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Factorial-like values in the balancing sequence

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Abstract. In this paper, we solve a few Diophantine equations linked to balancing numbers and factorials. The basic problem consists of solving the equation $B_y = x!$ in positive integers x, y, which has only one nontrivial solution $B_2 = 6 = 3!$, as a direct consequence of the theorem of F. Luca [5]. A more difficult problem is to solve $B_y = x_2!/x_1!$, but we were able to handle it under some conditions. Two related problems are also studied.

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1. Introduction

The balancing sequence $\{B_n\}$ is given by $B_0 = 0$ and $B_1 = 1$, and by the recursive rule $B_n = 6B_{n-1} - B_{n-2}$ for $n \geq 2$. The n^{th} element of the associate sequence of $\{B_n\}$ is denoted by C_n, which satisfies the recurrence relation $C_n = 6C_{n-1} - C_{n-2}$ $(n \geq 2)$, where the initial values are $C_0 = 2$ and $C_1 = 6$. The elements of the sequence $\{C_n\}$ are often called Lucas-balancing numbers. Note that

$$B_{2n} = B_n C_n, \tag{1}$$

and

$$C_n^2 - 32B_n^2 = 4. \tag{2}$$

These two identities can be obtained similarly to those for Fibonacci and Lucas numbers. Observe that

$$B_1 = 1 = 1!, \quad C_0 = 2 = 2!, \quad B_2 = C_1 = 6 = 3!, \tag{3}$$

so the question arises naturally whether there are other factorial values in $\{B_n\}$ or in $\{C_n\}$. More generally, one may claim the solutions to the Diophantine equations

$$B_y = x_1! \cdot x_2! \cdots x_r!, \qquad C_y = x_1! \cdot x_2! \cdots x_r!, \tag{4}$$

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in the positive integers $x_1, \ldots, x_r$, and y. Today it is an easy question since Luca ([5], Theorem 4) proved the following theorem. A Lucas sequence $\{u_n\}$ is a non-degenerate binary recurrence with the initial values $u_0 = 0$, $u_1 = 1$. Let $\mathcal{PF}$ be the set of all positive integers which can be written as a product of factorials.

Theorem 1 (see [5], Theorem 4). *Let $(u_n)_{n\geq 0}$ be a Lucas sequence. Let α and β denote the two roots of the characteristic equations. Suppose that $|\alpha| \geq |\beta|$. If $|u_n| \in \mathcal{PF}$, then*

$$y \leq \max\{12, 2e|\alpha| + 1\}. \tag{5}$$

The same upper bound is true for the associate sequence of $\{u_n\}$. Luca used deep algebraic number theoretical considerations and the Baker method.

In the case of balancing a sequence and its associate sequence, the zeros of the characteristic polynomial $x^2 - 6x + 1$ are $\alpha = 3 + 2\sqrt{2}$ and $\beta = 3 - 2\sqrt{2}$. Thus, by (5) we obtain $y \leq 32$. Using a brute force algorithm, computer search provides only (3) as all the solutions to (4). (All of them are single factorial terms.)

For the Fibonacci sequence given by $F_0 = 0$, $F_1 = 1$, and $F_n = F_{n-1} + F_{n-2}$, Luca's bound is only 53, and $u_y = x_1! \cdot x_2! \cdots x_r!$ is fulfilled in the cases

$$F_1 = F_2 = 1!, \quad F_3 = 2!, \quad F_6 = (2!)^3, \quad F_{12} = (2!)^2(3!)^2 = 3! \cdot 4!.$$

Consider now the Tribonacci sequence defined by $T_0 = 0$, $T_1 = T_2 = 1$ and by $T_n = T_{n-1} + T_{n-2} + T_{n-3}$. The equation

$$T_y = x!$$

was solved by Marques and Lengyel [4], and it showed that the only solutions are $(y, x) = (1, 1)$, $(2, 1)$, $(3, 2)$, $(7, 4)$. Their proof is based on the determination of the 2-adic order of Tribonacci numbers. The p-adic order of the non-zero integer n denoted by $\nu_p(n)$ is defined by the exponent of the highest power of prime p dividing n.

The main purpose of this paper is to investigate the solvability of three Diophantine equations linked to factorials and balancing numbers:

$$B_y = \frac{x_2!}{x_1!}, \qquad B_y = \frac{x_2!}{x_1}, \qquad B_y = x_1 x_2!,$$

in positive integers x_1, x_2 and y, under some conditions on x_1 and x_2.

One important argument, which will be used later, is the characterization of the 2-adic order of balancing numbers as it has been already obtained for Fibonacci numbers by Lengyel [3].

Theorem 2. *For $n \geq 1$, we have*

$$\nu_2(B_n) = \begin{cases} 0, & \text{if } n \equiv 1 \pmod 2 \\ 1, & \text{if } n \equiv 2 \pmod 4 \\ \nu_2(n), & \text{if } n \equiv 0 \pmod 4 \end{cases}.$$

Precise results are formulated in the next three theorems. We note that in Theorem 3 the value $\delta = 0.98$ and in Theorem 5 the value $K = 10^6$ were choosen to carry out precise calculations. Our method, at least in theory, works for arbitrary $0 < \delta < 1$ and $K \geq 1$.

Theorem 3. *Unless* $(x_1, x_2, y) = (1, 3, 2)$*, the Diophantine equation*

$$B_y = \frac{x_2!}{x_1!},$$

in positive integers y*,* x_1*,* x_2 *with* $x_1 + 2 \leq x_2$ *has no solution in the folowing cases:*

- $x_1 \leq 0.98x_2$*, or*
- $x_1 = x_2 - \eta$*, where* $\eta \in \{2, 3, 4\}$*.*

Theorem 4. *The only solutions of the Diophantine equation*

$$B_y = \frac{x_2!}{x_1}$$

in positive integers x_1*,* x_2 *and* y *with* $x_1 \leq x_2$ *are*

$$(x_1, x_2, y) = (1, 1, 1),\ (2, 2, 1),\ (1, 3, 2),\ (4, 4, 2).$$

Theorem 5. *The only solutions of the Diophantine equation*

$$B_y = x_1 x_2!$$

in positive integers x_1*,* x_2 *and* y *with the condition* $x_1 \leq 10^6 x_2$ *are*

$$\begin{aligned}(x_1, x_2, y) = \ & (1,1,1),\ (6,1,2),\ (3,2,2),\ (1,3,2),\ (35,1,3),\\ & (204,1,4),\ (102,2,4),\ (34,3,4),\ (1189,1,5),\ (6930,1,6),\\ & (3465,2,6),\ (1155,3,6),\ (40391,1,7),\ (235416,1,8),\\ & (117708,2,8),\ (39236,3,8),\ (9809,4,8),\ (1332869,3,10).\end{aligned}$$

2. Preliminaries

In this section, we present several lemmas which help us to prove the theorems.

Lemma 1. *If* $n \geq 2$*, then*

$$\alpha^{n-1} < B_n,$$

where α *is the larger zero in absolute value of the characteristic polynomial of the sequence* $\{B_n\}$*.*

Proof. See Lemma 4 in [1]. □

Theorem 2 is an immediate consequence of the following lemma; it states a bit more when 4 does not divide n.

Lemma 2. *Let n be a positive integer.*

1. $B_n \equiv n \pmod 4$.

2. *Let $n = 2^s r$ for some integers $s \geq 2$ and odd r. Then* $B_{2^s r} \equiv 2^s r \pmod{2^{s+1}}$.

Proof. (1) It is an easy consequence of considering the sequence of balancing numbers B_n modulo 4.

(2) We use induction on s. Assume $s = 2$. The balancing sequence modulo 8 begins with

$$0, 1, 6, 3, 4, 5, 2, 7, 0, 1, \ldots.$$

Clearly, the length of the period is 8 and $B_{4r} \equiv 4 \equiv 4r \pmod 8$ (note that here r is odd).

Suppose that the statement is true for some $s \geq 2$ and r odd, i.e. $B_{2^s r} = 2^s r + 2^{s+1} k$ holds for some positive integer k. It is easy to see that $C_n \equiv 2 \pmod 4$. Thus $C_n = 4u_n + 2$ for some positive integer sequence $\{u_n\}$. Applying identity (1), we have

$$\begin{aligned} B_{2^{s+1} r} &= C_{2^s r} B_{2^s r} = (4u_{2^s r} + 2)\left(2^s r + 2^{s+1} k\right) \\ &= 2^{s+1} r + 2^{s+2}\left(k + u_{2^s r} r + 2u_{2^s r} k\right) \\ &\equiv 2^{s+1} r \pmod{2^{s+2}}, \end{aligned}$$

and the proof of the lemma is complete (and Theorem 2 follows). □

Let $s_p(k)$ denote the sum of the base-p digits of the positive integer k.

Lemma 3 (Legendre)**.** *For any integer $k \geq 1$ and p prime, we have*

$$\nu_p(k!) = \frac{k - s_p(k)}{p-1}.$$

Proof. See [2]. □

The result of Legendre has the following consequence.

Corollary 1. *For any integer $k \geq 2$ and prime p the inequalities*

$$\frac{k}{p-1} - \frac{\log k}{\log p} - 1 \;\leq\; \nu_p(k!) \;\leq\; \frac{k-1}{p-1}$$

hold.

Proof. Consider the maximal and minimal values of $s_p(k)$, respectively. □

3. General approach to the proofs

This approach does not affect the problem $B_y = x_2!/x_1!$ with $x_2 - x_1 = c$, $c \in \{1, 2, 3\}$.

For a given positive integer r and the integer valued function $f(x_1, x_2, \dots, x_r)$ we would like to solve the Diophantine equation

$$B_y = f(x_1, x_2, \dots, x_r) \tag{6}$$

in the positive integers $y, x_1, \dots, x_r$. Recall Theorem 2 to remind us that the value $\nu_2(B_n)$ is rather small. If we are able to give a "good" lower bound for the "sufficiently large" $\nu_2(f(x_1, x_2, \dots, x_r))$, meanwhile we can provide a "good" upper bound for $f(x_1, x_2, \dots, x_r)$, then there is a chance to bound the variables. More precisely, Lemma 1 leads to

$$y < 1 + \frac{\log f(x_1, x_2, \dots, x_r)}{\log \alpha}$$

starting from (6). Theorem 2 implies

$$\nu_2(f(x_1, x_2, \dots, x_r)) = \nu_2(B_y) \le \nu_2(y) \le \frac{\log y}{\log 2}.$$

Combining the last two formulas, we obtain

$$\nu_2(f(x_1, x_2, \dots, x_r)) < \frac{1}{\log 2} \log \left(1 + \frac{\log f(x_1, x_2, \dots, x_r)}{\log \alpha} \right). \tag{7}$$

We succeed if the comparison of the two sides bounds the variables. This will happen in the following cases:

1. $f(x_1, x_2) = x_2!/x_1!$, with the condition $x_1 \le \delta x_2$ for some $0 < \delta < 1$,
2. $f(x_1, x_2) = x_2!/x_1$ with $x_1 \le x_2$,
3. $f(x_1, x_2) = x_1 x_2!$ with the restriction $x_1 \le K x_2$ for some positive integer $1 \le K$.

4. Proof of the theorems

4.1. Proof of Theorem 3

Case 1. $x_1 + 2 < x_2$ **and** $x_1 \le \delta x_2$ **with a fixed** $0 < \delta < 1$.

Assume that the positive integer solutions x_1, x_2 and y satisfy $x_1 + 2 < x_2$ and $x_1 \le \delta x_2$ with a fixed $0 < \delta < 1$.

Corollary 1 provides

$$\begin{aligned}\nu_2\left(\frac{x_2!}{x_1!}\right) &= \nu_2(x_2!) - \nu_2(x_1!) \ge x_2 - \frac{\log x_2}{\log 2} - 1 - (x_1 - 1) \\ &\ge (1 - \delta)x_2 - \frac{\log x_2}{\log 2}.\end{aligned}$$

On the other hand,

$$\frac{x_2!}{x_1!} \le x_2! \le \left(\frac{x_2}{2}\right)^{x_2}$$

follows where we applied the well-known identity $k! \le (k/2)^k$. The preparation till now enables us to apply (7). It leads to

$$(1-\delta)x_2 - \frac{\log x_2}{\log 2} < \frac{1}{\log 2}\log\left(1+\frac{x_2\log(x_2/2)}{\log\alpha}\right). \tag{8}$$

For fixed δ, it provides an upper bound for x_2. Indeed, if x_2 is large enough, the left-hand side of (8) is positive, further the leading term is linear, while the right-hand side is approximately logarithmic in x_2. For instance, if $\delta = 49/50$, then $x_2 \le 1102$. Making a simple computer verification in the range $3 < x_2 \le 1102$, $1 \le x_1 \le x_2 - 2$, $x_1 \le 49/50 x_2$, according to (2), we find a balancing number if

$$\sqrt{8\left(\frac{x_2!}{x_1!}\right)^2 + 1}$$

is an integer. It occurs only in the case $(x_1, x_2) = (1, 3)$, which gives $B_y = 6$, and then $y = 2$. Taking another example, say $\delta = 1 - 10^{-6}$, we obtain $x_2 < 5.5 \cdot 10^7$. This bound is too large, even to check possible cases by a computer!

Case 2. $x_1 = x_2 - 2$**.**

We have to solve $B_y = x_2\,(x_2 - 1)$. Put $z = C_n$. Then $z^2 = 32x_2^2(x_2-1)^2 + 4$ via $z_1 = z/2$ leads to the equation

$$z_1^2 = 8x_2^4 - 16x_2^3 + 8x_2^2 + 1.$$

To this equation, the Magma procedure

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IntegralQuarticPoints([8,-16,8,0,1]);
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determines the solutions

$$(x_2, z_1) = (-2, \pm 17),\ (0, \pm 1),\ (1, \pm 1),\ (3, \pm 17).$$

Only the last one provides solution to $B_y = x_2(x_2 - 1)$, namely $B_2 = 6 = 3 \cdot 2$, i.e. $(x_2, y) = (3, 2)$.

Case 3. $x_1 = x_2 - 3$**.**

Now, our task is to solve $B_y = x_2(x_2 - 1)(x_2 - 2)$. Let $z = C_n$ and $t = x_2 - 1$. Then we have

$$z^2 = 32(t-1)^2t^2(t+1)^2 + 4 = 32(t^2-1)^2t^2 + 4.$$

Applying $z = 2z_1$ and $t_1 = t^2$, and multiplying the equation by 3^6, together with $t_1 = (T-4)/6$, we arrive at the elliptic equation

$$(27z_1)^2 = T^3 - 108T + 1161. \tag{9}$$

We used Magma (`E:=EllipticCurve([-108,1161]);IntegralPoints(E);`) to solve (9), and we got

$$(T, 27z_1) = (-12, \pm 27),\ (-2, \pm 37),\ (6, \pm 27),\ (15, \pm 54),\ (60, \pm 459).$$

None of them gives a solution to $B_y = x_2(x_2-1)(x_2-2)$ with the given conditions.

Case 4. $x_1 = x_2 - 4$**.**

The corresponding equation is $B_y = x_2(x_2-1)(x_2-2)(x_2-3)$. Put $z = C_n$. Then $z^2 = 32x_2^2(x_2-1)^2(x_2-2)^2(x_2-3)^2+4$ via $z_1 = z/2$ and $t = x_2^2 - 3x_2 + 1$ leads to

$$z_1^2 = 8t^4 - 16t^2 + 9.$$

`IntegralQuarticPoints([8,0,-16,0,9]);` returns with

$$(t, z_1) = (\pm 6, \pm 99),\ (\pm 1, \pm 1),\ (0, \pm 3).$$

Clearly, none of them leads to a solution of $B_y = x_2(x_2-1)(x_2-2)(x_2-3)$.

4.2. Proof of Theorem 4

Here $f(x_1, x_2) = x_2!/x_1$ assuming $x_1 \le x_2$. Thus

$$\nu_2(B_y) = \nu_2(x_2!/x_1) = \nu_2(x_2!) - \nu_2(x_1) \ge x_2 - 1 - 2\frac{\log x_2}{\log 2}.$$

Further

$$\frac{x_2!}{x_1} \le x_2! \le \left(\frac{x_2}{2}\right)^{x_2}$$

follows. Putting them together to apply (7), we obtain

$$x_2 - 1 - 2\frac{\log x_2}{\log 2} < \frac{1}{\log 2}\log\left(1 + \frac{x_2\log(x_2/2)}{\log\alpha}\right).$$

It provides $2 \le x_2 \le 11$. Lastly, we checked the possible values of x_1 and x_2, and found three solutions.

4.3. Proof of Theorem 5

Now we study the function $f(x_1, x_2) = x_1x_2!$ with the restriction $x_1 \le Kx_2$, where $K = 10^6$.

$$\nu_2(B_y) = \nu_2(x_1x_2!) = \nu_2(x_1) + \nu_2(x_2!) \ge x_2 - 1 - \frac{\log x_2}{\log 2}$$

follows by Corollary 1. Also,

$$x_1x_2! \le x_1\left(\frac{x_2}{2}\right)^{x_2}$$

holds, so together with (7) we have

$$x_2 - 1 - \frac{\log x_2}{\log 2} < \frac{1}{\log 2}\log\left(1+\frac{\log x_1 + x_2\log(x_2/2)}{\log\alpha}\right)$$
$$\le \frac{1}{\log 2}\log\left(1+\frac{\log K + \log x_2 + x_2\log(x_2/2)}{\log\alpha}\right).$$

The solution of the inequality above for $K = 10^6$ is $x_2 \le 8$. A computer verification for $B_y = x_1 x_2!$ returns 18 solutions described in the theorem.

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