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Approximation by Bernstein-Chlodowsky operators of max-product kind

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Abstract. We define max-product (nonlinear) Bernstein-Chlodowsky operators and obtain some upper estimates of approximation error for some subclasses of functions. We also investigate the shape-preserving properties for these operators.

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Key words: nonlinear max-product operators, max-product Bernstein-Chlodowsky operators, degree of approximation, shape-preserving properties

1. Introduction

In the approximation theory, well-known Korovkin type theorems are constructed on linear positive approximating operators or functionals. The idea of constructing nonlinear positive operators by means of discrete linear approximating operators was proposed by B. Bede et al. [3, 4]. In these papers, the authors obtained nonlinear Shepard-type operators by replacing the operations sum and product by other pairs. They replaced a pair of operations sum-product with max-product in [3] and sum-product with max-min in [4], respectively. After that the max-product type Bernstein operators were introduced and an open problem was presented by S. G. Gal ([18], pp. 324-326, Open Problem 5.5.4). Due to this open problem the order of approximation of nonlinear approximating operators was investigated in [5]-[10]. In these papers, the order of approximation by nonlinear operators of max-product kind was studied and some shape-preserving properties were obtained and in [11] the authors collected all previous results concerning max-product operators. In recent years, several researchers have made significant contributions to this topic ([16, 20, 21]).

The Bernstein-Chlodowsky polynomials, which are the generalization of the classical Bernstein polynomials, are defined by

$$B_n\left(f;x\right)=\sum_{k=0}^{n}C_n^k\left(\frac{x}{b_n}\right)^k\left(1-\frac{x}{b_n}\right)^{n-k}f\left(\frac{b_nk}{n}\right),$$

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where $b_n \to \infty$ as $n \to \infty$ and $x \in [0, b_n]$, $\lim_{n\to\infty} \frac{b_n}{n} = 0$. The approximation properties of these operators were investigated for univariate and bivariate continuous functions in [1, 17, 19, 23]. Moreover, a q-generalization of Bernstein-Chlodowsky polynomials [15], Voronoskaja-type theorems related to these polynomials [12, 22] and Bezier variant of these polynomials [25] were discussed. Recently, some generalizations of bivariate Chlodowsky polynomials have been studied in [13, 14].

The aim of this paper to introduce nonlinear Bernstein-Chlodowsky operators of max-product kind and estimate the rate of pointwise convergence of these operators. In the present paper, we extend the results obtained in ([5, 6]) from a finite interval to the infinitely growing interval $[0, b_n]$, $b_n \to \infty$ as $n \to \infty$. Moreover, for some subclasses of functions we obtain a better degree of approximation by max-product type Bernstein-Chlodowsky operators than the degree of approximation for linear one given in ([23, 2]; pp. 347-348). In the last section, we investigate the shape-preserving properties of these operators regarding the approximating functions.

2. Preliminaries

We recall some notations and definitions that will be used in this study. The details can be found in [5, 6, 11]. A set of positive real numbers $\mathbb{R}_+$ has a semiring structure with operations "$\vee$" (maximum) and "." (product) and then $(\mathbb{R}_+, \vee, .)$ is called a max-product algebra. Let $I \subset \mathbb{R}$ be a finite or infinite interval, and let $CB_+(I)$ denote the space of all of continuous and bounded functions $f : I \to \mathbb{R}_+$.

A discrete max-product-type approximation operator $L_n : CB_+(I) \to CB_+(I)$ is defined by

$$L_n(f)(x) = \bigvee_{i=0}^{n} K_n(x, x_i).f(x_i),$$

or

$$L_n(f)(x) = \bigvee_{i=0}^{\infty} K_n(x, x_i).f(x_i),$$

where $n \in \mathbb{N}$, $f \in CB_+(I)$, $K_n(., x_i) \in CB_+(I)$ and $x_i \in I$, for all i. These operators are nonlinear positive operators having a pseudo-linearity property, i.e., for every $f, g \in CB_+(I)$ and for any $\alpha, \beta \in \mathbb{R}_+$

$$L_n(\alpha.f \vee \beta.g) = \alpha.L_n(f)(x) \vee \beta.L_n(g)(x).$$

Moreover, max-product operators are positive homogenous, that is, $L_n(\lambda f) = \lambda L_n(f)$ for all $\lambda \geq 0$.

In order to establish the next results, we give the following auxiliary lemma.

Lemma 1 (See [5])**.** *Let $I \subset \mathbb{R}$ be a bounded or unbounded interval and $f \in CB_+(I)$ and let $L_n : CB_+(I) \to CB_+(I), n \in \mathbb{N}$ be a sequence of operators satisfying the following properties:*

(*i*) *if $f, g \in CB_+(I)$ satisfy $f \leq g$, then $L_n(f) \leq L_n(g)$ for all $n \in \mathbb{N}$;*

(*ii*) *$L_n(f+g) \leq L_n(f) + L_n(g)$for all$f, g \in CB_+(I)$.*

Then for all $f,g\in CB_{+}(I)$, $n\in\mathbb{N}$ *and* $x\in I$ *we have*

$$|L_n(f)(x)-L_n(g)(x)|\leq L_n(|f-g|)(x).$$

Notice that max-product type operators satisfy the conditions $(i),(ii)$ in Lemma 1. In fact, for $\alpha=1,\ \beta=1$, they satisfy a pseudo-linearity property which is stronger than the above condition (ii).

3. Construction of the operators and auxiliary results

We define the nonlinear Bernstein-Chlodowsky operators of max-product type as follows:

$$C_n^{(M)}(f)(x)=\frac{\bigvee_{k=0}^{n}h_{n,k}(x)f(\frac{b_nk}{n})}{\bigvee_{k=0}^{n}h_{n,k}(x)}, \tag{1}$$

with

$$h_{n,k}(x)=\binom{n}{k}\left(\frac{x}{b_n}\right)^k\left(1-\frac{x}{b_n}\right)^{n-k},$$

where $0\leq x\leq b_n$ and (b_n) is a sequence of positive real numbers such that

$$\lim_{n\to\infty}b_n=\infty \quad \text{and} \quad \lim_{n\to\infty}\frac{b_n}{\sqrt{n}}=0.$$

The function $f:[0,b_n]\to\mathbb{R}_+$ is a continuous function. It is clear that $C_n^{(M)}(f)(x)$ is obtained from classical linear Bernstein-Chlodowsky operators, replacing the sum operation with the maximum one.

Note that

- For a continuous function $f:[0,b_n]\to\mathbb{R}_+$, the operators $C_n^{(M)}(f)(x)$ are positive and continuous on $[0,b_n]$.
- The operators $C_n^{(M)}(f)(x)$ satisfy the pseudo-linearity property and these operators also are positive homogenous.
- Since $C_n^{(M)}(f)(0)-f(0)=0$ for all n, we may suppose throughout the paper that $0<x\leq b_n$.

Furthermore, we provide an error estimate for the operators $C_n^{(M)}(f)$ given by (1) in terms of the modulus of continuity.

We need the following notations and lemmas for the proofs of the main results.

For each $k,j\in\{0,1,2,...,n\}$ and $x\in\left[\frac{b_nj}{n+1},\frac{b_n(j+1)}{n+1}\right]$, we will have the following

$$M_{k,n,j}(x)=\frac{h_{n,k}(x)\left|\frac{b_nk}{n}-x\right|}{h_{n,j}(x)},\ m_{k,n,j}(x)=\frac{h_{n,k}(x)}{h_{n,j}(x)}. \tag{2}$$

If $k \geq j+1$, then

$$M_{k,n,j}(x) = \frac{h_{n,k}(x)\left(\frac{b_n k}{n} - x\right)}{h_{n,j}(x)} \tag{3}$$

and if $k \leq j-1$, then

$$M_{k,n,j}(x) = \frac{h_{n,k}(x)\left(x - \frac{b_n k}{n}\right)}{h_{n,j}(x)}. \tag{4}$$

Also, for each $k, j \in \{0,1,2,...,n\}$, $k \geq j+2$ and $x \in \left[\frac{b_n j}{n+1}, \frac{b_n (j+1)}{n+1}\right]$, we will have the following

$$\bar{M}_{k,n,j}(x) = \frac{h_{n,k}(x)\left(\frac{b_n k}{n+1} - x\right)}{h_{n,j}(x)} \tag{5}$$

and for each $k, j \in \{0,1,2,...,n\}$, $k \leq j-2$ and $x \in \left[\frac{b_n j}{n+1}, \frac{b_n (j+1)}{n+1}\right]$, we will have the following

$$\hat{M}_{k,n,j}(x) = \frac{h_{n,k}(x)\left(x - \frac{b_n k}{n+1}\right)}{h_{n,j}(x)}. \tag{6}$$

Lemma 2. *Let* $x \in \left[\frac{b_n j}{n+1}, \frac{b_n (j+1)}{n+1}\right]$. *Then*

(*i*) *for all* $k, j \in \{0,1,2,...,n\}$, $k \geq j+2$, *we have*

$$\bar{M}_{k,n,j}(x) \leq M_{k,n,j}(x) \leq 3\bar{M}_{k,n,j}(x); \tag{7}$$

(*ii*) *for all* $k, j \in \{0,1,2,...,n\}$, $k \leq j-2$, *we have*

$$M_{k,n,j}(x) \leq \hat{M}_{k,n,j}(x) \leq 6M_{k,n,j}(x). \tag{8}$$

Proof. (i): By (3) and (5), it is obvious that $\bar{M}_{k,n,j}(x) \leq M_{k,n,j}(x)$.

Additionally,

$$\begin{aligned}\frac{M_{k,n,j}(x)}{\bar{M}_{k,n,j}(x)} &= \frac{\frac{b_n k}{n} - x}{\frac{b_n k}{n+1} - x} \leq \frac{\frac{b_n k}{n} - \frac{b_n j}{n+1}}{\frac{b_n k}{n+1} - \frac{b_n (j+1)}{n+1}} \\ &= \frac{b_n\left(kn + k - nj\right)}{nb_n\left(k-j-1\right)} = \frac{k-j}{k-j-1} + \frac{k}{n\left(k-j-1\right)} \leq 3,\end{aligned}$$

which implies (i).

(ii): By (4) and (6), it is obvious that $M_{k,n,j}(x) \leq \hat{M}_{k,n,j}(x)$.

Additionally,

$$\begin{aligned}\frac{\hat{M}_{k,n,j}(x)}{M_{k,n,j}(x)} &= \frac{x - \frac{b_n k}{n+1}}{x - \frac{b_n k}{n}} \leq \frac{\frac{b_n (j+1)}{n+1} - \frac{b_n k}{n+1}}{\frac{b_n j}{n+1} - \frac{b_n k}{n}} \\ &= \frac{nb_n\left(j+1-k\right)}{b_n\left(nj - nk - k\right)} \leq \frac{(n+1)\left(j+1-k\right)}{nj - nk - n} = \frac{n+1}{n} \cdot \frac{j+1-k}{j-k-1} \\ &\leq 2 \cdot \frac{j+1-k}{j-k-1} = 2\left(1 + \frac{2}{j-k-1}\right) \leq 6,\end{aligned}$$

which implies (ii). □

Lemma 3. *For all* $k,j\in\{0,1,2,...,n\}$ *and* $x\in\left[\frac{b_n j}{n+1},\frac{b_n(j+1)}{n+1}\right]$, *we have*

$$m_{k,n,j}(x)\leq 1.$$

Proof. We have two cases: 1) $k\geq j$ and 2)$k\leq j$.

Let $k\geq j$. Since the function $g(x)=\frac{b_n-x}{x}$ is nonincreasing on $\left[\frac{b_n j}{n+1},\frac{b_n(j+1)}{n+1}\right]$, it follows

$$\frac{m_{k,n,j}(x)}{m_{k+1,n,j}(x)}=\frac{k+1}{n-k}\cdot\frac{b_n-x}{x}\geq\frac{k+1}{n-k}\cdot\frac{b_n(n-j)}{b_n(j+1)}=\frac{k+1}{n-k}\cdot\frac{n-j}{j+1}\geq 1,$$

which implies

$$m_{j,n,j}(x)\geq m_{j+1,n,j}(x)\geq m_{j+2,n,j}(x)\geq ...\geq m_{n,n,j}(x).$$

We now turn to the case $k\leq j$.

$$\begin{aligned}\frac{m_{k,n,j}(x)}{m_{k-1,n,j}(x)}&=\frac{n-k+1}{k}\cdot\frac{x}{b_n-x}\geq\frac{n-k+1}{k}\cdot\frac{\frac{b_n j}{n+1}}{b_n-\frac{b_n j}{n+1}}\\&=\frac{n-k+1}{k}\cdot\frac{j}{n+1-j}\geq 1,\end{aligned}$$

which implies

$$m_{j,n,j}(x)\geq m_{j-1,n,j}(x)\geq m_{j-2,n,j}(x)\geq ...\geq m_{0,n,j}(x).$$

Since $m_{j,n,j}(x)=1$, the proof of the lemma is complete. □

Lemma 4. *Let* $x\in\left[\frac{b_n j}{n+1},\frac{b_n(j+1)}{n+1}\right]$.

(*i*) *If* $k\in\{j+2,j+3,...,n-1\}$ *is such that* $k-\sqrt{k+1}\geq j$, *then* $\bar{M}_{k,n,j}(x)\geq\bar{M}_{k+1,n,j}(x)$.

(*ii*) *If* $k\in\{1,2,...j-2\}$ *is such that* $k+\sqrt{k}\leq j$, *then* $\hat{M}_{k,n,j}(x)\geq\hat{M}_{k-1,n,j}(x)$.

Proof. (i): Let $k\in\{j+2,j+3,...,n-1\}$ with $k-\sqrt{k+1}\geq j$. Then we get

$$\frac{\bar{M}_{k,n,j}(x)}{\bar{M}_{k+1,n,j}(x)}=\frac{k+1}{n-k}\cdot\frac{b_n-x}{x}\cdot\frac{\frac{b_n k}{n+1}-x}{\frac{b_n(k+1)}{n+1}-x}.$$

Since the function $\mu(x)=\frac{b_n-x}{x}\cdot\frac{\frac{b_n k}{n+1}-x}{\frac{b_n(k+1)}{n+1}-x}$ is nonincreasing, it follows that

$$\mu(x)\geq\mu\left(\frac{b_n(j+1)}{n+1}\right)=\frac{n-j}{j+1}\cdot\frac{k-j-1}{k-j}\quad\text{for all}\quad x\in\left[\frac{b_n j}{n+1},\frac{b_n(j+1)}{n+1}\right].$$

Then, since the condition $k-\sqrt{k+1}\geq j$ implies $(k+1)(k-j-1)\geq(j+1)(k-j)$, we obtain

$$\frac{\bar{M}_{k,n,j}(x)}{\bar{M}_{k+1,n,j}(x)}\geq\frac{k+1}{n-k}\cdot\frac{n-j}{j+1}\cdot\frac{k-j-1}{k-j}\geq 1.$$

(ii): We get

$$\frac{\hat{M}_{k,n,j}(x)}{\hat{M}_{k-1,n,j}(x)}=\frac{n-k+1}{k}\cdot\frac{x}{b_n-x}\cdot\frac{x-\frac{b_nk}{n+1}}{x-\frac{b_n(k-1)}{n+1}}$$

Since the function $\eta(x)=\frac{x}{b_n-x}\cdot\frac{x-\frac{b_nk}{n+1}}{x-\frac{b_n(k-1)}{n+1}}$ is nondecreasing, it follows that

$$\eta(x)\geq\eta\left(\frac{b_nj}{n+1}\right)=\frac{j}{n+1-j}\cdot\frac{j-k}{j-k+1}\quad\text{for all}\quad x\in\left[\frac{b_nj}{n+1},\frac{b_n(j+1)}{n+1}\right].$$

Then, since the condition $k+\sqrt{k}\leq j$ implies $j(j-k)\geq k(j-k+1)$, we obtain

$$\frac{\hat{M}_{k,n,j}(x)}{\hat{M}_{k-1,n,j}(x)}\geq\frac{n-k+1}{k}\cdot\frac{j}{n+1-j}\cdot\frac{j-k}{j-k+1}\geq 1,$$

which proves the desired result. □

Lemma 5. *Denoting* $h_{n,k}(x)=\binom{n}{k}\left(\frac{x}{b_n}\right)^k\left(1-\frac{x}{b_n}\right)^{n-k}$, *we have*

$$\bigvee_{k=0}^{n}h_{n,k}(x)=h_{n,j}(x),\ \textit{for all}\ x\in\left[\frac{b_nj}{n+1},\frac{b_n(j+1)}{n+1}\right],\ j=0,1,...,n.$$

Proof. Our proof starts with the observation that for fixed $n\in\mathbb{N}$ and $0\leq k<k+1\leq n$, we have

$$0\leq h_{n,k+1}(x)\leq h_{n,k}(x)\quad\text{if and only if}\quad x\in[0,b_n(k+1)/n+1].$$

Let us evaluate

$$0\leq\binom{n}{k+1}\left(\frac{x}{b_n}\right)^{k+1}\left(1-\frac{x}{b_n}\right)^{n-k-1}\leq\binom{n}{k}\left(\frac{x}{b_n}\right)^{k}\left(1-\frac{x}{b_n}\right)^{n-k}.$$

After simplifications, the inequality is equivalent to

$$0\leq\frac{x}{b_n}\left[\binom{n}{k+1}+\binom{n}{k}\right]\leq\binom{n}{k}.$$

Since $\binom{n}{k+1}+\binom{n}{k}=\binom{n+1}{k+1}$, we get

$$0\leq x\leq\frac{b_n(k+1)}{n+1}.$$

By taking $k = 0, 1, ..., n$ in the inequality above, we get

$$h_{n,1}(x) \le h_{n,0}(x), \text{ if and only if } x \in [0, b_n/(n+1)],$$
$$h_{n,2}(x) \le h_{n,1}(x), \text{ if and only if } x \in [0, 2b_n/(n+1)],$$
$$h_{n,3}(x) \le h_{n,2}(x), \text{ if and only if } x \in [0, 3b_n/(n+1)],$$

and

$$h_{n,k+1}(x) \le h_{n,k}(x), \text{ if and only if } x \in [0, b_n (k+1)/(n+1)],$$

and finally

$$h_{n,n-2}(x) \le h_{n,n-3}(x), \text{ if and only if } x \in [0, b_n (n-2)/(n+1)],$$
$$h_{n,n-1}(x) \le h_{n,n-2}(x), \text{ if and only if } x \in [0, b_n (n-1)/(n+1)],$$
$$h_{n,n}(x) \le h_{n,n-1}(x), \text{ if and only if } x \in [0, b_n n/(n+1)].$$

Consequently, we obtain:

if $x \in [0, b_n/(n+1)]$ then $h_{n,k}(x) \le h_{n,0}(x)$, for all $k = 0, 1, ..., n$;
if $x \in [b_n/(n+1), 2b_n/(n+1)]$ then $h_{n,k}(x) \le h_{n,1}(x)$, for all $k = 0, 1, ..., n$;
if $x \in [2b_n/(n+1), 3b_n/(n+1)]$ then $h_{n,k}(x) \le h_{n,2}(x)$, for all $k = 0, 1, ..., n$;

and, in general

if $x \in [b_n n/(n+1), b_n]$ then $h_{n,k}(x) \le h_{n,n}(x)$, for all $k = 0, 1, ..., n$,

which implies the desired result. □

4. Degree of approximation by $C_n^{(M)}(f)(x)$

For estimating a degree of approximation of a function $f \in CB_+(I)$ we use the Shisha-Mond Theorem given for nonlinear max-product type operators in [5, 6].

Theorem 1. *If $f : [0, b_n] \to \mathbb{R}_+$ is a continuous function and $C_n^{(M)}(f)(x)$ are the max-product Bernstein-Chlodowsky operators defined in (1), then the following pointwise estimate holds:*

$$\left| C_n^{(M)}(f)(x) - f(x) \right| \le 12\omega_1\left(f; \frac{b_n}{\sqrt{n+1}} \right), \ \forall n \in \mathbb{N}, \ x \in [0, b_n],$$

where

$$\omega_1(f;\delta) = \sup\{|f(x) - f(y)|; x, y \in [0, b_n], |x-y| \le \delta\}.$$

Proof. Since $C_n^{(M)}(e_0)(x) = 1$, by using the Shisha-Mond Theorem, we get

$$\left| C_n^{(M)}(f)(x) - f(x) \right| \le \left(1 + \frac{1}{\delta_n} C_n^{(M)}(\varphi_x)(x) \right) \omega_1(f; \delta_n), \tag{9}$$

where $\varphi_x(t) = |t - x|$. Therefore, it is enough to estimate only the following term

$$E_n(x) := C_n^{(M)}(\varphi_x)(x) = \frac{\bigvee_{k=0}^{n} h_{n,k}(x) \left|\frac{b_n k}{n} - x\right|}{\bigvee_{k=0}^{n} h_{n,k}(x)}.$$

Let $x \in \left[\frac{b_n j}{n+1}, \frac{b_n (j+1)}{n+1}\right]$, where $j \in \{0, 1, ..., n\}$ is fixed and arbitrary. By Lemma 5 we have

$$E_n(x) = \bigvee_{k=0}^{n} M_{k,n,j}(x).$$

Since for $j = 0$ we get $E_n(x) \leq b_n/n$, for all $x \in [0, b_n/(n+1)]$, we may suppose that $j \in \{1, ..., n\}$. We will find an upper estimate for each $M_{k,n,j}(x)$, where $j \in \{0, 1, ..., n\}$ is fixed, $x \in \left[\frac{b_n j}{n+1}, \frac{b_n (j+1)}{n+1}\right]$ and $k \in \{0, 1, ..., n\}$. The proof will be divided into 3 cases:

$$1)\ k \in \{j-1, j, j+1\}, \quad 2)\ k \geq j+2, \quad \text{and} \quad 3)\ k \leq j-2.$$

Case 1): If $k = j$, then $M_{j,n,j}(x) = \left|\frac{b_n j}{n} - x\right|$. Since $x \in \left[\frac{b_n j}{n+1}, \frac{b_n (j+1)}{n+1}\right]$, it follows immediately that $M_{j,n,j}(x) \leq b_n/(n+1)$.

If $k = j+1$, then $M_{j+1,n,j}(x) = m_{j+1,n,j}(x)\left(b_n(j+1)/n - x\right)$. From Lemma 3, we have $m_{j+1,n,j}(x) \leq 1$, it implies

$$\begin{aligned} M_{j+1,n,j}(x) &\leq b_n(j+1)/n - x \leq b_n(j+1)/n - b_n j/n+1 \\ &= b_n(j+n+1)/n(n+1) \\ &\leq 3b_n/(n+1). \end{aligned}$$

If $k = j-1$, then $M_{j-1,n,j}(x) = m_{j-1,n,j}(x)\left(x - b_n(j-1)/n\right)$. Since by Lemma 3 we have $m_{j-1,n,j}(x) \leq 1$, it implies

$$\begin{aligned} M_{j-1,n,j}(x) &\leq x - b_n(j-1)/n \leq b_n(j+1)/(n+1) - b_n(j-1)/n \\ &= b_n(2n - (j-1))/n(n+1) \\ &\leq 2b_n/(n+1). \end{aligned}$$

Case 2):

Subcase (a): Let $k - \sqrt{k+1} < j$, then

$$\begin{aligned} \bar{M}_{k,n,j}(x) &= m_{k,n,j}(x)\left(\frac{b_n k}{n+1} - x\right) \leq \frac{b_n k}{n+1} - x \leq \frac{b_n k}{n+1} - \frac{b_n j}{n+1} \\ &\leq \frac{b_n k}{n+1} - \frac{b_n\left(k - \sqrt{k+1}\right)}{n+1} = \frac{b_n\sqrt{k+1}}{n+1} \leq \frac{b_n}{\sqrt{n+1}}. \end{aligned}$$

Subcase (b): Now let $k - \sqrt{k+1} \geq j$. Since the function $g(x) = x - \sqrt{x+1}$ is nondecreasing on the interval $[0, b_n]$, it follows that there exist $\bar{k} \in \{0, 1, 2, ..., n\}$ of maximum value, such that $\bar{k}-\sqrt{\bar{k}+1}<j$. Then for $k_1=\bar{k}+1$ we get $k_1-\sqrt{k_1+1}\geq j$ and

$$\bar{M}_{\bar{k}+1,n,j}(x)=m_{\bar{k}+1,n,j}(x)\left(\frac{b_n\left(\bar{k}+1\right)}{n+1}-x\right)\leq\frac{b_n\left(\bar{k}+1\right)}{n+1}-x$$

$$\leq\frac{b_n\left(\bar{k}+1\right)}{n+1}-\frac{b_nj}{n+1}\leq\frac{b_n\left(\bar{k}+1\right)}{n+1}-\frac{b_n\left(\bar{k}-\sqrt{\bar{k}+1}\right)}{n+1}$$

$$=\frac{b_n\left(\sqrt{\bar{k}+1}+1\right)}{n+1}\leq\frac{2b_n}{\sqrt{n+1}}.$$

Also, we have $k_1\geq j+2$. Indeed, this is a consequence of the fact that g is non decreasing and it is easy to see that $g(j+1)<j$.

By Lemma 4(i) it follows that $\bar{M}_{\bar{k}+1,n,j}(x)\geq\bar{M}_{\bar{k}+2,n,j}(x)\geq...\geq\bar{M}_{n,n,j}(x)$. We obtain $\bar{M}_{k,n,j}(x)\leq\frac{2b_n}{\sqrt{n+1}}$ for any $k\in\left\{\bar{k}+1,\bar{k}+2,...,n\right\}$.

Thus, in subcases (a) and subcases (b) we have $\bar{M}_{k,n,j}(x)\leq\frac{2b_n}{\sqrt{n+1}}$. So, from Lemma 2(i), we have $M_{k,n,j}(x)\leq\frac{6b_n}{\sqrt{n+1}}$.

Case 3):

Subcase (a): Let $k+\sqrt{k}\geq j$. Then

$$\hat{M}_{k,n,j}(x)=m_{k,n,j}(x)\left(x-\frac{b_nk}{n+1}\right)\leq\frac{b_n(j+1)}{n+1}-\frac{b_nk}{n+1}$$

$$\leq\frac{b_n\left(k+\sqrt{k}+1\right)}{n+1}-\frac{b_nk}{n+1}$$

$$=\frac{b_n\left(\sqrt{k}+1\right)}{n+1}\leq\frac{b_n(\sqrt{n}+1)}{n+1}\leq\frac{2b_n}{\sqrt{n+1}}.$$

Subcase (b): Now let $k+\sqrt{k}<j$. Let $\widetilde{k}\in\{0,1,2,...,n\}$ be the minimum value such that $\widetilde{k}+\sqrt{\widetilde{k}}\geq j$. Then $k_2=\widetilde{k}-1$ satisfies $k_2+\sqrt{k_2}<j$ and

$$\hat{M}_{\widetilde{k}-1,n,j}(x)=m_{\widetilde{k}-1,n,j}(x)\left(x-\frac{b_n\left(\widetilde{k}-1\right)}{n+1}\right)\leq\frac{b_n(j+1)}{n+1}-\frac{b_n\left(\widetilde{k}-1\right)}{n+1}$$

$$\leq\frac{b_n\left(\widetilde{k}+\sqrt{\widetilde{k}}+1\right)}{n+1}-\frac{b_n\left(\widetilde{k}-1\right)}{n+1}=\frac{b_n\left(\sqrt{\widetilde{k}}+2\right)}{n+1}\leq\frac{3b_n}{\sqrt{n+1}}.$$

Also, in this case we have $j\geq2$, which implies $k_2\leq j-2$.

By Lemma 4(ii), we get $\hat{M}_{\tilde{k}-1,n,j}(x) \geq \hat{M}_{\tilde{k}-2,n,j}(x) \geq ... \geq \hat{M}_{0,n,j}(x)$. Hence we obtain

$$\hat{M}_{k.n,j}(x) \leq \frac{3b_n}{\sqrt{n+1}} \text{ for any } k \leq j-2 \text{ and } x \in \left[\frac{b_n j}{n+1}, \frac{b_n(j+1)}{n+1}\right].$$

Thus, in subcases (a) and subcases (b) we have $\hat{M}_{k.n,j}(x) \leq \frac{3b_n}{\sqrt{n+1}}$. So from Lemma 2(ii), we have $M_{k.n,j}(x) \leq \frac{3b_n}{\sqrt{n+1}}$.

Collecting all the above estimates, we get

$$M_{k.n,j}(x) \leq \frac{6b_n}{\sqrt{n+1}}, \ \forall x \in \left[\frac{b_n j}{n+1}, \frac{b_n(j+1)}{n+1}\right], k = 0, 1, 2, ..., n,$$

which implies that

$$E_n(x) \leq \frac{6b_n}{\sqrt{n+1}}, \ \forall x \in [0, b_n], n \in \mathbb{N},$$

and taking $\delta_n = \frac{6b_n}{\sqrt{n+1}}$ in (9), we obtain the estimate

$$\left|C_n^{(M)}(f)(x) - (f)(x)\right| \leq 12\omega_1\left(f; \frac{b_n}{\sqrt{n+1}}\right), \ \forall n \in \mathbb{N}, \ x \in [0, b_n].$$

□

It is sufficient to make the following observation to obtain a better order of approximation for subclasses of functions f.

For any $k, j \in \{0, 1, 2, ..., n\}$, consider the functions $f_{k,n,j} : \left[\frac{b_n j}{n+1}, \frac{b_n(j+1)}{n+1}\right] \to \mathbb{R}$ defined by

$$f_{k,n,j}(x) = m_{k,n,j}(x) f\left(\frac{b_n k}{n}\right) = \frac{\binom{n}{k}}{\binom{n}{j}}\left(\frac{x}{b_n - x}\right)^{k-j} f\left(\frac{b_n k}{n}\right).$$

Thus, for any $j \in \{0, 1, 2, ..., n\}$ and $x \in \left[\frac{b_n j}{n+1}, \frac{b_n(j+1)}{n+1}\right]$, we can write

$$C_n^{(M)}(f)(x) = \bigvee_{k=0}^{n} f_{k,n,j}(x).$$

Lemma 6. *Let* $f : [0, b_n] \to [0, \infty)$ *such that*

$$C_n^{(M)}(f)(x) = \max\{f_{j,n,j}(x), f_{j+1,n,j}(x)\}, \ \forall x \in \left[\frac{b_n j}{n+1}, \frac{b_n(j+1)}{n+1}\right];$$

then

$$\left|C_n^{(M)}(f)(x) - f(x)\right| \leq 2\omega_1\left(f; \frac{b_n}{n}\right),$$

where $\omega_1(f; \delta) = \sup\{|f(x) - f(y)|; x, y \in [0, b_n], |x - y| \leq \delta\}$.

Proof. We have two cases:

Case (i): Let $x \in \left[\frac{b_n j}{n+1}, \frac{b_n (j+1)}{n+1}\right]$ be fixed such that $C_n^{(M)}(f)(x) = f_{j,n,j}(x)$. Since $-\frac{b_n}{n+1} \leq x - \frac{b_n j}{n} \leq \frac{b_n}{n+1}$ and $f_{j,n,j}(x) = f\left(\frac{b_n j}{n}\right)$, we have

$$\left|C_n^{(M)}(f)(x) - f(x)\right| = \left|f\left(\frac{b_n j}{n}\right) - f(x)\right| \leq \omega_1\left(f; \frac{b_n}{n+1}\right).$$

Case (ii): Let $x \in \left[\frac{b_n j}{n+1}, \frac{b_n (j+1)}{n+1}\right]$ be fixed such that $C_n^{(M)}(f)(x) = f_{j+1,n,j}(x)$. We have two subcases:

Subcase (a): If $C_n^{(M)}(f)(x) \leq f(x)$, then $f_{j,n,j}(x) \leq f_{j+1,n,j}(x) \leq f(x)$ and we get

$$\begin{aligned}\left|C_n^{(M)}(f)(x) - f(x)\right| &= \left|f_{j+1,n,j}(x) - f(x)\right| = f(x) - f_{j+1,n,j}(x) \\ &\leq f(x) - f_{j,n,j}(x) = f(x) - f\left(\frac{b_n j}{n}\right) \\ &\leq \omega_1\left(f; \frac{b_n}{n+1}\right).\end{aligned}$$

Subcase (b): If $C_n^{(M)}(f)(x) > f(x)$, then

$$\begin{aligned}\left|C_n^{(M)}(f)(x) - f(x)\right| &= f_{j+1,n,j}(x) - f(x) = m_{j+1,n,j}(x) . f\left(\frac{b_n (j+1)}{n}\right) - f(x) \\ &\leq f\left(\frac{b_n (j+1)}{n}\right) - f(x).\end{aligned}$$

Since

$$0 \leq \frac{b_n (j+1)}{n} - x \leq \frac{b_n (j+1)}{n} - \frac{b_n j}{n+1} = \frac{b_n j}{n(n+1)} + \frac{b_n}{n} < \frac{2b_n}{n},$$

then

$$f\left(\frac{b_n (j+1)}{n}\right) - f(x) \leq 2\omega_1\left(f; \frac{b_n}{n}\right),$$

which completes the proof. □

Lemma 7. *Let* $f : [0, b_n] \to [0, \infty)$ *such that*

$$C_n^{(M)}(f)(x) = \max\{f_{j,n,j}(x), f_{j-1,n,j}(x)\}, \ \forall x \in \left[\frac{b_n j}{n+1}, \frac{b_n (j+1)}{n+1}\right];$$

then

$$\left|C_n^{(M)}(f)(x) - f(x)\right| \leq 2\omega_1\left(f; \frac{b_n}{n}\right).$$

Proof. We have two cases:

Case (i): Let $C_n^{(M)}(f)(x) = f_{j,n,j}(x)$. We get, as in the proof of Lemma 6,

$$\left|C_n^{(M)}(f)(x) - f(x)\right| \leq \omega_1\left(f; \frac{b_n}{n+1}\right).$$

Case (ii): Let $x \in \left[\frac{b_n j}{n+1}, \frac{b_n(j+1)}{n+1}\right]$ be fixed such that $C_n^{(M)}(f)(x) = f_{j-1,n,j}(x)$. We have two subcases:

Subcase (a): If $C_n^{(M)}(f)(x) \leq f(x)$, then following the proof of Lemma 6, we get

$$C_n^{(M)}(f)(x) - f(x) \leq \omega_1\left(f; \frac{b_n}{n+1}\right).$$

Subcase (b): If $C_n^{(M)}(f)(x) > f(x)$, then

$$\begin{aligned}\left|C_n^{(M)}(f)(x) - f(x)\right| &= f_{j-1,n,j}(x) - f(x) = m_{j-1,n,j}(x) . f\left(\frac{b_n(j-1)}{n}\right) - f(x) \\ &\leq f\left(\frac{b_n(j-1)}{n}\right) - f(x).\end{aligned}$$

Since

$$0 \leq x - \frac{b_n(j-1)}{n} \leq \frac{b_n(j+1)}{n+1} - \frac{b_n(j-1)}{n} = \frac{-b_n j}{n(n+1)} + \frac{b_n}{n+1} < \frac{2b_n}{n},$$

then

$$f\left(\frac{b_n(j-1)}{n}\right) - f(x) \leq 2\omega_1\left(f; \frac{b_n}{n}\right).$$

This completes the proof. □

Lemma 8. *Let* $f : [0, b_n] \to [0, \infty)$ *such that*

$$C_n^{(M)}(f)(x) = \max\{f_{j-1,n,j}(x), f_{j,n,j}(x), f_{j+1,n,j}(x)\}, \ \forall x \in \left[\frac{b_n j}{n+1}, \frac{b_n(j+1)}{n+1}\right];$$

then

$$\left|C_n^{(M)}(f)(x) - f(x)\right| \leq 2\omega_1\left(f; \frac{b_n}{n}\right).$$

Proof. Let $x \in \left[\frac{b_n j}{n+1}, \frac{b_n(j+1)}{n+1}\right]$ be fixed such that $C_n^{(M)}(f)(x) = f_{j,n,j}(x)$ or $C_n^{(M)}(f)(x) = f_{j+1,n,j}(x)$. Then $C_n^{(M)}(f)(x) = \max\{f_{j,n,j}(x), f_{j+1,n,j}(x)\}$ and from Lemma 6 we get

$$\left|C_n^{(M)}(f)(x) - f(x)\right| \leq 2\omega_1\left(f; \frac{b_n}{n}\right).$$

If $C_n^{(M)}(f)(x) = f_{j-1,n,j}(x)$, then $C_n^{(M)}(f)(x) = \max\{f_{j,n,j}(x), f_{j-1,n,j}(x)\}$ and from Lemma 7 we get

$$\left|C_n^{(M)}(f)(x) - f(x)\right| \leq 2\omega_1\left(f; \frac{b_n}{n}\right),$$

which completes the proof. □

Lemma 9 (See [6])**.** *Let $f:[0,b_n]\to[0,\infty)$ be concave. Then the following two properties hold:*

(*i*) *the function $g:(0,b_n]\to[0,\infty)$, $g(x)=f(x)/x$ is nonincreasing;*

(*ii*) *the function $h:[0,b_n)\to[0,\infty)$, $h(x)=f(x)/(b_n-x)$ is nondecreasing.*

Proof. (i): Let $x,y\in(0,b_n]$ such that $x\le y$. Then

$$f(x)=f\left(\frac{x}{y}.y+\frac{y-x}{y}.0\right)\ge\frac{x}{y}f(y)+\frac{y-x}{y}f(0)\ge\frac{x}{y}f(y),$$

which implies $f(x)/x\ge f(y)/y$.

(ii): Let $x,y\in[0,b_n)$ such that $x\ge y$. Then

$$f(x)=f\left(\frac{b_n-x}{b_n-y}.y+\frac{x-y}{b_n-y}.b_n\right)\ge\frac{b_n-x}{b_n-y}f(y)+\frac{x-y}{b_n-y}f(b_n)\ge\frac{b_n-x}{b_n-y}f(y),$$

which implies $f(x)/(b_n-x)\ge f(y)/(b_n-y)$. □

Corollary 1. *Let $f:[0,b_n]\to[0,\infty)$ be a concave function. Then for all $x\in[0,b_n]$*

$$\left|C_n^{(M)}(f)(x)-f(x)\right|\le2\omega_1\left(f;\frac{b_n}{n}\right).$$

Proof. Let $x\in[0,b_n]$ and for $j\in\{0,1,2,...,n\}$, $x\in\left[\frac{b_nj}{n+1},\frac{b_n(j+1)}{n+1}\right]$. Let $k\in\{0,1,2,...,n\}$. If $k\ge j$, then

$$\begin{aligned}f_{k+1,n,j}(x)&=\frac{\binom{n}{k+1}}{\binom{n}{j}}\left(\frac{x}{b_n-x}\right)^{k+1-j}f\left(\frac{b_n(k+1)}{n}\right)\\&=\frac{\binom{n}{k}}{\binom{n}{j}}\frac{n-k}{k+1}\left(\frac{x}{b_n-x}\right)^{k-j}\frac{x}{b_n-x}f\left(\frac{b_n(k+1)}{n}\right).\end{aligned}$$

Since

$$f\left(\frac{b_n(k+1)}{n}\right)/\frac{b_n(k+1)}{n}\le f\left(\frac{b_nk}{n}\right)/\frac{b_nk}{n}$$

(that comes from Lemma 9 (i)), that is,

$$f\left(\frac{b_n(k+1)}{n}\right)\le\left(\frac{k+1}{k}\right)f\left(\frac{b_nk}{n}\right),$$

and since

$$\frac{x}{b_n-x}\le\frac{j+1}{n-j},$$

we get

$$\begin{aligned}f_{k+1,n,j}(x)&\le\frac{\binom{n}{k}}{\binom{n}{j}}\frac{n-k}{k+1}\left(\frac{x}{b_n-x}\right)^{k-j}\frac{j+1}{n-j}\frac{k+1}{k}f\left(\frac{b_nk}{n}\right)\\&=f_{k,n,j}(x)\frac{j+1}{k}\frac{n-k}{n-j}.\end{aligned}$$

By the above inequality for $k \geq j+1$ we get $f_{k,n,j}(x) \geq f_{k+1,n,j}(x)$. Hence

$$f_{j+1,n,j}(x) \geq f_{j+2,n,j}(x) \geq ... \geq f_{n,n,j}(x) \tag{10}$$

If $k \leq j$, then

$$\begin{aligned} f_{k-1,n,j}(x) &= \frac{\binom{n}{k-1}}{\binom{n}{j}} \left(\frac{x}{b_n - x}\right)^{k-1-j} f\left(\frac{b_n (k-1)}{n}\right) \\ &= \frac{\binom{n}{k}}{\binom{n}{j}} \frac{k}{n-k+1} \left(\frac{x}{b_n - x}\right)^{k-j} \frac{b_n - x}{x} f\left(\frac{b_n (k-1)}{n}\right). \end{aligned}$$

Since

$$f\left(\frac{b_n k}{n}\right) / \left(b_n - \frac{b_n k}{n}\right) \geq f\left(\frac{b_n (k-1)}{n}\right) / \left(b_n - \frac{b_n (k-1)}{n}\right)$$

(that comes from Lemma 9 (ii)), that is,

$$f\left(\frac{b_n k}{n}\right) \geq \left(\frac{n-k}{n-k+1}\right) f\left(\frac{b_n (k-1)}{n}\right).$$

Since

$$\frac{b_n - x}{x} \leq \frac{n+1-j}{j},$$

we get

$$\begin{aligned} f_{k-1,n,j}(x) &\leq \frac{\binom{n}{k}}{\binom{n}{j}} \frac{k}{n-k+1} \left(\frac{x}{b_n - x}\right)^{k-j} \frac{n+1-j}{j} \frac{n-k+1}{n-k} f\left(\frac{b_n k}{n}\right) \\ &= f_{k,n,j}(x) \frac{k}{j} \frac{n+1-j}{n-k}. \end{aligned}$$

By the above inequality, for $k \leq j-1$, we get $f_{k,n,j}(x) \geq f_{k-1,n,j}(x)$. Hence

$$f_{j-1,n,j}(x) \geq f_{j-2,n,j}(x) \geq ... \geq f_{0,n,j}(x). \tag{11}$$

We conclude from (10) and (11) that

$$C_n^{(M)}(f)(x) = \max\left\{f_{j-1,n,j}(x), f_{j,n,j}(x), f_{j+1,n,j}(x)\right\},$$

and finally from Lemma 8

$$\left|C_n^{(M)}(f)(x) - f(x)\right| \leq 2\omega_1\left(f; \frac{b_n}{n}\right),$$

which proves the corollary. □

Remark 1. *Compared with the approximation error* $\omega_1\left(f; \frac{b_n}{\sqrt{n}}\right)$ *in [23] given by linear Bernstein-Chlodowsky operators, from Corollary 1 it follows that the approximation order* $\omega_1\left(f; \frac{b_n}{n}\right)$*, given by nonlinear max-product Bernstein-Chlodowsky operators, is essentially better for concave functions.*

5. Shape-preserving properties

In this section, we deal with some shape-preserving properties for max-product type Bernstein-Chlodowsky operators.

As in the previous section, for any $k, j \in \{0, 1, 2, ..., n\}$ consider the functions $f_{k,n,j} : \left[\frac{b_n j}{n+1}, \frac{b_n(j+1)}{n+1}\right] \to \mathbb{R}$ defined by

$$f_{k,n,j}(x) = m_{k,n,j}(x) f\left(\frac{b_n k}{n}\right) = \frac{\binom{n}{k}}{\binom{n}{j}} \left(\frac{x}{b_n - x}\right)^{k-j} f\left(\frac{b_n k}{n}\right).$$

Thus, for any $j \in \{0, 1, 2, ..., n\}$ and $x \in \left[\frac{b_n j}{n+1}, \frac{b_n(j+1)}{n+1}\right]$, we can write

$$C_n^{(M)}(f)(x) = \bigvee_{k=0}^{n} f_{k,n,j}(x).$$

Lemma 10. *Let $f : [0, b_n] \to \mathbb{R}_+$ be a nondecreasing function; then for any $k, j \in \{0, 1, 2, ..., n\}$, $k \leq j$ and $x \in \left[\frac{b_n j}{n+1}, \frac{b_n(j+1)}{n+1}\right]$, we have $f_{k,n,j}(x) \geq f_{k-1,n,j}(x)$.*

Proof. By the proof of Lemma 3, case 2, $m_{k,n,j}(x) \geq m_{k-1,n,j}(x)$ for $k \leq j$. Since f is nondecreasing , it follows that

$$f\left(\frac{b_n k}{n}\right) \geq f\left(\frac{b_n(k-1)}{n}\right).$$

Hence, we get

$$m_{k,n,j}(x) f\left(\frac{b_n k}{n}\right) \geq m_{k-1,n,j}(x) f\left(\frac{b_n(k-1)}{n}\right).$$

This gives that $f_{k,n,j}(x) \geq f_{k-1,n,j}(x)$. □

Corollary 2. *Let $f : [0, b_n] \to \mathbb{R}_+$ be a nonincreasing function; then for any $k, j \in \{0, 1, 2, ..., n\}$, $k \geq j$ and $x \in \left[\frac{b_n j}{n+1}, \frac{b_n(j+1)}{n+1}\right]$, we have $f_{k,n,j}(x) \geq f_{k+1,n,j}(x)$.*

Proof. By the proof of Lemma 3 case 1, $m_{k,n,j}(x) \geq m_{k+1,n,j}(x)$ for $k \geq j$. Since f is nonincreasing, it follows that

$$f\left(\frac{b_n k}{n}\right) \geq f\left(\frac{b_n(k+1)}{n}\right).$$

Hence, we get

$$m_{k,n,j}(x) f\left(\frac{b_n k}{n}\right) \geq m_{k+1,n,j}(x) f\left(\frac{b_n(k+1)}{n}\right).$$

This implies the desired inequlity. □

Theorem 2. *If $f : [0, b_n] \to \mathbb{R}_+$ is a nondecreasing function, then $C_n^{(M)}(f)(x)$ is nondecreasing.*

Proof. Since $C_n^{(M)}(f)(x)$ is continuous on $[0, b_n]$, it is sufficient to show that on each subinterval of the form $\left[\frac{b_n j}{n+1}, \frac{b_n(j+1)}{n+1}\right]$, with $j \in \{0, 1, 2, ..., n\}$, $C_n^{(M)}(f)(x)$ is nondecreasing.

Let $j \in \{0, 1, 2, ..., n\}$ and $x \in \left[\frac{b_n j}{n+1}, \frac{b_n(j+1)}{n+1}\right]$. Since f is nondecreasing, from Lemma 10 we get $f_{j,n,j}(x) \geq f_{j-1,n,j}(x) \geq f_{j-2,n,j}(x) \geq ... \geq f_{0,n,j}(x)$.

For all $x \in \left[\frac{b_n j}{n+1}, \frac{b_n(j+1)}{n+1}\right]$, we can write

$$C_n^{(M)}(f)(x) = \bigvee_{k=j}^{n} f_{k,n,j}(x).$$

For $k \geq j$, since the function $f_{k,n,j}(x)$ is nondecreasing and $C_n^{(M)}(f)(x)$ can be written as the maximum of nondecreasing functions, then $C_n^{(M)}(f)(x)$ is nondecreasing. □

Corollary 3. *If $f : [0, b_n] \to \mathbb{R}_+$ is a nonincreasing function, then $C_n^{(M)}(f)(x)$ is nonincreasing.*

Proof. Since $C_n^{(M)}(f)(x)$ is continuous on $[0, b_n]$, it is sufficient to show that on each subinterval of the form $\left[\frac{b_n j}{n+1}, \frac{b_n(j+1)}{n+1}\right]$, with $j \in \{0, 1, 2, ..., n\}$, $C_n^{(M)}(f)(x)$ is nonincreasing.

Let $j \in \{0, 1, 2, ..., n\}$ and $x \in \left[\frac{b_n j}{n+1}, \frac{b_n(j+1)}{n+1}\right]$. Since f is nonincreasing, from Corollary 2 we get $f_{j,n,j}(x) \geq f_{j+1,n,j}(x) \geq f_{j+2,n,j}(x) \geq ... \geq f_{n,n,j}(x)$.

For all $x \in \left[\frac{b_n j}{n+1}, \frac{b_n(j+1)}{n+1}\right]$, we can write

$$C_n^{(M)}(f)(x) = \bigvee_{k=0}^{j} f_{k,n,j}(x).$$

For $k \leq j$, since the function $f_{k,n,j}(x)$ is nonincreasing and $C_n^{(M)}(f)(x)$ can be written as the maximum of nonincreasing functions, $C_n^{(M)}(f)(x)$ is nonincreasing. □

Remark 2 (See [6, 24])**.** *A continuous function f is quasiconvex on the bounded interval $[0, a]$ if there exists a point $c \in [0, a]$ such that f is nonincreasing on $[0, c]$ and nondecreasing on $[c, a]$.*

Corollary 4. *If $f : [0, b_n] \to \mathbb{R}_+$ is a continuous and quasiconvex function on $[0, b_n]$, then for all $n \in \mathbb{N}$, $C_n^{(M)}(f)(x)$ is quasiconvex on $[0, b_n]$.*

Proof. It is known that a continuous function f is quasiconvex on $[0, b_n]$ if there exists a point $c \in [0, b_n]$ such that f is nonincreasing on $[0, c]$ and nondecreasing on $[c, b_n]$.

If f is a nondecreasing or nonincreasing function on $[0, b_n]$, then by Theorem 2 or by Corollary 3, for all $n \in \mathbb{N}$, $C_n^{(M)}(f)(x)$ is nonincreasing or nondecreasing on $[0, b_n]$.

Now suppose that there exists a point $c \in (0, b_n)$ such that f is nonincreasing on $[0, c]$ and nondecreasing on $[c, b_n]$. The functions $F, G : [0, b_n] \to \mathbb{R}_+$ are defined by $F(x) = f(x)$ for all $x \in [0, c]$, $F(x) = f(c)$ for all $x \in [c, b_n]$ and $G(x) = f(c)$ for all $x \in [0, c]$, $G(x) = f(x)$ for all $x \in [c, b_n]$. It is obvious that F is nonincreasing and continuous on $[0, b_n]$, G is nondecreasing and continuous on $[0, b_n]$ and $f(x) = \max\{F(x), G(x)\}$, for all $x \in [0, b_n]$.

In addition, since $C_n^{(M)}(f)(x)$ is pseudo-linear, we can write for all $x \in [0, b_n]$

$$C_n^{(M)}(f)(x) = \max\left\{C_n^{(M)}(F)(x), C_n^{(M)}(G)(x)\right\}.$$

Hence by Corollary 3 and Theorem 2, $C_n^{(M)}(F)(x)$ is nonincreasing and continuous on $[0, b_n]$, $C_n^{(M)}(G)(x)$ is nondecreasing and continuous on $[0, b_n]$.

Now, we have two cases:

1) $C_n^{(M)}(F)(x)$ and $C_n^{(M)}(G)(x)$ do not intersect each other,

2) $C_n^{(M)}(F)(x)$ and $C_n^{(M)}(G)(x)$ intersect each other.

Case 1): For all $x \in [0, b_n]$,

$$\max\left\{C_n^{(M)}(F)(x), C_n^{(M)}(G)(x)\right\} = C_n^{(M)}(F)(x)$$

or

$$\max\left\{C_n^{(M)}(F)(x), C_n^{(M)}(G)(x)\right\} = C_n^{(M)}(G)(x).$$

Since the class of quasiconvex functions includes a class of nondecreasing functions and a class of nonincreasing functions, we get that $C_n^{(M)}(f)(x)$ is quasiconvex on $[0, b_n]$.

Case 2): If $C_n^{(M)}(F)(x)$ and $C_n^{(M)}(G)(x)$ intersect each other, then there exists a point $c \in [0, b_n]$ such that $C_n^{(M)}(f)(x)$ is nonincreasing on $[0, c]$ and nondecreasing on $[c, b_n]$, which implies that $C_n^{(M)}(f)(x)$ is quasiconvex on $[0, b_n]$. □

Note that, since the class of quasiconvex functions includes the class of nondecreasing functions, the class of nonincreasing functions and the class of convex functions on $[0, b_n]$, Corollary 4 shows that the shape-preserving holds in a wide class of functions.

Next, we illustrate the rate of convergence of the operators $C_n^{(M)}(f)(x)$ to certain functions by graphics. We also compare the convergence of the nonlinear operators $C_n^{(M)}(f)(x)$ and the Bernstein-Chlodowsky operators $B_n(f)(x)$ to a certain function.

Example 1. *Let* $(b_n) = (n^{1/3})$. *For* $n = 30$ *and* $n = 50$, *the convergence of the operators* $C_n^{(M)}(f)(x)$ *to*

$$f(x) = \begin{cases} 0, & \text{if } x = 0 \\ x^2 \sin\frac{1}{x}, & \text{if } x \in (0, b_n] \end{cases}$$

is illustrated in Figures 1 and 2, respectively.

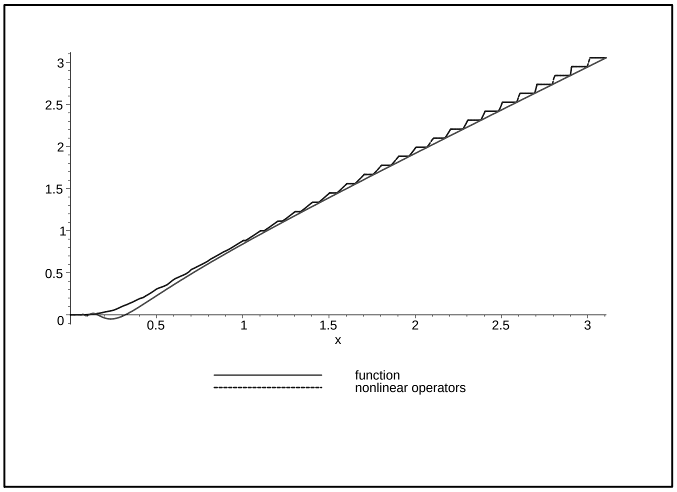

Figure 1:

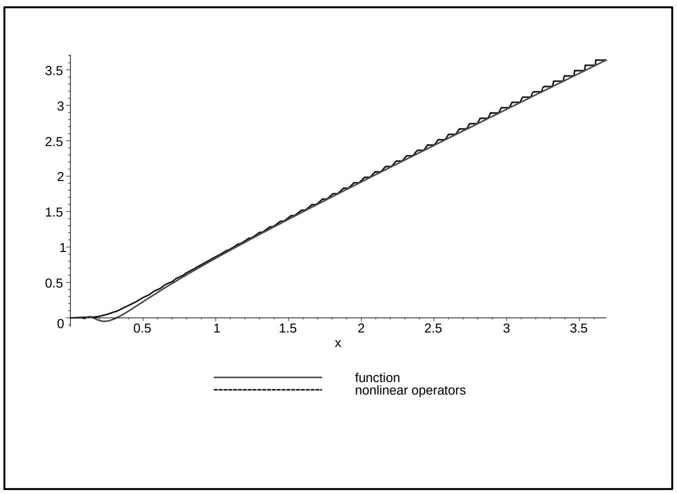

Figure 2:

Example 2. *Let* $(b_n) = (n^{1/5})$. *For* $n = 30$ *and* $n = 50$, *the convergence of the operators* $C_n^{(M)}(f)(x)$ *and* $B_n(f)(x)$ *to* $f(x) = \frac{1}{2} - \left| x - [x] - \frac{1}{2} \right|$ *is illustrated in Figures 3 and 4, respectively.*

From Figures 3 and 4 it is clearly seen that for the corresponding functions, the max-product Bernstein-Chlodowsky operators approximate much better than the linear Bernstein-Chlodowsky operators.

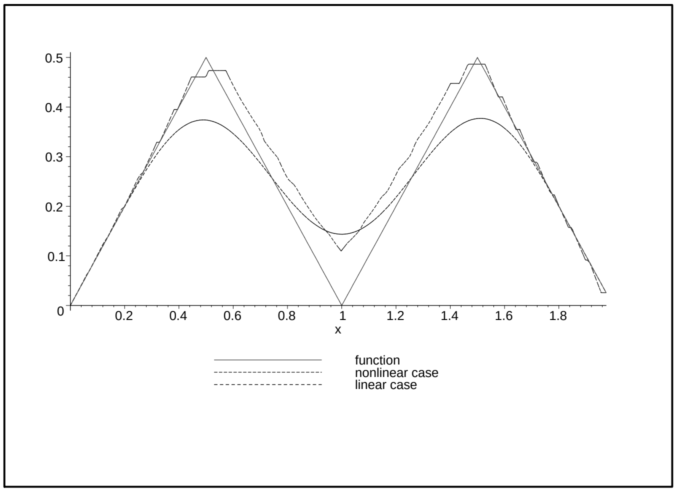

Figure 3:

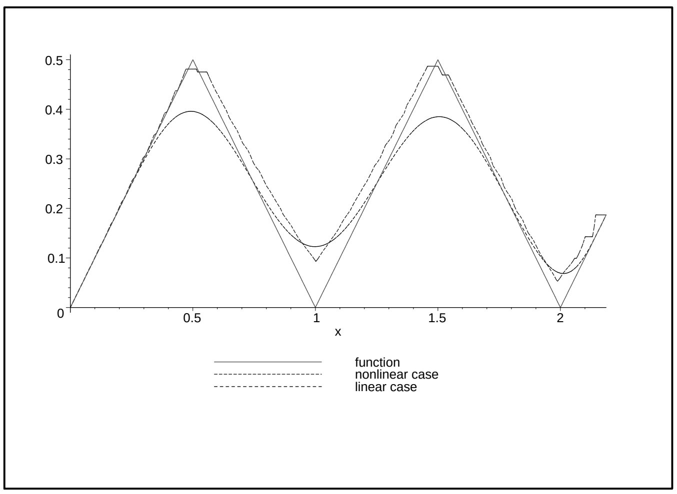

Figure 4:

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