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Subject review

Comparison of L_1 Norm and L_2 Norm Minimisation Methods in Trigonometric Levelling Networks

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Abstract: The most widely-used parameter estimation method today is the L_2 norm minimisation method known as the Least Squares Method (LSM). The solution to the L_2 norm minimisation method is always unique and is easily computed. This method distributes errors and is sensitive to outlying measurements. Therefore, a robust technique known as the Least Absolute Values Method (LAVM) might be used for the detection of outliers and for the estimation of parameters. In this paper, the formulation of the L_1 norm minimisation method will be explained and the success of the method in the detection of gross errors will be investigated in a trigonometric levelling network.

Keywords: linear programming; measurements with gross error; simplex method; trigonometric levelling networks

1 INTRODUCTION

Accuracy, precision, and reliability are important quality criteria used in geodetic networks. Accuracy is the degree of closeness of an estimated value to its true value and precision is the degree of closeness of observations to their mean values. Where the observations are only affected by inevitable random errors, accuracy and precision can be used interchangeably. Reliability is the resistance ability of a network to outliers. Redundant measurements are made to increase the accuracy of the computed results of a geodetic network and make it possible to adjust for the estimation of unknown parameters with greater precision and proper error analysis. Adjustments of the observations are usually performed using the method of least squares, i.e. L_2 norm minimisation that is based on the minimisation of the sum of the squares of the residuals, which permits the estimation of the most probable values of unknown parameters. The calculation algorithm of this method is easy and the solution is always unique, but it is vulnerable against outliers, i.e. its observations should be free of blunders and systematic errors. The network geometry may not be suitable for a successful outlier diagnosis and the outlier detection test may not recognise the gross error. The parameter estimation results obtained from the method of least squares are badly affected if the outliers are present in the observation data set, and the spreading effect of the method of least squares makes the diagnosis of outliers difficult by inspection of the residuals [1]. So, outlying observations must be detected and eliminated. For this purpose, robust estimation techniques can be used. The advantage of these methods is that the effects of blunders are minimised or eliminated from the adjustment results, and outliers can be easily detected. Robust techniques decrease the corrupt effect of outliers on the estimated parameters [2]. The most widely used robust estimation techniques are L_1 norm minimisation, M-estimation methods, the Least Median Squares (LMS) method, the Least Trimmed Squares (LTS) method, and the sign-constrained robust least squares method. For more information about robust estimation techniques, refer to [3-10].

Trigonometric height determination has been employed in rough terrain by measuring the zenith angle and slope distance between two points. This method is successful in the height determination of rough fields. The accuracy of trigonometric levelling may be enhanced by reducing the lines of sight and using simultaneous reciprocal observations to reduce the refraction effects [11].

In this paper, the robust L_1 norm minimisation method is performed in a trigonometric levelling network adjustment. The efficiency of the method in parameter estimation and outlier detection is demonstrated with a numerical example. As is well known, the method of least squares is the best linear unbiased estimator as it produces good results when unavoidable random errors affect our observations. The numerical example given in this paper shows that when observations are contaminated with gross errors, L_1 norm minimisation can yield parameter estimates like the results produced by least squares with gross error-free observations.

2 OUTLIER DETECTION

Gross errors are generally described as errors of large magnitude. Let us assume that the observation vector $\boldsymbol{l}$ is contaminated by a gross error vector $\nabla\boldsymbol{e}$. The effect of $\nabla\boldsymbol{e}$ on the estimated residual vector $\boldsymbol{v}$ is

$$\nabla\boldsymbol{v} = -\boldsymbol{R}\nabla\boldsymbol{e}, \tag{1}$$

where $\boldsymbol{R}$ is the redundancy matrix that portrays the network geometry.

The total residual vector because of both inevitable random errors and gross errors is

$$\tilde{\boldsymbol{v}} = \boldsymbol{v} + \nabla\boldsymbol{e}. \tag{2}$$

Therefore, the existence of gross errors in observation data can increase the magnitude of residuals in a least squares adjustment. So, an outlier is described as a residual that exceeds some boundary value that is based on stochastic features of the observations used in the adjustment [12].

In Geodesy, outlier detection methods can be divided into two broad categories, that is, tests for outliers and robust estimation methods. Outliers are generally searched in an iterative manner and after one outlier is found by applying a test method such as Baarda's data snooping technique, this observation is discarded and adjustment computations are repeated to find other possible outlying observations. Robust estimation methods may also be used to detect outliers and are advantageous in minimising or eliminating the effects of outliers on the adjustment results. The weights of observations are changed during the robust estimation procedure. However, some robust estimation methods such as the L1 norm minimisation or the LMS method try to minimise a given function of residuals [1, 10].

3 L2 NORM MINIMIZATION

In geodetic parameter estimation, the L_2 norm minimisation method also known as the LSM is commonly applied [13]. The L_2 norm minimisation method is a parameter estimation method that tries to minimise the sum of the squared residuals ($p[vv]$ = min). This method is widely used because the algorithm of the calculation is easy and no assumption about the distribution of the observations is needed, i.e., only the variance-covariance matrix of the observations must be constructed [14].

Random errors have a normal distribution with a specific standard deviation. For normally distributed measurement errors, the probability of being outside $\pm 3\sigma$ range is 0.003 and residuals larger than $\pm 3\sigma$ are treated as gross errors [15]. Parameter estimations with L_2 norm minimisation yield the best results in terms of minimum variance and maximum likelihood principles if blunder and systematic error-free observations are made. Any measurement whose residual exceeds a certain amount is an outlier. The sensitivity and geometry of the network also plays a role. Outliers can be detected using iterative methods such as Baarda, Pope etc. When one measurement is detected as an outlier, it is not used in the next adjustment. Unfortunately, L_2 norm minimisation yields deteriorated results in the presence of outliers. In parameter estimation, the difference between the estimated and true values is named as 'missing' and the aim is to minimise this difference [16]. When parameters are estimated by using the L_2 norm minimisation method, the minimum of the difference is named as 'the principle of minimum variance' and is one of the best aspects of the parameter estimation using the L_2 norm minimisation method.

The functional model of a linear or linearized geodetic parameter estimation problem is given as follows:

$$\boldsymbol{v} = \boldsymbol{A}\boldsymbol{x} - \boldsymbol{l}, \tag{3}$$

where $\boldsymbol{A}_{n\times u}$ is design matrix; $\boldsymbol{x}_{u\times 1}$ is the vector of unknown parameters; $\boldsymbol{l}_{n\times 1}$ is the vector of observations; and $\boldsymbol{v}_{n\times 1}$ is the vector of residuals. n is the number of observations and u is the number of unknowns. The stochastic part of the adjustment contains the weight matrix of observations $\boldsymbol{P}_{n\times n}$. The least squares solution of unknowns is accomplished using the following equation:

$$\boldsymbol{x} = (\boldsymbol{A}^{\mathrm{T}}\boldsymbol{P}\boldsymbol{A})^{-1}\boldsymbol{A}^{\mathrm{T}}\boldsymbol{P}\boldsymbol{l} = \boldsymbol{Q}_{xx}\boldsymbol{A}^{\mathrm{T}}\boldsymbol{P}\boldsymbol{l}. \tag{4}$$

Then, residuals are estimated, using Eq. (3). In the case of the free adjustment of geodetic networks, depending on the number of rank deficient (d), established $\boldsymbol{G}^{\mathrm{T}}$ matrix provides the condition as follows:

$$\boldsymbol{G}^{\mathrm{T}}\boldsymbol{x} = 0. \tag{5}$$

In this case, the unknown parameters vector is as follows:

$$\boldsymbol{x} = \left[(\boldsymbol{A}^{\mathrm{T}}\boldsymbol{P}\boldsymbol{A}^{-1} + \boldsymbol{G}\boldsymbol{G}^{\mathrm{T}})^{-1} - \boldsymbol{G}\boldsymbol{G}^{\mathrm{T}}\right]\boldsymbol{A}^{\mathrm{T}}\boldsymbol{P}\boldsymbol{l}. \tag{6}$$

The L_2 norm minimisation method smooths out blunders across the entire data set. Partial redundancy reflects the corrupt effect of outliers on the estimated parameters in any observation. Using the results of the free adjustment of network, partial redundancy is:

$$\boldsymbol{r}_i = \boldsymbol{P}\boldsymbol{Q}_{v_i v_i}, \tag{7}$$

where

$$\boldsymbol{Q}_{v_i v_i} = \boldsymbol{P}^{-1} - \boldsymbol{A}\boldsymbol{Q}_{xx}\boldsymbol{A}^{\mathrm{T}}. \tag{8}$$

In Eqs. (7) and (8), $\boldsymbol{A}_{n\times u}$ is the design matrix; $\boldsymbol{Q}_{xx_{u\times u}}$ is the cofactor matrix of unknown parameters; $\boldsymbol{P}_{n\times n}$ is the weight matrix of observations; $\boldsymbol{Q}_{vv_{u\times u}}$ is the cofactor matrix of residuals; and $\boldsymbol{r}_i$ is the partial redundancy value of the i^{th} observation. The partial redundancy value of a 'good' geodetic network is required to be close to 1.

The residual equation for the adjustment of trigonometric elevation networks is:

$$v_{Z_{ij}} = -l_{ij} + \frac{\sin^2 Z_{ij}^0}{S_{ij}}\rho\,\mathrm{d}H_i - \frac{\sin^2 Z_{ij}^0}{S_{ij}}\rho\,\mathrm{d}H_j, \tag{9}$$

where

$$Z_{ij}^0 = \operatorname{arccotg}\left(\frac{H_j^0 - H_i^0}{S_{ij}} - \frac{I_t - T_b}{S_{ij}} - \frac{1-k}{2R}S_{ij}\right), \tag{10}$$

$$-l_{ij} = -Z_{ij} + Z_{ij}^0. \tag{11}$$

The number of Eqs. (9) must be as the number of vertical angles measured. In the case of free adjustment of geodetic networks, the $\boldsymbol{G}^{\mathrm{T}}$ matrix is as follows:

$$\frac{\boldsymbol{G}^{\mathrm{T}}}{1,u} = \left[\frac{1}{\sqrt{p}}\frac{1}{\sqrt{p}}\frac{1}{\sqrt{p}}\cdots\frac{1}{\sqrt{p}}\right] \tag{12}$$

In these equations:
Z_{ij} - Vertical angle measured
Z_{ij}^0 - Vertical angle calculated
H_i - Approximate height of point on which the theodolite is set up
H_j - Approximate height of point on which the reflector is set up
I_t - The height of the theodolite

T_b - The height of the reflector
k - Refraction coefficient (0.13)
R - The radius of the earth (R = 6370 km)
S_{ij} - Horizontal distance between i and j points
p - Number of points in the trigonometric levelling network.

4 L1 NORM MINIMIZATION

In the classical Gauss-Markov model, the unknown parameters x for a linear (linearized) parametric adjustment are determined based on the following functional and stochastic models [17, 18];

$$\begin{aligned} &\boldsymbol{L}+\boldsymbol{v}=\boldsymbol{A x}, \\ &\boldsymbol{G}^{\mathrm{T}} \boldsymbol{x}=0, \\ &\boldsymbol{P}=\boldsymbol{Q}_l^{-1}=\sigma_0^2 \boldsymbol{C}_l^{-1}, \end{aligned} \tag{13}$$

where $\boldsymbol{v}_{n\times 1}$ is the vector of residuals; $\boldsymbol{l}_{n\times 1}$ is the vector of observations; $\boldsymbol{A}_{n\times u}$ is the rank deficient design matrix; and $\boldsymbol{P}_{n\times n}$ is the weight matrix of observations as mentioned previously. $\boldsymbol{G}_{u\times d}$ is the datum matrix of the network added to complete the rank deficiency of the design matrix; $\boldsymbol{0}_{d\times 1}$ is the zero vector; $\boldsymbol{C}_{l(n\times n)}$ is the covariance matrix of observations; $\boldsymbol{Q}_{l(n\times n)}$ is the cofactor matrix of observations; and σ_0^2 is a priori variance factor [6, 19].

As is known, the parameter estimation methods try to arrive at optimal values for unknowns by minimising a function of the residuals. This function is called an objective function. L1 norm minimisation is the estimation of parameters by minimising the sum of the absolute residuals [17]. In this method, the objective function is described as follows:

$$p^{\mathrm{T}} \,|\, v \,|=[p \,|\, v \,|]=\sum p \,|\, v \,|=\min . \tag{14}$$

The same mathematical model is used for both L2 norm minimisation and L1 norm minimisation. However, the objective functions of the two methods are different. To solve the problem using the L1 norm minimisation method the simplex method is used, transforming the linear programming [17].

To solve the problem using linear programming, there must be an objective function with an equation system which contains restrictions and has positive whole unknown parameters. This objective function is:

$$\begin{aligned} &\boldsymbol{C x}=\boldsymbol{d} \\ &\boldsymbol{f}=\boldsymbol{b}^{\mathrm{T}} \boldsymbol{x}=\min \end{aligned} \tag{15}$$

So, the problem according to Eq. (13) must be transformed according to Eq. (15). In the L1 norm minimisation method, the unknown parameters are calculated first, then the residuals. Here the unknown vector includes both the unknown parameters and residuals. In this case, the following equation systems must be solved under the principle of $[p|v|]$ = min:

$$\begin{aligned} &\boldsymbol{A x}-\boldsymbol{v}=\boldsymbol{l} \\ &\boldsymbol{G}^{\mathrm{T}} \boldsymbol{x}=0 \\ &\begin{bmatrix} \boldsymbol{A}_{n\times u} & -\boldsymbol{I}_{n\times n} \\ \boldsymbol{G}_{d\times u}^{\mathrm{T}} & \boldsymbol{0}_{d\times d} \end{bmatrix}\begin{bmatrix} \boldsymbol{x} \\ \boldsymbol{v} \end{bmatrix}=\begin{bmatrix} \boldsymbol{l} \\ \boldsymbol{0} \end{bmatrix} \end{aligned} \tag{16}$$

To solve Eq. (16) under the principle of $[p|v|]$ = min according to linear programming, the unknown parameters and residuals must be positive [17, 20]. For this purpose, x unknown parameters and each residual are reformulated as the difference of the new unknown parameters and the new residuals which are derived as positive or negative:

$$\boldsymbol{x}=\boldsymbol{x}^{+}-\boldsymbol{x}^{-}\boldsymbol{x}^{+},\ \boldsymbol{x}^{-} \geq 0 \tag{17}$$

$$\boldsymbol{v}=\boldsymbol{v}^{+}-\boldsymbol{v}^{-}\boldsymbol{v}^{+},\ \boldsymbol{v}^{-} \geq 0 \tag{18}$$

With this reformulation, the Eq. (16) is rewritten as follows:

$$\underbrace{\begin{bmatrix} \boldsymbol{A}_{n\times u} & -\boldsymbol{A}_{n\times u} & -\boldsymbol{I}_{n\times n} & \boldsymbol{I}_{n\times n} \\ \boldsymbol{G}_{d\times u}^{\mathrm{T}} & -\boldsymbol{G}_{d\times u}^{\mathrm{T}} & \boldsymbol{0}_{d\times n} & \boldsymbol{0}_{d\times n} \end{bmatrix}}_{\boldsymbol{C}} \underbrace{\begin{bmatrix} x^{+} \\ x^{-} \\ v^{+} \\ v^{-} \end{bmatrix}}_{\boldsymbol{x}}=\underbrace{\begin{bmatrix} \boldsymbol{l} \\ \boldsymbol{0} \end{bmatrix}}_{\boldsymbol{d}} \tag{19}$$

$$\boldsymbol{C x}=\boldsymbol{d}$$

In this case, the number of unknown parameters and residuals increases twice. In Eq. (19), there are equations of $n + d$ numbers and unknown parameters of $2(n + u)$ numbers. In the L1 norm minimisation method, $[p|v|]$ = min the principle which requires linear programming is transformed $\boldsymbol{f} = \boldsymbol{b}^{\mathrm{T}}\boldsymbol{x}$ (objective function). In this case, the L1 norm minimisation method Eq. (14) is rewritten as follows:

$$p^{\mathrm{T}} \,|\, v \,|= p^{\mathrm{T}} \,|\, v^{+}-v^{-} \,|=\left[p \,|\, v^{+}-v^{-} \,|\right]=\sum p \,|\, v^{+}+v^{-} \,|=\min$$

where, $v+$ and v^{-} are equal to zero, $\left[p \,|\, v^{+}-v^{-} \,|\right]$ $=\sum p \,|\, v^{+}+v^{-} \,|$ can be written. The matrix form of the objective function as the unknown function is written as follows:

$$\boldsymbol{f}=\boldsymbol{b}^{\mathrm{T}} \boldsymbol{x}=[p \,|\, v \,|]=\min \tag{20}$$

$$\boldsymbol{b}=\left[\boldsymbol{0}_u^{\mathrm{T}} \boldsymbol{0}_u^{\mathrm{T}} \boldsymbol{P}_n^{\mathrm{T}} \boldsymbol{P}_n^{\mathrm{T}}\right] \tag{21}$$

$$\boldsymbol{x}^{\mathrm{T}}=\left[\boldsymbol{x}^{+} \boldsymbol{x}^{-} \boldsymbol{v}^{+} \boldsymbol{v}^{-}\right] \tag{22}$$

$$\boldsymbol{0}^{\mathrm{T}}=[0\ 0\ 0\ 0\ \ldots\ 0]^{\mathrm{T}} \tag{23}$$

zero vector with n elements

$$\boldsymbol{p}^{\mathrm{T}}=\left[p_1 p_2 p_3\ \cdots\ p_n\right]^{\mathrm{T}} \tag{24}$$

weight vector with n elements.

Considering Eq. (19) $\boldsymbol{Cx} = \boldsymbol{d}$ constraints and in Eq. (20) $\boldsymbol{f} = \boldsymbol{b}^{\mathrm{T}}\boldsymbol{x}$ (objective function), the solution to the linear programming problem is calculated using a simplex algorithm as previously described. For this purpose, the subroutine 'linprog.m' of MATLAB software has been used. The solution to the problem, having a rank defect, can be found without the need for extra processing (such as pseudo inverse) using the L_1 norm minimisation method [21, 22]. This is one of the advantages of using the L_1 norm minimisation method [23].

5 NUMERICAL APPLICATION

To compare L_1 norm and L_2 norm minimisation methods, an application was made in a trigonometric levelling network (Fig. 1). Zenith angles and distances were observed (Tab. 1). The accuracy of distance (D) and angle observations are ±(2+2 ppm D) mm and 2^{cc}, respectively and the number of repetitions is 2. In adjusting the zenith angles, point heights were taken as unknown. As all the zenith angles were made by the same survey team, using the same equipment and equal precision on the whole points, the weights were taken as 1, i.e. $\boldsymbol{P}$ is a unit matrix.

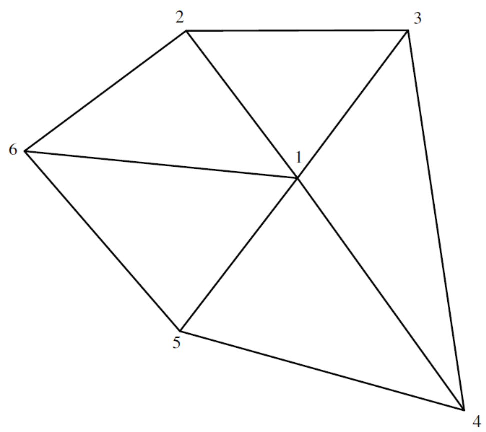

Figure 1 Trigonometric levelling network

Firstly, the free network adjustment is carried out and the outliers are investigated using the Pope Method, but no outliers were detected. Partial redundancies of the zenith angles were calculated using equations (7-8) and then the measurements were adjusted using L1 and L2 norm minimisation methods. In the adjustment, the numbers of measurements were 20 (n = 20), and the numbers of unknowns were 5 (u = 5).

After, the first and 13th measurements were burdened virtually with gross errors, the amounts of which were -20^c and $+10^c$, respectively, the L_1 and L_2 norm minimisation methods were applied with this measurement including gross errors. In this example, the following matrices were used: $\boldsymbol{C}$ in Eq. (19) was a 50 × 20 matrix, $\boldsymbol{d}$ in Eq. (19) was a 20 × 1 matrix and $\boldsymbol{b}$ in equation (21) was a 50 × 1 matrix. At the solution of the L_1 norm minimisation method using linear programming, the subroutine 'linprog.m' in MATLAB was used, and at the results of this solution, the solution vector ($\boldsymbol{x}$) of 50 × 1 size was obtained. As mentioned, this vector included $\boldsymbol{x}^+$, $\boldsymbol{x}^-$, $\boldsymbol{v}^+$, $\boldsymbol{v}^-$ sub-vectors. These sub-vectors, residuals ($\boldsymbol{v}$) and adjusted heights ($\boldsymbol{x}$) were obtained by using Eqs. (17)-(19) (Tabs. 2-5).

Table 1 Observations of zenith angle and distance with instrument heights (I_t) and target heights (T_b)

Points		Meas. zenith	Distances	I_t	T_b
From	To	angles (^{g})	(m)	(m)	(m)
1	2	96,3458	1495,636	1,56	2,05
1	3	101,1175	1647,062	1,56	2,05
1	4	101,5039	2317,524	1,56	2,05
1	5	98,7084	1562,958	1,56	2,05
1	6	97,0743	2194,196	1,56	2,05
2	3	104,1033	1774,170	1,54	2,05
2	1	103,6255	1495,636	1,54	2,05
2	6	99,4769	1875,414	1,54	2,05
3	4	100,5255	3134,617	1,54	2,05
3	1	98,8626	1647,062	1,54	2,05
3	2	95,8797	1774,170	1,54	2,05
4	5	97,6675	2362,309	1,41	2,05
4	1	98,4901	2317,524	1,41	2,05
4	3	99,4806	3134,617	1,41	2,05
5	6	97,7006	1924,506	1,46	2,05
5	1	101,2643	1562,958	1,46	2,05
5	4	102,324	2362,309	1,46	2,05
6	2	100,5085	1875,414	1,56	2,05
6	1	102,9191	2194,196	1,56	2,05
6	5	102,2823	1924,506	1,56	2,05

Table 2 Corrections for L_1 and L_2 norm minimisation methods and partial redundancies

Obs. No	Points		L_1 norm v^{cc}	L_2 norm v^{cc}	Partial redundancies r
	From	To			
1	**1**	**2**	**0,00**	**3,93**	**0,718**
2	1	3	−7,81	−22,46	0,717
3	1	4	−1,84	−9,22	0,771
4	1	5	−30,88	−39,82	0,697
5	1	6	0,25	−4,73	0,820
6	2	3	−0,24	−17,13	0,726
7	2	1	−7,64	−11,57	0,718
8	2	6	0,00	−8,97	0,730
9	3	4	−8,09	−5,85	0,837
10	3	1	−36,43	−21,77	0,717
11	3	2	−40,61	−23,73	0,726
12	4	5	−2,39	−1,07	0,755
13	**4**	**1**	**−47,00**	**−39,62**	**0,771**
14	4	3	−13,92	−16,16	0,837
15	5	6	−25,14	−23,57	0,730
16	5	1	0,00	8,93	0,697
17	5	4	−38,51	−39,84	0,755
18	6	2	−30,38	−21,41	0,730
19	6	1	−27,61	−22,63	0,820
20	6	5	6,46	4,89	0,730
					$\Sigma r_i = 15$

Table 3 Adjusted heights for fix and new points

Points	Apr. heights H (m)	L_1 norm solution		L_2 norm solution	
		Res. v (cm)	Adj. heights (m)	Res. v (cm)	Adj. heights (m)
1	1000,00	-	1000,0000	-	1000,0000
2	1085,60	0,67	1085,6067	−0,25	1085,5975
3	970,80	0,05	970,8005	3,85	970,8385
4	945,20	−7,40	945,1260	−4,71	945,1529
5	1031,60	−13,30	1031,4670	−11,11	1031,4889
6	1100,80	−5,28	1100,7472	−3,56	1100,7644

As can be seen in Tab. 4, when the measurements with gross errors were adjusted, the residual had a -20^c gross error calculated as $19^c,92$ using the L_1 norm minimisation method and $14^c,11$ using the L_2 norm minimisation method. The residual had $+10^c$ gross error calculated as $-10^c,49$ using the L_1 norm minimisation method and $-7^c,53$ using the L_2 norm minimisation method. Also, it is seen that gross errors at the first and thirteenth rows affected both their measurements and the others using the L_2 norm minimisation method. In the results, in detecting the gross errors it is seen that the L_1 norm minimisation method provides more successful results than the L_2 norm minimisation method.

Table 4 Measurements with gross errors and corrections using the L_1 and L_2 norm minimisation methods

Obs. No	Points		Meas. zenith angles (^{g})	L_1 norm v^{cc}	L_2 norm v^{cc}
	From	To			
1	**1**	**2**	**96,1458**	**1992,36**	**1410,87**
2	1	3	101,1175	−14,49	−291,46
3	1	4	101,5039	0,00	−295,38
4	1	5	98,7084	−30,88	−214,52
5	1	6	97,0743	−4,96	−199,84
6	2	3	104,1033	0,00	233,35
7	2	1	103,6255	0,00	581,50
8	2	6	99,4769	0,00	236,24
9	3	4	100,5255	−3,21	−76,10
10	3	1	98,8626	−29,74	247,23
11	3	2	95,8797	−40,86	−274,20
12	4	5	97,6675	−4,20	164,02
13	**4**	**1**	**98,5901**	**−1048,84**	**−753,47**
14	4	3	99,4806	−18,80	54,09
15	5	6	97,7006	−31,09	−104,30
16	5	1	101,2643	0,00	183,64
17	5	4	102,3240	−36,70	−204,93
18	6	2	100,5085	−30,38	−266,62
19	6	1	102,9191	−22,39	172,49
20	6	5	102,2823	12,41	85,62

Table 5 Adj. heights for fix and new points by gross errors

Points	Apr. heights H (m)	L_1 norm solution		L_2 norm solution	
		Res. v (cm)	Adj. heights (m)	Res. v (cm)	Adj. heights (m)
1	1000,00	-	1000,0000	-	1000,0000
2	1085,60	2,47	1085,6247	139,31	1086,9931
3	970,80	1,78	970,8178	73,45	971,5345
4	945,20	−8,06	945,1194	99,48	946,1948
5	1031,60	−13,30	1031,4670	31,79	1031,9179
6	1100,80	−3,48	1100,7652	63,76	1101,4376

6 CONCLUSIONS

The most commonly applied parameter estimation method for geodetic networks is the L_2 norm minimisation method. The solutions provided using this method are easy and unique. Using this method, it is possible to calculate precision and reliability criteria of unknown parameters and their functions. However, the L_2 norm minimisation method is sensitive against gross errors. Because of the corruption effect of errors, when measurements include gross errors the method does not provide the correct results.

The L_1 norm minimisation method is an important estimation method and the advantages are that it is less sensitive against measurements with gross errors and is less affected from these measurements than the L_2 norm minimisation method. However, the L_1 norm minimisation method does not always provide a solution, and where there is a solution, the residuals of at least u (number of unknown parameters) and the number of measurements are always zero. If some of the residuals equal zero, it is a contradictious situation to the theory of error.

A parameter estimation method must be able to detect the gross errors and not distribute the effects of these errors to other measurements.

In this paper, an application was performed in a trigonometric levelling network to compare L_1 and L_2 norm minimisation methods. When the application results were evaluated, it obtained the following results:

- L_1 and L_2 norm minimisation methods give close results when measurements have only random errors,
- The L_1 norm minimisation method gives better results than the L_2 norm minimisation method in terms of blunder detection. If Tab. 4 is examined, gross errors given to the first and thirteenth measurements are reflected in the residuals of these measurements. Gross errors, which are given to the first and thirteenth measurements as -20^c, and 10^c respectively, affect the residuals of these measurements as $19^c,92$ and $-10^c,49$ respectively. However, in the solution using the L_2 norm minimisation method, the amounts of the effect of errors are $14^c,11$ and $-7^c,53$ for the first and thirteenth measurements, respectively,

In the trigonometric levelling network used in the application, the partial redundancy values are 0,718 for the first measurement and 0,771 for the thirteenth measurement. The sum of the redundancy values is equal to the degree of freedom of the network. The residuals of the first and thirteenth measurements using the L_2 norm minimisation method are equal to the multiplication of gross errors which is given to measurements artificially with redundancy values, and these residuals are smaller than the given gross errors. However, using the L_1 norm minimisation method, the residuals' measurements are almost as much as the amount of gross errors which are given to the measurements artificially. In this case, it is seen that the L_1 norm minimisation method produces more successful results than the L_2 norm minimisation method.

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