n_1},\alpha_{n_2}}((t-x)^2|s-y|;q_{n_1},q_{n_2},x,y)\\
&\qquad+\delta_{n_2}^{-1}D_{n_1,n_2}^{\alpha_{n_1},\alpha_{n_2}}(|t-x|(s-y)^2;q_{n_1},q_{n_2},x,y)\\
&\qquad+\delta_{n_1}^{-1}\delta_{n_2}^{-1}D_{n_1,n_2}^{\alpha_{n_1},\alpha_{n_2}}((t-x)^2(s-y)^2;q_{n_1},q_{n_2},x,y)\Big)\omega_{mixed}(D_Bf;\delta_{n_1},\delta_{n_2})\\
&\quad\le 3||D_Bf||_\infty\sqrt{D_{n_1,n_2}^{\alpha_{n_1},\alpha_{n_2}}((t-x)^2(s-y)^2;q_{n_1},q_{n_2},x,y)}\\
&\qquad+\Big(\sqrt{D_{n_1,n_2}^{\alpha_{n_1},\alpha_{n_2}}((t-x)^2(s-y)^2;q_{n_1},q_{n_2},x,y)}\\
&\qquad+\delta_{n_1}^{-1}\sqrt{D_{n_1,n_2}^{\alpha_{n_1},\alpha_{n_2}}((t-x)^4(s-y)^2;q_{n_1},q_{n_2},x,y)}\\
&\qquad+\delta_{n_2}^{-1}\sqrt{D_{n_1,n_2}^{\alpha_{n_1},\alpha_{n_2}}((t-x)^2(s-y)^4;q_{n_1},q_{n_2},x,y)}\\
&\qquad+\delta_{n_1}^{-1}\delta_{n_2}^{-1}D_{n_1,n_2}^{\alpha_{n_1},\alpha_{n_2}}((t-x)^2(s-y)^2;q_{n_1},q_{n_2},x,y)\Big)\omega_{mixed}(D_Bf;\delta_{n_1},\delta_{n_2}).
\end{aligned}$$

We observe that for $(x,y),(t,s)\in I$ and $i,j\in\{1,2\}$

$$D_{n_1,n_2}^{\alpha_{n_1},\alpha_{n_2}}((t-x)^{2i}(s-y)^{2j};q_m,q_n,x,y)=D_{n_1}^{\alpha_1}((t-x)^{2i};q_m,x,y)D_{n_2}^{\alpha_2}((s-y)^{2j};q_n,x,y).$$

Now taking

$$\delta_{n_1}=\delta_{n_1,q_{n_1}}^{\alpha_{n_1}}(x)=\frac{C_1}{[n_1]_{q_{n_1}}^{1/2}},\;\;\delta_{n_2}=\delta_{n_2,q_{n_2}}^{\alpha_{n_2}}(y)=\frac{C_2}{[n_2]_{q_{n_2}}^{1/2}},$$

we obtain the desired result. □

Acknowledgement

The first author is grateful to the Ministry of Human Resource and Development, India for financial assistance to carry out the above research and the second author was financed by Lucian Blaga University of Sibiu research grants LBUS-IRG-2017-03.

References

[1] T. Acar, A. Aral, S. A. Mohiuddine, *Approximation by bivariate (p, q)-Bernstein-Kantorovich operators*, Iranian J. Sci. Tech. Trans. A: Sci. (2016), doi: 10.1007/s40995-016-0045-4.

[2] G. A. Anastassiou, S. Gal, *Approximation Theory: Moduli of Continuity and Global Smoothness Preservation*, Birkhäuser, Boston, 2000.

[3] A. Aral, V. Gupta, P. Agarwal Ravi, *Application of q-Calculus in Operator Theory*, Springer, New York, 2013.

[4] C. Badea, *K-functionals and moduli of smoothness of functions defined on compact metric spaces*, Comput. Math. Appl. **30**(1995), 23–31.

[5] C. Badea, C. Cottin, *Korovkin-type theorems for Generalised Boolean Sum operators*, Colloquia Mathematica Societatis János Bolyai, Approximation Theory, Kecskemét (Hungary) **58**(1990), 51–67.

[6] C. Badea, I. Badea, H. Gonska, *A test function theorem and approximation by pseudopolynomials*, Bull. Austral. Math. Soc. **34**(1986), 53–64.

[7] C. Badea, I. Badea, H. Gonska, *Notes on the degree of approximation of $B-$continuous and $B-$differentiable functions*, J. Approx. Theory Appl. **4**(1988), 95–108.

[8] D. Barbosu, *GBS operators of Schurer-Stancu type*, Annals of University of Craiova, Math. Comp. Sci. Ser. **30**(2003), 34–39.

[9] D. Barbosu, C. V. Muraru, *Approximating B-continuous functions using GBS operators of Bernstein-Schurer-Stancu type based on q-integers*, App. Math. Comput. **259**(2015), 80–87.

[10] I. Barsan, P. Braica, M. Farcas, *About approximation of B-continuous functions of several variables by generalized Boolean sum operators of Bernstein type on a simplex*, Creat. Math. Inform. **20**(2011), 20–23.

[11] K. Bögel, *Mehrdimensionale Differentiation von Functionen mehrerer Veränderlicher*, J. Reine Angew. Math. **170**(1934), 197–217.

[12] K. Bögel, *Über die mehrdimensionale differentiation, integration und beschränkte variation*, J. Reine Angew.Math. **173**(1935), 5–29.

[13] K. Bögel, *Über die mehrdimensionale differentiation*, Jahresber. Deutsch. Math.-Verein. **65**(1962), 45–71.

[14] C. Cottin, *Mixed K-functionals: A measure of smoothness for blending-type approximation*, Math. Z. **204**(1990), 69–83.

[15] E. Dobrescu, I. Matei, *The approximation by Bernstein type polynomials of bidimensionally continuous functions*, An. Univ. Timişoara Ser. Sti. Mat.-Fiz. **4**(1966), 85–90. (in Romanian)

[16] A. Erencin, G. B. Tunca, F. Taşdelen, *Kantorovich type q-Bernstein-Stancu operators*, Stud. Univ. Babeş-Bolyai Math. **57**(2012), 89–105.

[17] M. D. Farcas, *About approximation of B-continuous and B-differentiable functions of three variables by GBS operators of Bernstein type*, Creat. Math. Inform. **17**(2008), 20–27.

[18] M. D. Farca, *About approximation of B-continuous functions of three variables by GBS operators of Bernstein type on a tetrahedron*, Acta Univ. Apulensis Math. Inform. **16**(2008), 93–102.

[19] M. D. Farca, *About approximation of B-continuous and B-differential functions of three variables by GBS operators of Bernstein-Schurer type*, Bul. Ştiinţ. Univ. Politeh. Timis. Ser. Mat. Fiz. **66**(2007), 13–22.

[20] Z. Finta, V. Gupta, *Approximation by q-Durrmeyer operators*, J Appl. Math. Comput. **29**(2009), 401–415.

[21] V. Gupta, *Some approximation properties of q-Durrmeyer operators*, Appl. Math. Comput. **197**(2008), 172–178.

[22] V. Kac, P. Cheung, *Quantum Calculus*, Universitext, Springer, 2002.

[23] A. Lupas, *A q-analogue of the Bernstein operators*, Seminar on Numerical and Statistical Calculas, University of Cluj-Napoca **9**(1987), 85–92.

[24] T. Neer, P. N. Agrawal, S. Araci, *Stancu-Durrmeyer operators based on q-integers*, Appl. Math. Inf. Sci. **11**(2017), 1–9.

[25] N. I. Mahmudov, H. Kaffaoğlu, *On q-Szász-Durrmeyer operators*, Cent. Eur. J. Math. **8**(2010), 399–409.

[26] G. Nowak, *Approximation properties for generalized q-Bernstein polynomials*, J. Math. Anal. Appl. **350**(2009), 50–55.

[27] G. Nowak, V. Gupta, *The rate of pointwise approximation of positive linear operators based on q-integer*, Ukrainian Math. J. **63**(2011), 403–415.

[28] G. M. Phillips, *Bernstein polynomials based on the q-integers*, Ann. Numer. Math. **4**(1997), 511–518.

[29] O. T. Pop, *Voronovskaja-type theorem for certain GBS operators*, Glas. Mat. Ser. III **43**(2008), 179–194.

[30] O. T. Pop, *Approximation of B-continuous and B-differentiable functions by GBS operators defined by infinite sum*, JIPAM J. Inequal. Pure Appl. Math. **10**(2009), Article 7, 8 pp.

[31] O. T. Pop, M. D. Farcas, *About the bivariate operators of Kantorovich type*, Acta Math. Univ. Comenian. (N.S.) **78**(2009), 43–52.

[32] C. Radu, *Statistical approximation properties of Kantorovich operators based on q-integers*, Creat. Math. Inform. **17**(2008), 75–84.

[33] M. Sidharth, N. Ispir, P. N. Agrawal, *GBS operators of Bernstein-Schurer-Kantorovich type based on q-integers*, Appl. Math. Comput. **269**(2015), 558–568.

[34] D. D. Stancu, *Approximation of functions by a new class of linear polynomial operators*, Rev. Roum. Math. Pures Appl. **13**(1968), 1173–1194.

[35] V. I. Volkov, *On the convergence of sequences of linear positive operators in the space of continuous functions of two variables*, Dokl. Akad. Nauk SSSR (NS) **115**(1957), 17–19. (in Russian)

[36] X. M. Zeng, U. Abel, M. Ivan, *Approximation by a Kantorovich variant of the Bleimann, Butzer and Hahn operators*, Math. Inequal. Appl, **11**(2008), 317–325.