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Characteristic function of the order statistics of the Student's t distribution

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Abstract. An exact expression is established for the characteristic function of the order statistics of the Student's t distribution. The expression is a single infinite sum of terms involving the modified Bessel function of the second kind. It is simpler and yet more general than previously known expressions.

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1. Introduction

The Student's t distribution is one of the most popular distributions in statistics. It has been used to model various phenomena, especially in finance. It has been a popular model in particular for stock returns. In stock modeling, the primary interest is in the order statistics of stock returns (for example, largest stock return, smallest stock return, etc). Moments and the characteristic function of order statistics are of interest for various reasons:

1. Suppose stock returns are recorded daily. Then the average of the largest stock returns observed over say periods of 1000 days will need the first moment of the largest order statistic; the variability of the largest stock returns observed over say periods of 1000 days will need the second moment of the largest order statistic; the skewness of the largest stock returns observed over say periods of 1000 days will need the third moment of the largest order statistic; the kurtosis of the largest stock returns observed over say periods of 1000 days will need the fourth moment of the largest order statistic; and so on.

2. Estimation of models for financial data are often based on the empirical characteristic function. Some examples include estimation of affine asset pricing models ([20]); estimation of stochastic conditional duration models ([15]); estimation of Markov models ([12]); estimation of value at risk ([24]); semi-

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nonparametric estimation of independently and identically repeated first-price auctions ([3]).

3. Hypothesis testing involving financial data is also often based on the empirical characteristic function. Some examples include goodness of fit tests based on kernel density estimators ([7]); a test of the martingale difference hypothesis ([16]); generalized spectral tests for conditional mean models in time series ([10]); nonparametric tests for conditional symmetry in dynamic models ([6]); a simulation-based specification test for diffusion processes ([2]); a martingale approach to testing diffusion models ([21]); tests of short memory ([1]); tests for conditional distributions in count data ([3]); integrated conditional moment tests for conditional distributions ([4]); testing for the Markov property in time series ([5]); tests for conditional ellipticity in multivariate GARCH models ([9]); testing stationarity of time series ([11]).

4. Various financial risk measures are based on the order statistics and their moments (for example, expected shortfall).

The first motivation directly involves moments of order statistics. If data are available on the order statistics of stock returns, then the remaining motivations will need moments and characteristic functions of the order statistics. Note that the moments of order statistics can be easily derived if their characteristic functions were known.

There has not been much work on moments of order statistics from the Student's t distribution: Fisher and Healy (1956) tabulated numerical values of the expected order statistics; Kabir and Rahman (1974) derived bounds for the expected order statistics; Vaughan (1992) gave expressions for expected values, variances and co-variances of order statistics for the Student's t distribution with two degrees of freedom.

Most recently, Nadarajah (2007, 2008) derived expressions for moments of the Student's t order statistics. But the expressions involved the generalized Kampé de Fériet function, which is defined as a multiple infinite sum of elementary functions. The number of infinite sums increases as the sample size increases. Hence, the computation becomes formidable for large sample sizes. Besides, we are aware of no packages containing in-built routines for computing the generalized Kampé de Fériet function.

The aim of this note is to derive a simple expression for the characteristic function of the Student's t order statistics. The expression derived is a single infinite sum of terms which are elementary, except for the modified Bessel function of the second kind. In-built routines for the modified Bessel function of the second kind are widely available, even in the freely downloadable R software ([19]).

Let T denote a Student's t random variable with ν degrees of freedom. Its probability density function is

$$f(x) = \frac{1}{\sqrt{\nu}\, \mathrm{B}\left(\frac{\nu}{2}, \frac{1}{2}\right)} \left(1 + \frac{x^2}{\nu}\right)^{-\frac{\nu+1}{2}} \tag{1}$$

for $x \in \mathbb{R}$, where B denotes the beta function defined by

$$\mathrm{B}(p,q) = \int_0^1 x^{p-1} \ (1-x)^{q-1} \ \mathrm{d}x$$

for $\min\left[\Re(p), \Re(q)\right] > 0$. We write $T \sim t(\nu)$ throughout. The corresponding cumulative distribution function is ([13], page 364, equation (28.4a))

$$F(x) = 1 - \frac{1}{2} \ I_{\sigma(x)} \left(\frac{\nu}{2}, \frac{1}{2} \right) \tag{2}$$

for $\sigma(x) = \frac{\nu}{\nu + x^2}$ and $x \geq 0$; other values can be obtained by symmetry. Also I denotes the regularized incomplete beta function defined by

$$I_z(p,q) = \frac{\mathrm{B}_z(p,q)}{\mathrm{B}(p,q)},$$

where

$$\mathrm{B}_z(p,q) = \int_0^z \ x^{p-1} \ (1-x)^{q-1} \ \mathrm{d}x.$$

Hence, the probability density function is an even function, the corresponding characteristic function, $\phi_T(t) = \mathsf{E}\mathrm{e}^{\mathrm{i}tT} = 2\mathsf{E}\cos(tT)$, is in fact the Fourier cosine transform of $f(x)$. It is well known that (see [13])

$$\phi_T(t) = \frac{2}{\Gamma\left(\frac{\nu}{2}\right)} \ \left(\frac{t\sqrt{\nu}}{2} \right)^{\frac{\nu}{2}} \ K_{\frac{\nu}{2}} \left(t\sqrt{\nu} \right)$$

for $\nu > 0$ and $t \in \mathbb{R}$, where K_μ denotes the modified Bessel function of the second kind of order μ and Γ denotes the gamma function defined by

$$\Gamma(s) = \int_0^\infty x^{s-1} \ \mathrm{e}^{-x} \ \mathrm{d}x$$

for $\Re(s) > 0$. Note that $\mathrm{B}(p,q) \ \Gamma(p+q) = \Gamma(p) \ \Gamma(q)$.

Suppose $T_1, \ldots, T_n$ are independent copies of $T \sim t(\nu)$. Let $T_{(1)} < \cdots < T_{(n)}$ denote the corresponding order statistics. Theorem 1 derives an explicit expression for the characteristic function of $T_{(r)}$. The expression is a Neumann series of the modified Bessel function of the second kind K_μ, where μ contains all summation indices.

If $T_1, \ldots, T_n$ are independent $t(\nu)$ random variables with location parameter μ and scale parameter σ, then the characteristic function of $T_{(r)}$ is $\phi_r(\sigma t) \exp\left(\mathrm{i}t\mu\right)$, where $\phi_r(\cdot)$ is given by Theorem 1.

2. Main results

The main result is Theorem 1.

Theorem 1. *For all $t \in \mathbb{R}$, $\nu > 0$ and for all $r = 1, \ldots, n$, the characteristic function of $T_{(r)}$ is*

$$\begin{aligned}\phi_r(t) = & \frac{2\sqrt{\pi}\ n\binom{n-1}{r-1}\ \left(\frac{t\sqrt{\nu}}{2}\right)^{\frac{\nu}{2}(n-r+1)}}{\nu^{n-r}\left[\mathrm{B}\left(\frac{\nu}{2}, \frac{1}{2}\right)\right]^{n-r+1}} \sum_{k=1}^{r-1} \frac{(-1)^k \binom{r-1}{k}}{\nu^k\ \mathrm{B}^k\left(\frac{\nu}{2}, \frac{1}{2}\right)} \left(\frac{t\sqrt{\nu}}{2}\right)^{\frac{\nu}{2}k} \\ & \times \sum_{j \geq 0} \sum_{j_2=0}^{j} \cdots \sum_{j_{n-r+k}=0}^{j-j_2-\cdots-j_{n-r+k-1}} \frac{K_{\frac{\nu}{2}(n-r+k+1)+j+\frac{1}{2}}(t\sqrt{\nu})}{\Gamma\left(\frac{\nu}{2}(n-r+k+1)+j+\frac{1}{2}\right)} \\ & \times \prod_{s=1}^{n-r+k} \frac{\left(\frac{1}{2}\right)_{j_s}\left(\frac{\nu}{2}\right)_{j_s}}{\left(\frac{\nu}{2}+1\right)_{j_s}} \frac{\left(\frac{t\sqrt{\nu}}{2}\right)^{j_s}}{j_s!}.\end{aligned}$$

Proof. By (1) and (2), the probability density function of $T_{(r)}$ is

$$f_r(x) = \frac{n\binom{n-1}{r-1}\mathrm{B}_\sigma(x)\left(\frac{\nu}{2}, \frac{1}{2}\right)}{\sqrt{\nu}2^{n-r}\left[\mathrm{B}\left(\frac{\nu}{2}, \frac{1}{2}\right)\right]^{n-r+1}} \left(1 + \frac{x^2}{\nu}\right)^{-\frac{\nu+1}{2}} \left[1 - \frac{1}{2}\ I_{\sigma(x)}\left(\frac{\nu}{2}, \frac{1}{2}\right)\right]^{r-1}$$

for $x \geq 0$. Since we will use the cosine Fourier transform, it is sufficient to consider the probability density and cumulative distribution functions for non-negative values. That is,

$$\begin{aligned}\phi_r(t) &= 2\ \mathsf{E}\cos\left(tT_{(r)}\right) = 2\int_0^\infty \cos(tx)\ f_r(x)\ \mathrm{d}x \\ &= \frac{2n\binom{n-1}{r-1}}{\sqrt{\nu}\ 2^{n-r}\left[\mathrm{B}\left(\frac{\nu}{2}, \frac{1}{2}\right)\right]^{n-r+1}} \\ &\quad \times \int_0^\infty \frac{\cos(tx)\ \mathrm{B}_\sigma(x)\left(\frac{\nu}{2}, \frac{1}{2}\right)}{\left(1 + \dfrac{x^2}{\nu}\right)^{\frac{\nu+1}{2}}} \left[1 - \frac{1}{2}\ I_{\sigma(x)}\left(\frac{\nu}{2}, \frac{1}{2}\right)\right]^{r-1} \mathrm{d}x. \qquad (3)\end{aligned}$$

Substituting $x\ \sqrt{\nu} \mapsto x$ and writing the shorthand

$$C_k = \frac{(-1)^k n\binom{n-1}{r-1}\binom{r-1}{k}}{2^{n-r+k-1}\left[\mathrm{B}\left(\frac{\nu}{2}, \frac{1}{2}\right)\right]^{n-r+k+1}},$$

(3) can be rewritten as

$$\phi_r(t) = \sum_{k=1}^{r-1} C_k \int_0^\infty \frac{\cos\left(t\sqrt{\nu}\ x\right)}{(1+x^2)^{\frac{\nu+1}{2}}} \left[\int_0^{\left(1+x^2\right)^{-1}} u^{\frac{\nu}{2}-1}(1-u)^{-\frac{1}{2}}\ \mathrm{d}u\right]^m \mathrm{d}x,$$

where $m = n - r + k$. Expanding the binomial term into a Maclaurin series in the inner u integral and expressing the result in terms of Pochhammer symbols, we obtain

$$\phi_r(t) = \sum_{k=1}^{r-1} C_k\ \left(\tfrac{2}{\nu}\right)^m \sum_{j_1,\ldots,j_m \geq 0} \prod_{s=1}^{m} \frac{\left(\frac{1}{2}\right)_{j_s}\left(\frac{\nu}{2}\right)_{j_s}}{\left(\frac{\nu}{2}+1\right)_{j_s} \cdot j_s!} \int_0^\infty \frac{\cos\left(t\sqrt{\nu}\ x\right)}{(1+x^2)^{\mu+\frac{1}{2}}}\ \mathrm{d}x, \qquad (4)$$

where

$$\mu = \tfrac{\nu}{2}(m+1) + j_1 + \cdots + j_m.$$

By [23], page 172, Section 6.16, equation (1),

$$\int_0^\infty \frac{\cos(sx)}{(a^2+x^2)^{\mu+\frac{1}{2}}}\,\mathrm{d}x = \frac{\sqrt{\pi}}{\Gamma\left(\mu+\frac{1}{2}\right)}\left(\frac{s}{2a}\right)^{\mu} K_\mu(a\ s) \tag{5}$$

holds for all $\Re(a) > 0$ and $2\Re(\mu) + 1 > 0$. Hence,

$$\begin{aligned}\phi_r(t) &= \sqrt{\pi}\sum_{k=1}^{r-1} C_k \left(\frac{2}{\nu}\right)^m \left(\frac{t\sqrt{\nu}}{2}\right)^{\frac{\nu}{2}(m+1)} \sum_{j_1,\ldots,j_m\geq 0} \frac{K_{\mu+\frac{1}{2}}\left(t\sqrt{\nu}\right)}{\Gamma\left(\mu+\frac{1}{2}\right)} \\ &\quad\times \prod_{s=1}^{m} \frac{\left(\frac{1}{2}\right)_{j_s}\left(\frac{\nu}{2}\right)_{j_s}}{\left(\frac{\nu}{2}+1\right)_{j_s}} \frac{\left(\frac{t\sqrt{\nu}}{2}\right)^{j_s}}{j_s!}.\end{aligned}$$

Further reduction gives the expression

$$\begin{aligned}\phi_r(t) &= \frac{2\sqrt{\pi}\ n\binom{n-1}{r-1}\left(\frac{t\sqrt{\nu}}{2}\right)^{\frac{\nu}{2}(n-r+1)}}{\nu^{n-r}\left[\mathrm{B}\left(\frac{\nu}{2},\frac{1}{2}\right)\right]^{n-r+1}} \sum_{k=1}^{r-1} \frac{(-1)^k\binom{r-1}{k}}{\nu^k\ \mathrm{B}^k\left(\frac{\nu}{2},\frac{1}{2}\right)}\left(\frac{t\sqrt{\nu}}{2}\right)^{\frac{\nu}{2}k} \\ &\quad\times \sum_{j_1,\ldots,j_m\geq 0} \frac{K_{\frac{\nu}{2}(m+1)+j_1+\cdots+j_m+\frac{1}{2}}\left(t\sqrt{\nu}\right)}{\Gamma\left(\frac{\nu}{2}(m+1)+j_1+\cdots+j_m+\frac{1}{2}\right)} \\ &\quad\times \prod_{s=1}^{m} \frac{\left(\frac{1}{2}\right)_{j_s}\left(\frac{\nu}{2}\right)_{j_s}}{\left(\frac{\nu}{2}+1\right)_{j_s}} \frac{\left(\frac{t\sqrt{\nu}}{2}\right)^{j_s}}{j_s!},\end{aligned}$$

which is equivalent to the asserted result. □

Corollary 1 is the special case of Theorem 1 for $n = r = 3$. Corollary 2 is the special case for $n = 3$ and $r = 1$.

Corollary 1. *For all $t \in \mathbb{R}$ and $\nu > 0$, the characteristic function of $T_{(3)}$ for $n = 3$ is*

$$\begin{aligned}\phi_3(t) &= \frac{12\ \sqrt{\pi}\ \left(\frac{t\sqrt{\nu}}{2}\right)^{\frac{\nu}{2}}}{\mathrm{B}\left(\frac{\nu}{2},\frac{1}{2}\right)}\left\{\frac{K_{\frac{\nu}{2}}\left(t\sqrt{\nu}\right)}{2\Gamma\left(\frac{\nu}{2}\right)} - \frac{\left(\frac{t\sqrt{\nu}}{2}\right)^{\frac{\nu}{2}}}{\nu\mathrm{B}\left(\frac{\nu}{2},\frac{1}{2}\right)\Gamma(\nu)}\right. \\ &\quad\times \sum_{n\geq 0} \frac{\left(\frac{1}{2}\right)_n\left(\frac{\nu}{2}\right)_n\ K_{\nu+n}\left(t\sqrt{\nu}\right)}{\left(\frac{\nu}{2}+1\right)_n(\nu)_n\ n!}\left(\frac{t\sqrt{\nu}}{2}\right)^n + \frac{\left(\frac{t\sqrt{\nu}}{2}\right)^{\nu}}{\nu^2\left[\mathrm{B}\left(\frac{\nu}{2},\frac{1}{2}\right)\right]^2\Gamma\left(\frac{3\nu}{2}\right)} \\ &\quad\left.\times \sum_{j\geq 0}\sum_{n=0}^{j} \frac{\left(\frac{1}{2}\right)_{j-n}\left(\frac{\nu}{2}\right)_{j-n}\left(\frac{1}{2}\right)_n\left(\frac{\nu}{2}\right)_n\ K_{\frac{3\nu}{2}+j}\left(t\sqrt{\nu}\right)}{\left(\frac{\nu}{2}+1\right)_{j-n}\left(\frac{\nu}{2}+1\right)_n\ \left(\frac{3\nu}{2}\right)_j\ (j-n)!\ n!}\left(\frac{t\sqrt{\nu}}{2}\right)^j\right\}.\end{aligned}$$

Proof. The probability density function of $T_{(3)}$ is

$$f_3(x) = \frac{3\ [\sigma(x)]^{\frac{\nu+1}{2}}}{\sqrt{\nu}\ \mathrm{B}\left(\frac{\nu}{2}, \frac{1}{2}\right)} \left[1 - \frac{1}{2}\ I_{\sigma(x)}\left(\frac{\nu}{2}, \frac{1}{2}\right)\right]^2$$

for $x \in \mathbb{R}$. Therefore,

$$\phi_3(t) = \frac{6}{\mathrm{B}\left(\frac{\nu}{2}, \frac{1}{2}\right)} \int_0^\infty \frac{\cos(tx)}{\left(1 + \dfrac{x^2}{\nu}\right)^{\frac{\nu+1}{2}}} \left[1 - \frac{1}{2}\ I_{\sigma(x)}\left(\frac{\nu}{2}, \frac{1}{2}\right)\right]^2 \mathrm{d}x.$$

This expression can be written as the sum of three integrals say $\mathscr{I}_1 + \mathscr{I}_2 + \mathscr{I}_3$. Since $\nu > 0$, by (5),

$$\mathscr{I}_1 = \frac{6}{\mathrm{B}\left(\frac{\nu}{2}, \frac{1}{2}\right)} \int_0^\infty \frac{\cos\left(t\sqrt{\nu}\ x\right)}{(1+x^2)^{\frac{\nu+1}{2}}}\ \mathrm{d}x = \frac{6\ \sqrt{\pi}}{\mathrm{B}\left(\frac{\nu}{2}, \frac{1}{2}\right)\Gamma\left(\frac{\nu}{2}\right)} \left(\frac{t\sqrt{\nu}}{2}\right)^{\frac{\nu}{2}} K_{\frac{\nu}{2}}\left(t\sqrt{\nu}\right). \quad (6)$$

Next, by transforming the binomial term in the integrand of $B_{\sigma(x)}$, changing legitimately the order of summation and integration, and using (5),

$$\begin{aligned}
\mathscr{I}_2 &= -\frac{6}{\left[\mathrm{B}\left(\frac{\nu}{2}, \frac{1}{2}\right)\right]^2} \int_0^\infty \frac{\cos(tx)}{(1+x^2)^{\frac{\nu+1}{2}}} \int_0^{\left(1+x^2\right)^{-1}} u^{\frac{\nu}{2}-1}(1-u)^{-\frac{1}{2}}\ \mathrm{d}u\ \mathrm{d}x \\
&= -\frac{12}{\nu\left[\mathrm{B}\left(\frac{\nu}{2}, \frac{1}{2}\right)\right]^2} \sum_{n\ge 0} \frac{\left(\frac{1}{2}\right)_n \left(\frac{\nu}{2}\right)_n}{\left(\frac{\nu}{2}+1\right)_n\ n!} \int_0^\infty \frac{\cos\left(t\sqrt{\nu}\ x\right)}{(1+x^2)^{\nu+n+\frac{1}{2}}}\ \mathrm{d}x \\
&= -\frac{12\ \sqrt{\pi}\ \left(\frac{t\sqrt{\nu}}{2}\right)^{\nu}}{\nu\left[\mathrm{B}\left(\frac{\nu}{2}, \frac{1}{2}\right)\right]^2 \Gamma(\nu)} \sum_{n\ge 0} \frac{\left(\frac{1}{2}\right)_n \left(\frac{\nu}{2}\right)_n\ K_{\nu+n}\left(t\sqrt{\nu}\right)}{\left(\frac{\nu}{2}+1\right)_n (\nu)_n\ n!} \left(\frac{t\sqrt{\nu}}{2}\right)^n. \qquad (7)
\end{aligned}$$

Similarly,

$$\begin{aligned}
\mathscr{I}_3 &= \frac{3}{2\left[\mathrm{B}\left(\frac{\nu}{2}, \frac{1}{2}\right)\right]^3} \int_0^\infty \frac{\cos(tx)}{(1+x^2)^{\frac{\nu+1}{2}}} \int_0^{\left(1+x^2\right)^{-1}} \int_0^{\left(1+x^2\right)^{-1}} \frac{(yz)^{\frac{\nu}{2}-1}\ \mathrm{d}y\ \mathrm{d}z}{\sqrt{(1-y)(1-z)}}\ \mathrm{d}x \\
&= \frac{12\ \sqrt{\pi}\ \left(\frac{t\sqrt{\nu}}{2}\right)^{\frac{3\nu}{2}}}{\nu^2\left[\mathrm{B}\left(\frac{\nu}{2}, \frac{1}{2}\right)\right]^3 \Gamma\left(\frac{3\nu}{2}\right)} \\
&\quad \times \sum_{m,n\ge 0} \frac{\left(\frac{1}{2}\right)_m \left(\frac{\nu}{2}\right)_m \left(\frac{1}{2}\right)_n \left(\frac{\nu}{2}\right)_n\ K_{\frac{3\nu}{2}+m+n}\left(t\sqrt{\nu}\right)}{\left(\frac{\nu}{2}+1\right)_m \left(\frac{\nu}{2}+1\right)_n\ \left(\frac{3\nu}{2}\right)_{m+n}\ m!\ n!} \left(\frac{t\sqrt{\nu}}{2}\right)^{m+n}. \qquad (8)
\end{aligned}$$

The sum of (6), (7) and (8) gives the asserted expression. □

Corollary 2. *For all $t \in \mathbb{R}$ and $\nu > 0$, the characteristic function of $T_{(1)}$ for $n = 3$ is*

$$\phi_1(t) = \frac{12\,\sqrt{\pi}\,\left(\frac{t\sqrt{\nu}}{2}\right)^{\frac{3\nu}{2}}}{\nu^2\left[\mathrm{B}\left(\frac{\nu}{2},\frac{1}{2}\right)\right]^3\Gamma\left(\frac{3\nu}{2}\right)} \times \sum_{j\geq 0}\sum_{n=0}^{j}\frac{\left(\frac{1}{2}\right)_{j-n}\left(\frac{\nu}{2}\right)_{j-n}\left(\frac{1}{2}\right)_{n}\left(\frac{\nu}{2}\right)_{n}\,K_{\frac{3\nu}{2}+j}\left(t\sqrt{\nu}\right)}{\left(\frac{\nu}{2}+1\right)_{j-n}\left(\frac{\nu}{2}+1\right)_{n}\left(\frac{3\nu}{2}\right)_{j}\,(j-n)!\,n!}\left(\frac{t\sqrt{\nu}}{2}\right)^{j}.$$

Proof. The probability density function of $T_{(1)}$ is

$$f_1(x) = \frac{3\left[\sigma(x)\right]^{\frac{\nu+1}{2}}}{4\sqrt{\nu}\,\left[\mathrm{B}\left(\frac{\nu}{2},\frac{1}{2}\right)\right]^3}\,\mathrm{B}^2_{\sigma(x)}\left(\frac{\nu}{2},\frac{1}{2}\right)$$

for $x \in \mathbb{R}$. The corresponding characteristic function is actually equal to $\mathscr{I}_3$. $\square$

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References

[1] C. Amsler, P. Schmidt, *Tests of short memory with thick-tailed errors*, J. Bus. Econom. Statist. **30**(2012), 381–390.
[2] G. Bhardwaj, V. Corradi, N. R. Swanson, *A simulation-based specification test for diffusion processes*, J. Bus. Econom. Statist. **26**(2008), 176–193.
[3] H. J. Bierens, H. Song, *Semi-nonparametric estimation of independently and identically repeated first-price auctions via an integrated simulated moments method*, J. Econometrics **168**(2012), 108–119.
[4] H. J. Bierens, L. Wang, *Integrated conditional moment tests for parametric conditional distributions*, Econometric Theory **28**(2012), 328–362.
[5] B. Chen, Y. M. Hong, *Testing for the Markov property in time series*, Econometric Theory **28**(2012), 130–178.
[6] M. A. Delgado, J. C. Escanciano, *Nonparametric tests for conditional symmetry in dynamic models*, J. Econometrics **141**(2007), 652–682.
[7] Y. G. Fan, *Goodness-of-fit tests based on kernel density estimators with fixed smoothing parameter*, Econometric Theory **14**(1998), 604–621.
[8] R. A. Fisher, M. J. R. Healy, *New tables of Behrens' test of significance*, J. R. Stat. Soc. Ser. B Stat. Methodol. **18**(1956), 212–216.
[9] C. Francq, M. D. Jimenez-Gamero, S. G. Meintanis, *Tests for conditional ellipticity in multivariate GARCH models*, J. Econometrics **196**(2017), 305–319.
[10] Y. M. Hong, Y. J. Lee, *Generalized spectral tests for conditional mean models in time series with conditional heteroscedasticity of unknown form*, Rev. Econ. Stud. **72**(2005), 499–541.
[11] Y. M. Hong, X. Wang, S. Y. Wang, *Testing strict stationarity with applications to macroeconomic time series*, Internat. Econom. Rev. **58**(2017), 1227–1277.

[12] G. J. Jiang, J. L. Knight, *ECF estimation of Markov models where the transition density is unknown*, Econom. J. **13**(2010), 245–270.

[13] N. L. Johnson, S. Kotz, N. Balakrishnan, *Continuous Univariate Distributions*, Volume 2, Wiley Series in Probability and Mathematical Statistics, John Wiley and Sons, Inc., New York, 1995.

[14] A. B. M. L. Kabir, M. Rahman, *Bounds for expected values of order statistics*, Comm. Stat. **3**(1974), 557–566.

[15] J. Knight, C. Q. Ning, *Estimation of the stochastic conditional duration model via alternative methods*, Econom. J. **11**(2008), 593–616.

[16] C. M. Kuan, W. M. Lee, *A new test of the martingale difference hypothesis*, Stud. Nonlinear Dyn. Econom. **8**(2004), Article 1.

[17] S. Nadarajah, *Explicit expressions for moments of t order statistics*, C. R. Math. **345**(2007), 523–526.

[18] S. Nadarajah, *Explicit expressions for moments of order statistics*, Statist. Probab. Lett. **78**(2008), 196–205.

[19] R Development Core Team, *R: A Language and Environment for Statistical Computing*, R Foundation for Statistical Computing, Vienna, Austria, 2017.

[20] K. J. Singleton, *Estimation of affine asset pricing models using the empirical characteristic function*, J. Econometrics **102**(2001), 111–141.

[21] Y. G. Song, *A martingale approach for testing diffusion models based on infinitesimal operator*, J. Econometrics **162**(2011), 189–212.

[22] D. C. Vaughan, *Expected values, variances and covariances of order statistics for Student's t distribution with two degrees of freedom*, Comm. Statist. Simulation Comput. **21**(1992), 391–404.

[23] G. N. Watson, *A Treatise on the Theory of Bessel Functions*, Cambridge University Press, Cambridge, 1922.

[24] D. H. Xu, T. S. Wirjanto, *An empirical characteristic function approach to VaR under a mixture-of-normal distribution with time-varying volatility*, J. Deriv. **18**(2010), 39–58.