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On solving operator equations by Galerkin's method with Gabor frame

Fatemeh Zarmehi[1] and Ali Tavakoli[1,2,*]

[1] *Department of Mathematics, Vali-e-Asr University of Rafsanjan, P. O. Box 518, 77 188 97 111, Rafsanjan, Iran*
[2] *Department of Mathematics, University of Mazandaran, 57 416 13 534, Babolsar, Iran*

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Abstract. This paper deals with solving boundary value problems by Galerkin's method in which we use Gabor frames as trial and test functions. We show that the preconditioned stiffness matrix resulting from discretization is compressible and its sparsity pattern involves a bounded polyhedron structure. Moreover, we introduce an adaptive Richardson iterative method to solve the resulting system and we also show that by choosing a suitable smoothing parameter, the method would be convergent.
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1. Introduction

Wavelets successfully find applications to general problems such as compression and denoising [2, 15, 22]. They also have broad applications in numerical analysis. For example, the matrices that result from elliptic operator equations involve bounded condition numbers that make numerical solving of such equations efficient. Moreover, they can be applied to derive adaptive numerical schemes guaranteed to converge with optimal order [9, 10, 11]. Of course, in order to solve numerically a boundary value problem by Galerkin's method, using a wavelet basis on this domain is a hard mission [20]. In fact, the construction of these wavelets requires certain matching conditions, which can be difficult to satisfy in practical implementations [30]. One method to cut down on these problems is to use frames instead of wavelet bases, because the frame functions do not necessarily form a basis. Frame theory, especially wavelet frames, were developed a long time (see e.g. [12, 19]). In [5, 11], it has been shown that all advantages of wavelet methods outlined above can be satisfied by the frames. In addition to signal processing, today frame theory plays an important role in various applied areas [3, 4]. Frames are still a highly active field of research in the area of solving boundary value problems [1, 5, 11, 25].

Using a wavelet or a wavelet frame in Galerkin's method can follow linear systems along with (bi-)infinite coefficient matrices. The preconditioned forms of these matrices are compressible and have finger or multi-diagonal structure patterns [17].

*Corresponding author. *Email addresses:* a.tavakoli@umz.ac.ir (A. Tavakoli), f.zarmehi@mail.vru.ac.ir (F. Zarmehi)

In frames, the functions that produce the solution space, do not necessarily form a basis for the space. This redundancy may cause problems in numerical applications since it gives rise to a singular stiffness matrix [10]. However, construction of (wavelet) frames is easier than that of (wavelet) Riesz bases [10]. Hence, frames are better in this regard.

Gabor frames have been studied in time-frequency analysis over the last 30 years. However, most of the applications of Gabor frames are observed in the field of signal processing [12, 14, 16, 19, 21, 24, 26]. In this paper, we use Gabor frames as trial and test functions of Galerkin's method for solving boundary value problems. In Section 3, we show that the linear system resulting from this discretization, can follow the (quad-)infinite coefficient matrices. In other words, a general form of the matrix is $G = (g_{i,j})_{i,j=\ldots,-1,0,1,\ldots}$. In Section 4, we prove that the preconditioned form of these coefficient matrices would be compressible. Also, in this section, we show that the sparsity pattern of the compressible preconditioned matrix involves a bounded polyhedron structure. This implies that the linear system generated by Gabor frames is solved simpler than that of wavelets and wavelet frames. In Section 5, we present an adaptive Richardson iterative scheme to solve the infinite linear system. The convergence and computational complexity of the proposed method is also discussed in Section 5. Finally, in Section 6, two numerical examples are given to support our theoretical results.

Throughout this paper, $< .,. >$ denotes the well-known inner product in the L_2 space and $|X|$ denotes the cardinal number of the set X. The norm of an operator $L : L^2(\mathbb{R}) \longrightarrow L^2(\mathbb{R})$ is defined as follows:

$$\|L\| = \sup_{u \neq 0} \tfrac{\|Lu\|}{\|u\|}.$$

Also, letting $a = (a_1, a_2, \ldots) \in \ell_2$, where $a_i \in \mathbb{R}$, the norm of a, $\|a\|$, is defined as $\|a\| = \sum\limits_{i=1}^{\infty} |a_i|^2$. Moreover, the notation $A \lesssim B$ indicates $A \leq cB$ with a constant $c > 0$, independent of A and B.

2. Preliminaries

Galerkin's method is one of the most powerful approaches for solving boundary value problems. We first shortly explain Galerkin's method:
A variational form of the operator equation $Lu = f$ with given boundary conditions is to find $u \in H = span\{\phi_1, \phi_2, \ldots, \phi_n\}$ (the space of trial functions) such that

$$\langle Lu, v\rangle = \langle f, v\rangle_{L^2(\Omega)}, \qquad \forall v \in H = span\{\phi_1, \phi_2, \ldots, \phi_n\}, \text{(the space of test functions)}$$

where

$$< f, g >= \int_{\Omega} f(x)\bar{g}(x)dx.$$

So,

$$< f, \phi_j >=< Lu, \phi_j >=< L \sum_{i=1}^{n} c_i\phi_i, \phi_j >= \sum_{i=1}^{n} c_i < L\phi_i, \phi_j >, \quad j = 1, \ldots, n.$$

This discretization yields a system $AC = b$, where $A = (L\phi_i, \phi_j)_{1\leq i,j\leq n}$, $C = [c_1, \ldots, c_n]^T$ and $b = [< f, \phi_1 >, \ldots, < f, \phi_n >]^T$.
Solving this system yields $u \approx \sum_{i=1}^{n} c_i\phi_i$.

Now, we shortly study the frame theory. Frames for Hilbert spaces were introduced by Duffin and Schaeffer [13] as part of their research in non-harmonic Fourier series. Now, we introduce the concept of frame [8].

Definition 1. *A family $\mathbf{\Psi} = (\psi_\lambda)_{\lambda\in\Lambda}$ in a Hilbert space $\mathbb{H}$ is a frame for $\mathbb{H}$ if there exist constants $0 < A \leq B < \infty$ such that*

$$A||f||^2 \leq \sum_{\lambda\in\Lambda} | < f, \psi_\lambda > |^2 \leq B||f||^2, \quad \forall f \in \mathbb{H},$$

and the frame operator of $(\psi_\lambda)_{\lambda\in\Lambda}$ is defined by

$$S : \mathbb{H} \longrightarrow \mathbb{H}$$

$$Sf = \sum_{\lambda\in\Lambda} < f, \psi_\lambda > \psi_\lambda, \quad \forall f \in \mathbb{H}.$$

For every frame $(\psi_\lambda)_{\lambda\in\Lambda}$ there exists a dual frame $(\tilde{\psi}_\lambda)_{\lambda\in\Lambda}$ such that

$$f = \sum_{\lambda\in\Lambda} < f, \tilde{\psi}_\lambda > \psi_\lambda, \quad \forall f \in \mathbb{H}.$$

If $\tilde{\psi} = S^{-1}\psi$, this dual is called a canonical dual and other duals are called alternate duals. Therefore, the representation of f is not enforced to be unique. Also, frames may not form a basis which in numerical applications implies the singularity of the stiffness matrix.

Gabor frames are the result of taking a base function, and applying translations and modulations to generate a sequence of functions forming a frame. Modulation and translation operators on $L^2(\mathbb{R})$ are defined by:

$$E_b : L^2(\mathbb{R}) \longrightarrow L^2(\mathbb{R}), \quad (E_b f)(x) = e^{2\pi ibx} f(x), \qquad b \in \mathbb{R},$$

and

$$T_a : L^2(\mathbb{R}) \longrightarrow L^2(\mathbb{R}), \quad (T_a f)(x) = f(x - a), \qquad a \in \mathbb{Z},$$

respectively. Moreover, Gabor introduced the Gabor frame as follows [7, 8, 18]:

Definition 2. *A Gabor frame is a frame for $L^2(\mathbb{R})$ of the form $\{E_{mb}T_{na}g\}_{m,n\in\mathbb{Z}}$ for $g \in L^2(\mathbb{R})$ and $a, b > 0$. In other words, these functions have the form*

$$E_{mb}T_{na}g = e^{2\pi imbx} g(x - na). \tag{1}$$

The following theorem gives a necessary condition in order to have $\{E_{mb}T_{na}g\}_{m,n\in\mathbb{Z}}$ as a frame.

Theorem 1 (see [1]). *Let $g \in L^2(\mathbb{R})$ and $a, b > 0$ be given, and assume that $\{E_{mb}T_{na}g\}_{m,n\in\mathbb{Z}}$ is a frame with bounds A, B. Then*

$$bA \leq \sum_{n\in\mathbb{Z}} |g(x - na)|^2 \leq bB, \qquad x \in \mathbb{R}.$$

Moreover, if $ab > 1$, then $\{E_{mb}T_{na}g\}_{m,n\in\mathbb{Z}}$ can not be a frame for $L^2(\mathbb{R})$.

B-splines are very appropriate functions to play the rule of g in the definition of the Gabor frame. One reason is that B-splines have compact support. Another important feature is that a basis is generated only by translations of one B-spline function and for the B-spline of degree N we have $\sum_{k\in\mathbb{Z}} B_N(x - k) = 1$.

Let us recall that the B-spline function is defined as follows [6, 27]:

Definition 3. *With a strictly increasing sequence $\xi = \{\xi_k\}_{k=i}^{i+s}$, the B-spline basis functions are defined recursively starting with piecewise constants for $N = 1$:*

$$B_{i,1,\xi}(x) = \begin{cases} 1, & \xi_i \leq x < \xi_{i+1}, \\ 0, & otherwise. \end{cases}$$

For $N \geq 2$, the ith B-spline basis function of degree N is defined by

$$B_{i,N,\xi}(x) = \tfrac{x-\xi_i}{\xi_{i+N-1}-\xi_i} B_{i,N-1,\xi}(x) + \tfrac{\xi_{i+N}-x}{\xi_{i+N}-\xi_{i+1}} B_{i+1,N-1,\xi}(x).$$

The aim of this paper is to solve numerically the operator equation $Lu = f$ on the interval $[\alpha, \beta]$ with certain boundary conditions, where $L : \mathbb{H} \to \mathbb{H}$ is a boundedly invertible and self adjoint operator defined on a separable Hilbert space $\mathbb{H}$. Our numerical scheme involves using Galerkin's method with Gabor frames as trial and test functions. Galerkin's method computes the best approximation to the true solution from a given finite dimensional subspace.

3. Preconditioning

The operator equation $Lu = f$ with a boundary condition on the interval $[\alpha, \beta]$ can be solved numerically by Galerkin's method with Gabor frame as trial and test functions.

A choice for the function g in (1) is B_N, the B-spline function of degree N. Let

$$\xi_i = \alpha + \frac{\beta - \alpha}{\mu} i, \quad i \in \mathbb{Z}, \quad p \in \mathbb{N},$$

be the nodal points of B-spline functions. We consider those B-splines whose support contain the interval $[\alpha, \beta]$. The solution space can be defined by

$$H := span\Big\{ E_{mb}T_{\xi_n}B_N : [\alpha, \beta] \to \mathbb{C} \;\Big|\; m \in \mathbb{Z}, n \in \mathcal{J} \Big\},$$

where $\mathcal{J}$ is the index set of B-spline.
For instance, in a partial differential equation with Dirichlet boundary conditions

(i.e., a PDE whose solution is known on the boundary) on domain $[0,1]$, there exist $\mu - N$ B-spline basis functions whose compacts support contains the interval $[0,1]$. In this case, $\mathcal{J} = \{0,1,\ldots,\mu-N-1\}$.

In order to solve the elliptic operator equation $Lu = f$ with given boundary conditions via Galerkin's method, we use Gabor frames as trial and test functions. In this case, a system

$$K\mathbf{u} = F,$$

appears, where $F = (f_{(m,n)})$ and $K = (k_{(m,n),(p,q)})$ are defined by

$$f_{(m,n)} = \langle f, E_{mb}T_{\xi_n}B_N\rangle,$$

and

$$k_{(m,n),(p,q)} = \left\langle LE_{mb}T_{\xi_n}B_N, E_{pb}T_{\xi_q}B_N\right\rangle,$$

for $m,p \in \mathbb{Z}$ and $n,q \in \mathcal{J}$.

The coefficient matrix K is a quad-infinite and dense matrix; therefore, to construct an approximately sparse matrix [28], we define the preconditioned matrix by

$$G = D^{-1}KD^{-1}, \tag{2}$$

where

$$\begin{aligned} D &= diag\left(\ldots,\overbrace{2^{|-1|b},\ldots,2^{|-1|b}}^{p-N\ times},\overbrace{2^{|0|b},\ldots,2^{|0|b}}^{p-N\ times},\overbrace{2^{|1|b},\ldots,2^{|1|b}}^{p-N\ times},\ldots\right) \\ &= diag\left(\ldots,\overbrace{2^{b},\ldots,2^{b}}^{p-N\ times},\overbrace{1,\ldots,1}^{p-N\ times},\overbrace{2^{b},\ldots,2^{b}}^{p-N\ times},\ldots\right). \end{aligned}$$

Therefore, the preconditioned system is $G\mathbf{v} = \mathbf{f}$, where $\mathbf{v} = D\mathbf{u}$ and $\mathbf{f} = D^{-1}F$.

Remark 1. *The preconditioned matrix G can be derived alternatively. In fact, it is enough to replace the trial and test Gabor functions $\{E_{mb}T_{\xi_n}B_N\}_{m\in\mathbb{Z},n\in\mathcal{J}}$ by $\{2^{-|m|b}E_{mb}T_{\xi_n}B_N\}_{m\in\mathbb{Z},n\in\mathcal{J}}$ in Galerkin's method.*

In the next section, we describe how to numerically solve the preconditioned system.

4. Compressibility

In this section, first the compressibility of a matrix is defined, and then it is shown that the preconditioned matrix G is a compressible matrix.

Definition 4 (see [29]). *A matrix $A = (a_{ij})_{i,j}$ is called compressible if for any $n \in \mathbb{N}$ there exists a sequence $\alpha = (\alpha_n)_{n\in\mathbb{N}} \in \ell_1(\mathbb{N})$, a matrix $A_n = (a_{ij}^n)_{i,j}$ defined by*

$$a_{ij}^n = \begin{cases} a_{ij}, & |a_{ij}| > \alpha_n 2^{-n}, \\ 0, & otherwise, \end{cases}$$

and a positive constant C_A such that

$$\|A - A_n\| \leq C_A \alpha_n 2^{-n}.$$

Let $G = (g_{(m,n),(p,q)})$, where $m, p \in \mathbb{Z}$ and $n, q \in \mathcal{J}$. The neighbourhood entries of the entry $[(m,n),(p,q)]$ are shown in Table 1.

$[(m,1),(p,1)]$		$[(m,1),(p,q-1)]$	$[(m,1),(p,q)]$	$[(m,1),(p,q+1)]$		$[(m,1),(p,\|\mathcal{J}\|)]$
$\vdots$	$\cdots$	$\vdots$	$\vdots$	$\vdots$	$\cdots$	$\vdots$
$[(m,n-1),(p,1)]$	$\cdots$	$[(m,n-1),(p,q-1)]$	$[(m,n-1),(p,q)]$	$[(m,n-1),(p,q+1)]$	$\cdots$	$[(m,n-1),(p,\|\mathcal{J}\|)]$
$[(m,n),(p,1)]$	$\cdots$	$[(m,n),(p,q-1)]$	$[(m,n),(p,q)]$	$[(m,n),(p,q+1)]$	$\cdots$	$[(m,n),(p,\|\mathcal{J}\|)]$
$[(m,n+1),(p,1)]$	$\cdots$	$[(m,n+1),(p,q-1)]$	$[(m,n+1),(p,q)]$	$[(m,n+1),(p,q+1)]$	$\cdots$	$[(m,n+1),(p,\|\mathcal{J}\|))]$
$\vdots$	$\cdots$	$\vdots$	$\vdots$	$\vdots$	$\cdots$	$\vdots$
$[(m,\|\mathcal{J}\|),(p,1)]$		$[(m,\|\mathcal{J}\|),(p,q-1)]$	$[(m,\|\mathcal{J}\|),(p,q)]$	$[(m,\|\mathcal{J}\|),(p,q+1)]$		$[(m,\|\mathcal{J}\|),(p,\|\mathcal{J}\|)]$

Table 1: *The neighbourhood entries of $[(m,n),(p,q)]$ in the stiffness matrix G*

Since L is a linear continuous operator, by Remark 1 for $m, p \in \mathbb{Z}$ and $n, q \in \mathcal{J}$ one can write:

$$\begin{aligned}
\left|g_{(m,n),(p,q)}\right| &= \left|\left\langle L(2^{-|m|b} E_{mb} T_{\xi_n} B_N), 2^{-|p|b} E_{pb} T_{\xi_q} B_N \right\rangle\right| \\
&\leq \left\|L(2^{-|m|b} E_{mb} T_{\xi_n} B_N)\right\| \left\|2^{-|p|b} E_{pb} T_{\xi_q} B_N)\right\| \\
&\leq \left\|L\right\| \left\|2^{-|m|b} E_{mb} T_{\xi_n} B_N\right\| \left\|2^{-|p|b} E_{pb} T_{\xi_q} B_N\right\| \qquad (3) \\
&\leq 2^{-(|m|+|p|)b} \left\|L\right\| \left\|T_{\xi_n} B_N\right\| \left\|T_{\xi_q} B_N\right\| \\
&\lesssim 2^{-(|m|+|p|)b}.
\end{aligned}$$

In the above inequalities, we used this reality that $|e^{2imbx}| = |e^{2ipbx}| = 1$ and $\|L\| < \infty$. Since $b > 0$, relation (3) guarantees some decay with respect to m and p. It turns out that the matrix G can be approximated by a sparse matrix.
To continue, we will show that the matrix G is compressible. In order to do so, we need the following well-known Schur lemma [9]:

Lemma 1. *Let A be a matrix and there exist a sequence $(\omega_i)_i$ and a constant $0 < \eta < \infty$ such that*

$$\sum_i \omega_i |a_{ij}| \leq \eta \omega_j, \qquad \sum_j \omega_j |a_{ij}| \leq \eta \omega_i.$$

Then the matrix A is bounded.

Theorem 2. *The matrix G given by (2) is compressible.*

Proof. Let m, p, n, q and $\mathcal{J}$ be given as before. For any fixed integer number $\bar{m}$, one can define the matrix $G_{\bar{m}} = \left(g^{(\bar{m})}_{(m,n),(p,q)}\right)$ as follows:

$$g^{(\bar{m})}_{(m,n),(p,q)} = \begin{cases} g_{(m,n),(p,q)}, & |g_{(m,n),(p,q)}| > \alpha_{\bar{m}} 2^{-\bar{m}}, \\ 0, & otherwise, \end{cases}$$

where the sequence $(\alpha_i)_i$ belongs to $\ell_1(\mathbb{N})$ and $\alpha_i \neq 0$ for all i. For example. one can take $\alpha_i = 2^{-i}$. We define $\Delta := \alpha_{\bar{m}}^{-1} 2^{\bar{m}} (G - G_{\bar{m}}) = \left(\delta_{(m,n),(p,q)}\right)$ such that

$$\delta_{(m,n),(p,q)} = \alpha_{\bar{m}}^{-1} 2^{\bar{m}} \begin{cases} 0, & |g_{(m,n),(p,q)}| > \alpha_{\bar{m}} 2^{-\bar{m}}, \\ g_{(m,n),(p,q)}, & |g_{(m,n),(p,q)}| \leq \alpha_{\bar{m}} 2^{-\bar{m}}. \end{cases}$$

Now, it is enough to show that the matrix Δ satisfies the Schur Lemma. For this sake, we define the sequence $(\omega_{(i,j)})$ in the Schur Lemma by $\omega_{(i,j)} = 2^{-b|i|}$ for $i \in \mathbb{Z}, j \in \mathcal{J}$ and $0 < b < 1$. Hence, for $n, q = 0, 1, \dots, |\mathcal{J}| - 1$, $\omega_{(m,n)} = 2^{-b|m|}, \omega_{(p,q)} = 2^{-b|p|}$ and in view of (3) we have

$$\begin{aligned} \omega^{-1}_{(p,q)} \sum_{m\in\mathbb{Z}, n\in\mathcal{J}} \omega_{(m,n)}\ \alpha_{\bar{m}}^{-1} 2^{\bar{m}} |g_{(m,n),(p,q)}| & \\ \lesssim \quad & \alpha_{\bar{m}}^{-1} 2^{b|p|} 2^{\bar{m}} \sum_{m\in\mathbb{Z}, n\in\mathcal{J}} 2^{-b|m|} 2^{-(|m|+|p|)b} \\ = \quad & \alpha_{\bar{m}}^{-1} 2^{b|p|} 2^{\bar{m}} 2^{-b|p|} \sum_{m\in\mathbb{Z}, n\in\mathcal{J}} 2^{-2b|m|} \\ = \quad & \alpha_{\bar{m}}^{-1} 2^{\bar{m}} \sum_{m\in\mathbb{Z}, n\in\mathcal{J}} 2^{-2b|m|} \\ = \quad & \alpha_{\bar{m}}^{-1} 2^{\bar{m}} |\mathcal{J}| \sum_{m\in\mathbb{Z}} 2^{-2b|m|} \\ = \quad & \alpha_{\bar{m}}^{-1} 2^{\bar{m}} |\mathcal{J}| \left(\sum_{m=-\infty}^{-1} 2^{-2b|m|} + \sum_{m=0}^{\infty} 2^{-2b|m|} \right) \\ = \quad & \alpha_{\bar{m}}^{-1} 2^{\bar{m}} |\mathcal{J}| \left(\frac{1}{2^{2b} - 1} + \frac{2^{2b}}{2^{2b} - 1} \right) \\ = \quad & \alpha_{\bar{m}}^{-1} 2^{\bar{m}} (\mu - N) \left(\frac{1}{2^{2b} - 1} + \frac{2^{2b}}{2^{2b} - 1} \right) \lesssim 1. \end{aligned} \tag{4}$$

In an analogous way, it is seen that

$$\omega^{-1}_{(m,n)} \sum_{p\in\mathbb{Z}, q\in\mathcal{J}} \omega_{(p,q)} \alpha_{\bar{m}}^{-1} 2^{\bar{m}} |g_{(m,n),(p,q)}| \lesssim 1.$$

Thus according to Schur Lemma, the matrix Δ is bounded, i.e., $\|\alpha_{\bar{m}}^{-1} 2^{\bar{m}} (G - G_{\bar{m}})\| \lesssim 1$ or $\|G - G_{\bar{m}}\| \lesssim \alpha_{\bar{m}} 2^{-\bar{m}}$, and that proves the theorem. □

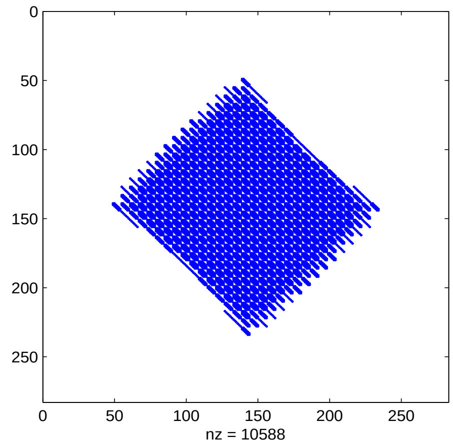

Figure 1: *Bounded polyhedron structure of the compressed matrix* $G_{\bar{m}}$

Remark 2. *According to (3),* $|g_{(m,n),(p,q)}| \longrightarrow 0$ *as* m *or* $p \longrightarrow \infty$*. Hence, the compressed matrix* $G_{\bar{m}}$ *is sparse. Moreover, the sparsity pattern of matrix* $G_{\bar{m}}$ *is a bounded polyhedron structure, (see Figure 1), while that of wavelet and wavelet frames are unbounded finger structures (see Figure 2).*

By (3), there exists a constant value C such that

$$|g_{(m,n),(p,q)}| \leq C2^{-(|m|+|p|)b}.$$

On the other hand, if $2^{-(|m|+|p|)b} \leq \alpha_{\bar{m}} 2^{-\bar{m}}$, then $|m| + |p| \geq \frac{1}{b}(\bar{m} + \log_2(\alpha_{\bar{m}}^{-1}))$. Now, take

$$M := \lceil \frac{1}{b}(\bar{m} + \log_2(\alpha_{\bar{m}}^{-1})) \rceil, \tag{5}$$

where $\lceil x \rceil$ denotes the smallest integer number greater than or equal to x. It turns out that if $|m|+|p| \geq M$, then we will have $g_{(m,n),(p,q)} = 0$. The following algorithm generates the compressed matrix $G_{\bar{m}}$ and the corresponding right-hand side $\mathbf{f}^{(\bar{m})}$ as follows:

Algorithm: $[G_{\bar{m}}, \mathbf{f}_{\bar{m}}] = STIFF - RHS[f, \bar{m}, N, b, \mathcal{J}, L, \alpha_{\bar{m}}]$

1. $G_{\bar{m}} = (g_{(m,n),(p,q)}) :=$ Zero infinite dimensional matrix;
2. $\mathbf{f}^{(\bar{m})} = (\mathbf{f}_{(m,n)}) :=$ Zero infinite dimensional vector;
3. Compute M from (5);
4. For $m = -M, \ldots, M$ do
5.   For $n \in \mathcal{J}$ do
6.   Compute $X = 2^{-|m|b} E_{mb} T_{\xi_n} B_N$;
7. $\mathbf{f}_{(m,n)} =< f, X >$;   % Generating the right-hand side vector.
8.   For $p = -M, \ldots, M$ do
9.   For $q \in \mathcal{J}$ do
10.   Compute $Y = L(2^{-|p|b} E_{pb} T_{\xi_q} B_N)$;

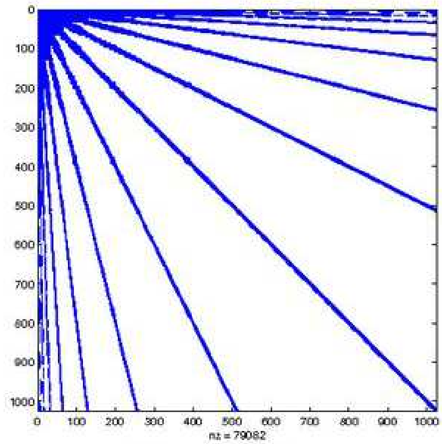

Figure 2: *Finger structure of the stiffness matrix generated by Galerkin's method with wavelet basis functions*

$g_{(m,n),(p,q)} =< Y, X >$; % Generating the preconditioned matrix.
EndFor
EndFor
EndFor
EndFor

Based on the discussion just before the STIFF-RHS Algorithm, if one of the cases $m < -M, p < -M, m > M$ or $p > M$ happens, then $g_{(m,n),(p,q)} = 0$ and $\mathbf{f}_{(m,n)} = 0$.

5. Convergence and computational complexity

Assume that $\mathbf{v} = (v_{(m,n)})_{m\in\mathbb{Z},n\in\mathcal{J}}$ is the exact solution of $G\mathbf{v} = \mathbf{f}$. As noted in Section 3, we know that the value of $\sum_{m\in\mathbb{Z}}\sum_{n\in\mathcal{J}} 2^{|m|b} v_{(m,n)} E_{mb} T_{\xi_n} B_N(x)$ approximates the exact solution of the operator equation $Lu = f$ with given boundary condition(s), where $\mathbf{v} = D\mathbf{u}$. So, we show that the value of

$$\sum_{m\in\mathbb{Z}}\sum_{n\in\mathcal{J}} 2^{|m|b} v_{(m,n)} E_{mb} T_{\xi_n} B_N(x)$$

is real for each $x \in \mathbb{R}$.

Lemma 2. *Let G and $\mathbf{f}$ be the same as the ones we have dealt with before and let $\mathbf{v} = (v_{(m,n)})_{m\in\mathbb{Z},n\in\mathcal{J}}$ be the exact solution of $G\mathbf{v} = \mathbf{f}$. Then, the value of*

$$\sum_{m\in\mathbb{Z}}\sum_{n\in\mathcal{J}} 2^{|m|b} v_{(m,n)} E_{mb} T_{\xi_n} B_N(x)$$

is real for each $x \in \mathbb{R}$.

Proof. It is enough to show that

$$v_{(m,n)} = \overline{v}_{(-m,n)}, \qquad m \in \mathbb{Z}, \quad n \in \mathcal{J},$$

because

$$\begin{aligned}
\sum_{m\in\mathbb{Z}}\sum_{n\in\mathcal{J}} 2^{|m|b} v_{(m,n)} E_{mb} T_{\xi_n} B_N(x) &= \sum_{m\in\mathbb{Z}}\sum_{n\in\mathcal{J}} 2^{|-m|b} \overline{v}_{(-m,n)} \overline{E}_{-mb} T_{\xi_n} B_N(x) \\
&= \overline{\sum_{m\in\mathbb{Z}}\sum_{n\in\mathcal{J}} 2^{|-m|b} v_{(-m,n)} E_{-mb} T_{\xi_n} B_N(x)} \\
&= \overline{\sum_{m\in\mathbb{Z}}\sum_{n\in\mathcal{J}} 2^{|m|b} v_{(m,n)} E_{mb} T_{\xi_n} B_N(x)},
\end{aligned}$$

which yields that the value of $\sum_{m\in\mathbb{Z}}\sum_{n\in\mathcal{J}} 2^{|m|b} v_{(m,n)} E_{mb} T_{\xi_n} B_N(x)$ is real for each $x \in \mathbb{R}$.

Assume that $G = (g_{(m,n),(p,q)})$ and $\mathbf{f} = (f_{(m,n)})$ are given for $m, p \in \mathbb{Z}$ and $n, q \in \mathcal{J}$. The system $G\mathbf{v} = \mathbf{f}$ can be split as follows:

$$\begin{bmatrix} G^{(1)} \\ \cdots\cdots\cdots\cdots \\ G^{(2)} \\ \cdots\cdots\cdots\cdots \\ G^{(3)} \end{bmatrix} \begin{bmatrix} \mathbf{v}^{(1)} \\ \cdots \\ \mathbf{v}^{(2)} \\ \cdots \\ \mathbf{v}^{(3)} \end{bmatrix} = \begin{bmatrix} \mathbf{f}^{(1)} \\ \cdots \\ \mathbf{f}^{(2)} \\ \cdots \\ \mathbf{f}^{(3)} \end{bmatrix},$$

where

$$\begin{aligned}
&G^{(1)} = (g^{(1)}_{(m,n),(p,q)}),\ m \in \mathbb{Z}^+, p \in \mathbb{Z}, n \in \mathcal{J}, q \in \mathcal{J}, \\
&G^{(2)} = (g^{(2)}_{(m,n),(p,q)}),\ m = 0, p \in \mathbb{Z}, n \in \mathcal{J}, q \in \mathcal{J}, \\
&G^{(3)} = (g^{(3)}_{(m,n),(p,q)}),\ m \in \mathbb{Z}^-, p \in \mathbb{Z}, n \in \mathcal{J}, q \in \mathcal{J}, \\
&\mathbf{f}^{(1)} = (f^{(1)}_{(m,n)}), m \in \mathbb{Z}^+, n \in \mathcal{J}, \\
&\mathbf{f}^{(2)} = (f^{(2)}_{(m,n)}), m = 0, n \in \mathcal{J}, \\
&\mathbf{f}^{(3)} = (f^{(3)}_{(m,n)}), m \in \mathbb{Z}^-, n \in \mathcal{J}.
\end{aligned}$$

On the one hand, by reordering (row-wise and column-wise), the above system can be given equivalently as

$$\begin{bmatrix} G^{(3)} \\ \cdots\cdots\cdots\cdots \\ G^{(2)} \\ \cdots\cdots\cdots\cdots \\ G^{(1)} \end{bmatrix} \begin{bmatrix} \mathbf{v}^{(3)} \\ \cdots \\ \mathbf{v}^{(2)} \\ \cdots \\ \mathbf{v}^{(1)} \end{bmatrix} = \begin{bmatrix} \mathbf{f}^{(3)} \\ \cdots \\ \mathbf{f}^{(2)} \\ \cdots \\ \mathbf{f}^{(1)} \end{bmatrix}.$$

On the other hand, $\overline{G} = [G^{(3)}, G^{(2)}, G^{(1)}]^T$ and $\overline{\mathbf{f}} = [\mathbf{f}^{(3)}, \mathbf{f}^{(2)}, \mathbf{f}^{(1)}]^T$, where $\overline{G}$ and $\overline{\mathbf{f}}$ denote the conjugate of G and $\mathbf{f}$, respectively. This can be described as follows:

$$\begin{aligned}
\overline{g}_{(m,n),(p,q)} &= \overline{< L(2^{-|m|b} E_{mb} T_{\xi_n} B_N), 2^{-|p|b} E_{pb} T_{\xi_q} B_N >} \\
&= < \overline{L(2^{-|m|b} E_{mb} T_{\xi_n} B_N)}, \overline{2^{-|p|b} E_{pb} T_{\xi_q} B_N} > \\
&= < L(2^{-|m|b} \overline{E}_{mb} T_{\xi_n} B_N), 2^{-|p|b} \overline{E}_{pb} T_{\xi_q} B_N > \\
&= < L(2^{-|m|b} E_{-mb} T_{\xi_n} B_N), 2^{-|p|b} E_{-pb} T_{\xi_q} B_N > \\
&= < L(2^{-|-m|b} E_{-mb} T_{\xi_n} B_N), 2^{-|-p|b} E_{-pb} T_{\xi_q} B_N > \\
&= g_{(-m,n),(-p,q)}.
\end{aligned}$$

Similarly, it is seen that

$$\overline{\mathbf{f}}_{(m,n)} = \mathbf{f}_{(-m,n)}.$$

Then, we have

$$\overline{G}\mathbf{v}' = \overline{\mathbf{f}}, \tag{6}$$

where $\mathbf{v}' = (\mathbf{v}^{(3)}, \mathbf{v}^{(2)}, \mathbf{v}^{(1)})^T$. Also, the system $G\mathbf{v} = \mathbf{f}$ is used to obtain the system

$$\overline{G}\overline{\mathbf{v}} = \overline{\mathbf{f}}. \tag{7}$$

Now, in view of (6) and (7)

$$\begin{aligned}
\mathbf{v}^{(3)} &= \overline{\mathbf{v}}^{(1)}, \\
\mathbf{v}^{(2)} &= \overline{\mathbf{v}}^{(2)}, \\
\mathbf{v}^{(1)} &= \overline{\mathbf{v}}^{(3)},
\end{aligned}$$

since $\overline{G}$ is a nonsingular matrix. This completes the proof. □

5.1. Convergence

Let $g(r,:)$ and $g(:,s)$ be the rth row and the sth column of the compressed matrix $G_{\bar{m}}$, respectively. By (5), there exists an integer number M and an index set $\mathcal{J}$ such that for $n \in \mathcal{J}$, $\|g((m,n),:)\| = \|g(:,(m,n))\| = 0$ if $m < -M$ or $m > M$. Assume that $\mathbf{u}$ is an arbitrary infinite dimensional vector in $\ell_2(\mathbb{Z} \times \mathcal{J})$. We define $\tilde{\mathbf{u}}$ as follows:

$$\tilde{\mathbf{u}}_{(m,n)} = \begin{cases} \mathbf{u}_{(m,n)}, & -M \le m \le M, \\ 0, & \textit{otherwise}. \end{cases}$$

We have:

$$\begin{aligned}
\|G\mathbf{u} - G_{\bar{m}}\tilde{\mathbf{u}}\| &\le \|G\|\|\mathbf{u} - \tilde{\mathbf{u}}\| + \|G - G_{\bar{m}}\|\|\tilde{\mathbf{u}}\| \\
&\le C\|\mathbf{u} - \tilde{\mathbf{u}}\| + C_G \alpha_{\bar{m}} 2^{-\bar{m}} \|\tilde{\mathbf{u}}\| =: R_{\bar{m}},
\end{aligned}$$

where C is an upper bound for $\|G\|$ and C_G is the constant appearing in the definition of a compressible matrix. Now, by the following algorithm [11, 29], one can find an approximation for $\|G\mathbf{u} - G_{\bar{m}}\tilde{\mathbf{u}}\|$.

Algorithm: $\Pi_{\bar{m}} = APPLY[G, \mathbf{u}, \epsilon]$

1. $\bar{m} := 1$;
2. While $R_{\bar{m}} > \epsilon$ do
3. $\bar{m} := \bar{m} + 1$;
4. End
5. $\Pi_{\bar{m}} := G_{\bar{m}}\tilde{\mathbf{u}}$.

The APPLY Algorithm shows that the smaller ϵ, the sparser compressed matrix $G_{\bar{m}}$ and the smaller the surface of the dense submatrix of $G_{\bar{m}}$.

Suppose that $\mathbf{f}_{\delta_k}$ is an approximation of $\mathbf{f}$ that satisfies the following:

$$\|\mathbf{f} - \mathbf{f}_{\delta_k}\| \leq \delta_k, \tag{8}$$

in which δ_k is a tolerance. We recall that the Richardson iterative method [23, 29] to solve the system

$$G_{\bar{m}}\mathbf{w} = \mathbf{f}_{\delta_k} \tag{9}$$

is defined as follows:

$$\mathbf{w}^{(k+1)} = \mathbf{w}^{(k)} + \theta(\mathbf{f}_{\delta_k} - G_{\bar{m}}\mathbf{w}^{(k)}), \qquad k = 0, 1, \dots \tag{10}$$

where θ is a nonnegative scalar. Taking into account the exact solution $\mathbf{w}$ of (9), it is seen that

$$\mathbf{w}^{(k+1)} - \mathbf{w} = (I - \theta G_{\bar{m}})(\mathbf{w}^{(k)} - \mathbf{w}).$$

The following algorithm is an adaptive Gabor-Richardson scheme given by the Richardson iterative method with an initial guess $\mathbf{w}^{(0)}$:

Algorithm: $\mathbf{w} = ITERATIVE[G, \mathbf{w}^{(0)}, \mathbf{f}, \delta_k, \theta]$

INPUT: Given $\mathbf{w}^{(0)} \in \ell_2(\mathbb{Z})$ as an initial guess with finite support, $(\delta_k)_{k\geq 0}$, $\delta_k > 0$ as a sequence of tolerances, the nonnegative scalar θ, the coefficient matrix G and the right-hand side vector $\mathbf{f}$.

1. Repeat until convergence.

2. Compute $\mathbf{w}^{(k+1)} = \mathbf{w}^{(k)} - \theta APPLY[G, \mathbf{w}^{(k)}, \delta_k] + \theta\mathbf{f}_{\delta_k}$, where $\mathbf{f}_{\delta_k}$ is an approximation of $\mathbf{f}$ such that $\|\mathbf{f} - \mathbf{f}_{\delta_k}\| \leq \delta_k$.

3. End

4. $\mathbf{w} := \mathbf{w}^{(k+1)}$.

To continue, we prove that $\|\mathbf{w}^{(k+1)} - \mathbf{v}\|$ vanishes as k tends to infinity. For this sake, one can write

$$\|\mathbf{w}^{(k+1)} - \mathbf{v}\| \leq \|\mathbf{w}^{(k+1)} - \mathbf{v}^{(k+1)}\| + \|\mathbf{v}^{(k+1)} - \mathbf{v}\|,$$

where $\mathbf{v}^{(k+1)}$ denotes the Richardson iteration of $G\mathbf{v} = \mathbf{f}$, namely

$$\mathbf{v}^{(k+1)} = (I - \theta G)\mathbf{v}^{(k)} + \theta \mathbf{f}, \qquad \mathbf{v}^{(0)} := \mathbf{w}^{(0)}. \tag{11}$$

By (11), it is readily seen that

$$\|\mathbf{v}^{(k+1)} - \mathbf{v}\| \leq \|I - \theta G\|^{k+1} \|\mathbf{v}^{(0)} - \mathbf{v}\|. \tag{12}$$

Also, we have

$$\begin{aligned}
\mathbf{w}^{(k+1)} - \mathbf{v}^{(k+1)} &= \mathbf{w}^{(k)} - \theta APPLY[G, \mathbf{w}^{(k)}, \delta_k] + \theta \mathbf{f}_{\delta_k} - \mathbf{v}^{(k)} + \theta G \mathbf{v}^{(k)} - \theta \mathbf{f} \\
&= \mathbf{w}^{(k)} - \mathbf{v}^{(k)} - \theta\Big(APPLY[G, \mathbf{w}^{(k)}, \delta_k] - G\mathbf{v}^{(k)}\Big) + \theta\Big(\mathbf{f}_{\delta_k} - \mathbf{f}\Big) \\
&= \mathbf{w}^{(k)} - \mathbf{v}^{(k)} - \theta\Big(APPLY[G, \mathbf{w}^{(k)}, \delta_k] - G\mathbf{w}^{(k)}\Big) \\
&+ \theta G\Big(\mathbf{v}^{(k)} - \mathbf{w}^{(k)}\Big) + \theta\Big(\mathbf{f}_{\delta_k} - \mathbf{f}\Big) = \Big(I - \theta G\Big)\Big(\mathbf{w}^{(k)} - \mathbf{v}^{(k)}\Big) \\
&- \theta\Big(APPLY[G, \mathbf{w}^{(k)}, \delta_k] - G\mathbf{w}^{(k)}\Big) + \theta\Big(\mathbf{f}_{\delta_k} - \mathbf{f}\Big).
\end{aligned}$$

Now, the APPLY Algorithm and inequality (8) imply that

$$\begin{aligned}
\|\mathbf{w}^{(k+1)} - \mathbf{v}^{(k+1)}\| &\leq \|I - \theta G\| \|\mathbf{w}^{(k)} - \mathbf{v}^{(k)}\| + 2\theta\delta_k \\
&\leq \|I - \theta G\|^2 \|\mathbf{w}^{(k-1)} - \mathbf{v}^{(k-1)}\| + 2\theta\Big(\delta_k + \|I - \theta G\|\delta_{k-1}\Big) \\
&\;\vdots \\
&\leq \|I - \theta G\|^{k+1} \|\mathbf{w}^{(0)} - \mathbf{v}^{(0)}\| + 2\theta \sum_{p=0}^{k} \|I - \theta G\|^{k-p} \delta_p \\
&= 2\theta \sum_{p=0}^{k} \|I - \theta G\|^{k-p} \delta_p,
\end{aligned} \tag{13}$$

since $\mathbf{w}^{(0)} = \mathbf{v}^{(0)}$. The parameter $\theta \in \mathbb{R}^+$ is selected in such a way that the algorithm converges, i.e.,

$$mu = \|I - \theta G\| < 1.$$

Now, we assume that the sequence of tolerances δ_p is chosen to be small enough such that $\gamma := \{\gamma_p = \mu^{-p}\delta_p\}_{p \in \mathbb{Z}^+} \in \ell_1(\mathbb{Z}^+)$. By using relations (12) and (13) we have

$$\begin{aligned}
\|\mathbf{w}^{(k+1)} - \mathbf{v}\| \leq &\quad 2\theta \sum_{p=0}^{k} \mu^{k-p}\delta_p + \mu^{k+1} \|\mathbf{v}^{(0)} - \mathbf{v}\| \\
= &\quad \mu^k \Big(2\theta \sum_{p=0}^{k} \mu^{-p}\delta_p + \mu \|\mathbf{v}^{(0)} - \mathbf{v}\|\Big) \\
\leq &\quad \mu^k \Big(2\theta \sum_{p=0}^{\infty} \mu^{-p}\delta_p + \mu \|\mathbf{v}^{(0)} - \mathbf{v}\|\Big) \\
= &\quad \mu^k \Big(2\theta \|\gamma\|_{\ell_1(\mathbb{N})} + \mu \|\mathbf{v}^{(0)} - \mathbf{v}\|\Big) \\
\longrightarrow &\; 0 \quad (\text{as} \;\; k \longrightarrow \infty),
\end{aligned}$$

which proves the claim. □

5.2. Computational complexity

Let the general form of the stiffness matrix $G_{\bar{m}}$ be as follows:

$$G_{\bar{m}} = \left[\begin{array}{c|c|c} \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \hline \mathbf{0} & \tilde{G}_{\bar{m}} & \mathbf{0} \\ \hline \mathbf{0} & \mathbf{0} & \mathbf{0} \end{array}\right],$$

where $\tilde{G}_{\bar{m}}$ is a square matrix with a dimension $(2M+1)|\mathcal{J}|$ so that M satisfies (5). We note that by (5), the value of M depends on $\bar{m}$. The appropriate value of $\bar{m}$ is selected by the APPLY Algorithm. Hence, after running the Adaptive ITERATIVE code that uses the APPLY algorithm, the value of $\bar{m}$ and hence M is fixed. Moreover, suppose that $\mathbf{u}$ is an infinite dimensional vector such that $|supp\,\mathbf{u}| = N$ and $\mathbf{u}_{(m,n)} = 0$ for $m < -M$ and $m > M$. Multiplying each row of $G_{\bar{m}}$ by $\mathbf{u}$ includes at most N multiplication and $N-1$ addition. Hence, $G_{\bar{m}}\mathbf{u}$ includes at most $(2M+1)|\mathcal{J}|N$ multiplication and $(2M+1)|\mathcal{J}|(N-1)$ addition. If the dimension of $\tilde{G}_{\bar{m}}$ is proportional to the magnitude of the support of $\mathbf{u}$, then the number of operations for computing $G_{\bar{m}}\mathbf{u}$ would be the order of $O(N^2)$.
Because of $\|\tilde{\mathbf{u}}\|$ and $\|\mathbf{u}-\tilde{\mathbf{u}}\|$ in the definition of $R_{\bar{m}}$ in the APPLY Algorithm, the order of each iteration for this algorithm would clearly be $O(N^2)$.

6. Numerical experiments

In this section, we present two numerical examples to confirm the theoretical results given in the previous sections. For both examples, we consider the stop criteria by $\|u-\bar{u}\|_{L^2([0,1])} \leq 0.001$, where u and $\bar{u}$ denote the exact and approximated solutions, respectively. In addition, we take $\delta_k = 2^{-k}$ for both examples.

Example 1. *Consider the problem*

$$\begin{aligned} -u^{''} &= -6x, \quad in \quad \Omega = (0,1), \\ u(0) &= u(1) = 0, \end{aligned}$$

where the exact solution is $u = x^3 - x$. *Let*

$$0 = \xi_0 < \xi_1 = \frac{1}{8} < \ldots < \xi_7 = \frac{7}{8} < \xi_8 = 1,$$

be the nodal points of B-spline basis functions of degree $N = 2$, *defined on* $[0,1]$. *The* *trial and test functions are taken as*‡

$$H := span\Big\{2^{-|m|b}E_{mb}T_{\xi_n}B_N,\ m \in \mathbb{Z}, n \in \mathcal{J}\Big\} \subset H_0^1([0,1]).$$

Hence, by (2), the preconditioned stiffness matrix G *is given by*

$$g_{(m,n),(p,q)} = \Big\langle (2^{-|m|b}E_{mb}T_{\xi_n}B_N)', (2^{-|p|b}E_{pb}T_{\xi_q}B_N)' \Big\rangle,$$

where $n, q \in \mathcal{J} = \{0, 1, \ldots, \mu - N - 1 = 5\}$. *Figure 3 shows a comparison between the exact and approximate solutions. Also, the sparsity pattern of the sample compressed matrix* G_{47} *is shown in Figure 4. Table 1 shows that the adaptive Richardson-Gabor frame method converges after* 17 *iterations.*

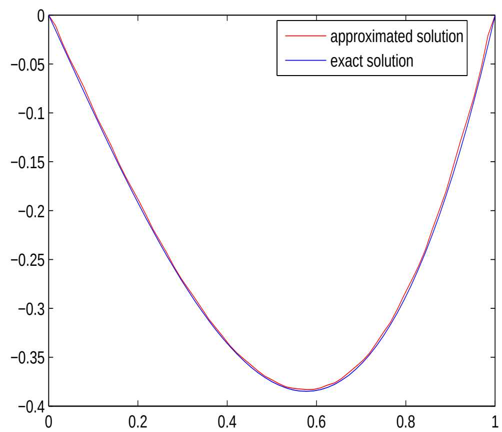

Figure 3: *The exact and approximate solutions for Example 1*

b	θ	Iteration	$\mathbf{w}^{(0)}$	CPU(sec.)
0.5	0.07	17	$\mathbf{0}$	21.87

Table 2: *Data of Example 1*

Example 2. *We construct the right-hand side* f *of problem*

$$\begin{aligned} -u'' + 0.1u &= f, \qquad x \in (0,1), \\ u(0) &= u(1), \\ u'(0) &= u'(1), \end{aligned}$$

‡Let $\Omega \subset \mathbb{R}$ be a bounded domain. The space $H_0^1(\Omega)$ is defined by:

$$H_0^1(\Omega) = \Big\{f:\ f \in L^2(\Omega), DF \in L^2(\Omega)\ and\ f\Big|_{\partial\Omega} = 0\Big\},$$

where Df is the (weak) derivative of f and $\partial\Omega$ is the boundary of Ω.

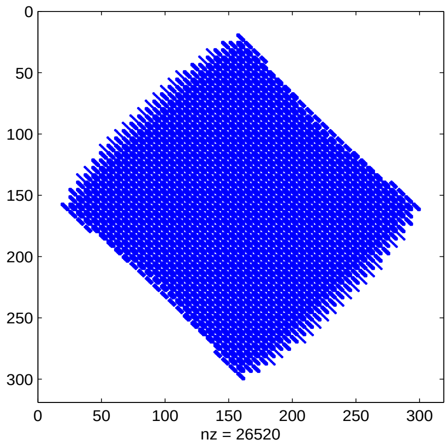

Figure 4: *The sparsity pattern of the compressed matrix* G_{47}

such that the exact solution is

$$u(x) = e^{-100x^2(1-x)^2}.$$

In this case, the trial and test functions are taken as

$$H := span\Big\{2^{-|m|b} E_{mb} T_{\xi_n} B_N,\ m \in \mathbb{Z}, n \in \mathcal{J}\Big\} \subset H^1([0,1]),$$

where B_N *is the B-spline of order* 2 *with*

$$0 = \xi_0 < \xi_1 = \frac{1}{8} < \ldots < \xi_7 = \frac{7}{8} < \xi_8 = 1,$$

as nodal points and $\mathcal{J} = \{0, 1, \ldots, 9 = \mu + N - 1\}$. *A variational problem is to find* $u_h \in H$ *such that*

$$a(u_h, v_h) = < f, v_h >_{L^2[0,1]} \qquad \forall v_h \in H,$$

where the bilinear form $a : H \times H \longrightarrow \mathbb{C}$ *is defined by*

$$\begin{aligned} a(u_h, v_h) &= < u_h', v_h' >_{L^2[0,1]} + < u_h, v_h >_{L^2[0,1]} \\ &= \int_0^1 u_h'(x)\bar{v}_h'(x)dx + \int_0^1 u_h(x)\bar{v}_h(x)dx. \end{aligned}$$

The preconditioned stiffness matrix G *is given by*

$$\begin{aligned} g_{(m,n),(p,q)} &= \Big\langle (2^{-|m|b} E_{mb} T_{\xi_n} B_N)', (2^{-|p|b} E_{pb} T_{\xi_q} B_N)' \Big\rangle \\ &\quad + \Big\langle (2^{-|m|b} E_{mb} T_{\xi_n} B_N), (2^{-|p|b} E_{pb} T_{\xi_q} B_N) \Big\rangle, \end{aligned}$$

where $n, q \in \mathcal{J}$. *Figure 5 shows a comparison between the exact and approximate solutions. Also, the sparsity pattern of the sample compressed matrix* G_{32} *is shown in Figure 6. Table 2 shows that the adaptive Richardson-Gabor frame method converges after* 25 *iterations.*

b	θ	Iteration	$\mathbf{w}^{(0)}$	CPU(sec.)
0.3	0.06	25	$\mathbf{0}$	35.27

Table 3: *Data of Example 2*

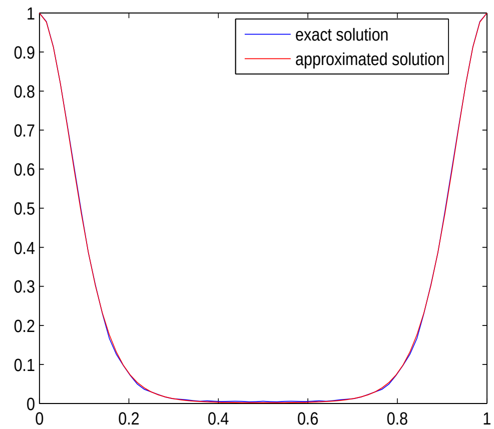

Figure 5: *The exact and approximate solutions*

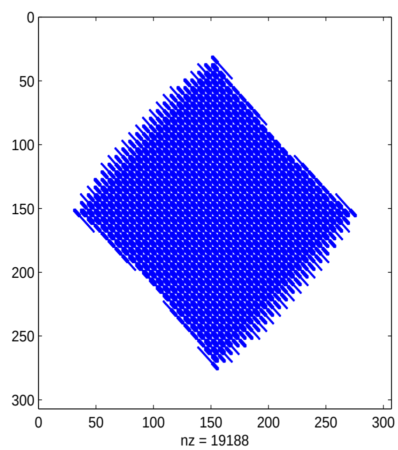

Figure 6: *The sparsity pattern of the compressed matrix* G_{32}

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