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SOME RESULTS ON q-HERMITE BASED HYBRID POLYNOMIALS

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Abstract. In this article, a hybrid class of the q-Hermite based Apostol type Frobenius-Euler polynomials is introduced by means of generating function and series representation. Several important formulas and recurrence relations for these polynomials are derived via different generating function methods. Further, the $2D$ q-Hermite based Apostol-Bernoulli, Apostol-Euler and Apostol-Genocchi polynomials are introduced and important relations for these polynomials are also established. Finally, a new class of the $2D$ q-Hermite based Appell polynomials is investigated as the generalization of the above polynomials. The determinant definitions for the $2D$ q-Hermite based Appell and related polynomials are also explored.

1. Introduction and preliminaries

The subject of q-calculus started appearing in the nineteenth century due to its applications in various fields of mathematics, physics and engineering. The definitions and notations of q-calculus reviewed here are taken from [2].

The q-analogue of the shifted factorial $(a)_n$ is given by

$$(a;q)_0 = 1, (a;q)_n = \prod_{m=0}^{n-1}(1-q^m a),\ n \in \mathbb{N}.$$

The q-analogues of a complex number a and of the factorial function are given by

$$[a]_q = \frac{1-q^a}{1-q},\ q \in \mathbb{C}-\{1\};\ a \in \mathbb{C},$$

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$$[n]_q! = \prod_{m=1}^{n} [m]_q = [1]_q[2]_q \cdots [n]_q = \frac{(q;q)_n}{(1-q)^n},\ q \neq 1;\ n \in \mathbb{N},$$

$$[0]_q! = 1,\ q \in \mathbb{C};\ 0 < q < 1.$$

The Gauss q-binomial coefficient $\begin{bmatrix} n \\ k \end{bmatrix}_q$ is given by

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{[n]_q!}{[k]_q![n-k]_q!} = \frac{(q;q)_n}{(q;q)_k(q;q)_{n-k}},\ k = 0, 1, \ldots, n.$$

The q-analogue of the function $(x+y)^n$ is given by

$$(x+y)_q^n := \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q q^{k(k-1)/2} x^{n-k} y^k,\ n \in \mathbb{N}_0. \tag{1.1}$$

The q-analogues of exponential functions are given by

$$e_q(x) = \sum_{n=0}^{\infty} \frac{x^n}{[n]_q!} := \frac{1}{((1-q)x;q)_\infty},\ 0 < |q| < 1;\ |x| < |1-q|^{-1},$$

$$E_q(x) = \sum_{n=0}^{\infty} q^{n(n-1)/2} \frac{x^n}{[n]_q!} := (-(1-q)x;q)_\infty,\ 0 < |q| < 1;\ x \in \mathbb{C}.$$

Moreover, the functions $e_q(x)$ and $E_q(x)$ satisfy the following properties:

$$D_q e_q(x) = e_q(x),\ D_q E_q(x) = E_q(qx), \tag{1.2}$$

where the q-derivative $D_q f$ of a function f at a point $0 \neq z \in \mathbb{C}$ is defined as follows:

$$D_q f(z) = \frac{f(qz) - f(z)}{qz - z},\ 0 < |q| < 1.$$

For any two arbitrary functions $f(z)$ and $g(z)$, the q-derivative operator D_q satisfies the following product and quotient relations:

$$D_{q,z}(f(z)g(z)) = f(z)D_{q,z}g(z) + g(qz)D_{q,z}f(z), \tag{1.3}$$

$$D_{q,z}\left(\frac{f(z)}{g(z)}\right) = \frac{g(qz)D_{q,z}f(z) - f(qz)D_{q,z}g(z)}{g(z)g(qz)}. \tag{1.4}$$

Recently, extensive investigations related to the q-Bernoulli polynomials $B_{n,q}(x)$, q-Euler polynomials $E_{n,q}(x)$ and q-Genocchi polynomials $G_{n,q}(x)$ and their generalizations in two variables x and y are considered, see for example [16, 18–20, 17, 10, 11, 25, 24]. We recall the following definitions.

Definition 1.1. *The q-Apostol-Bernoulli polynomials $\mathfrak{B}_{n,q}^{(\alpha)}(x,y;\lambda)$ of order α ($q \in \mathbb{C}$, $\alpha \in \mathbb{N}$, $0 < |q| < 1$) in x and y are defined by [19]*

$$\left(\frac{t}{\lambda e_q(t) - 1}\right)^{\alpha} e_q(xt)E_q(yt) = \sum_{n=0}^{\infty} \mathfrak{B}_{n,q}^{(\alpha)}(x,y;\lambda) \frac{t^n}{[n]_q!},\ |t + \log\lambda| < 2\pi, \tag{1.5}$$

where $\mathfrak{B}_{n,q}^{(\alpha)}(\lambda) := \mathfrak{B}_{n,q}^{(\alpha)}(0,0;\lambda)$ are the q-Apostol-Bernoulli numbers.

DEFINITION 1.2. *The q-Apostol-Euler polynomials $\mathfrak{E}^{(\alpha)}_{n,q}(x,y;\lambda)$ of order α ($q \in \mathbb{C}$, $\alpha \in \mathbb{N}$, $0 < |q| < 1$) in x and y are defined by [19]*

$$\left(\frac{2}{\lambda e_q(t)+1}\right)^{\alpha} e_q(xt)E_q(yt) = \sum_{n=0}^{\infty} \mathfrak{E}^{(\alpha)}_{n,q}(x,y;\lambda)\frac{t^n}{[n]_q!}, \ |t+\log(-\lambda)| < \pi, \tag{1.6}$$

where $\mathfrak{E}^{(\alpha)}_{n,q}(\lambda) := \mathfrak{E}^{(\alpha)}_{n,q}(0,0;\lambda)$ are the q-Apostol-Euler numbers.

DEFINITION 1.3. *The q-Apostol-Genocchi polynomials $\mathcal{G}^{(\alpha)}_{n,q}(x,y;\lambda)$ of order α ($q \in \mathbb{C}$, $\alpha \in \mathbb{N}$, $0 < |q| < 1$) in x and y are defined by [19]*

$$\left(\frac{2t}{\lambda e_q(t)+1}\right)^{\alpha} e_q(xt)E_q(yt) = \sum_{n=0}^{\infty} \mathcal{G}^{(\alpha)}_{n,q}(x,y;\lambda)\frac{t^n}{[n]_q!}, \ |t+\log(-\lambda)| < \pi, \tag{1.7}$$

where $\mathcal{G}^{(\alpha)}_{n,q}(\lambda) := \mathcal{G}^{(\alpha)}_{n,q}(0,0;\lambda)$ are the q-Apostol-Genocchi numbers.

Several unified forms of the Apostol-type polynomials are introduced and studied by many authors, for this see [7,21,23,22,27,8,9,12,15]. We recall the definition of the Apostol type q-Frobenius-Euler polynomials $\mathcal{H}^{(\alpha)}_{n,q}(x,y;u;\lambda)$ introduced and studied by Kurt in [13].

DEFINITION 1.4. *The Apostol type q-Frobenius-Euler polynomials of order α $\mathcal{H}^{(\alpha)}_{n,q}(x,y;u;\lambda)$ ($q \in \mathbb{C}$, $\alpha \in \mathbb{N}$, $0 < |q| < 1$) in x and y are defined by*

$$\left(\frac{1-u}{\lambda e_q(t)-u}\right)^{\alpha} e_q(xt)E_q(yt) = \sum_{n=0}^{\infty} \mathcal{H}^{(\alpha)}_{n,q}(x,y;u;\lambda)\frac{t^n}{[n]_q!}, \tag{1.8}$$

where $\mathcal{H}^{(\alpha)}_{n,q}(u;\lambda) := \mathcal{H}^{(\alpha)}_{n,q}(0,0;u;\lambda)$ are the Apostol type q-Frobenius-Euler numbers *defined by*

$$\left(\frac{1-u}{\lambda e_q(t)-u}\right)^{\alpha} = \sum_{n=0}^{\infty} \mathcal{H}^{(\alpha)}_{n,q}(0,0;u;\lambda)\frac{t^n}{[n]_q!}. \tag{1.9}$$

• For $y = 0$, the polynomials $\mathcal{H}^{(\alpha)}_{n,q}(x,y,u;\lambda)$ reduce to the q-Apostol type Frobenius-Euler polynomials $\mathcal{H}^{(\alpha)}_{n,q}(x,u;\lambda)$ in one variable [26].

• By letting $q \to 1^-$ and $\lambda = 1$, the polynomials $\mathcal{H}^{(\alpha)}_{n,q}(x,u;\lambda)$ reduce to the Frobenius-Euler polynomials $\mathcal{H}^{(\alpha)}_{n}(x,u)$ [14,3].

Very recently a new type of q-Hermite polynomial is considered in [4], which is a particular member of the q-Appell family [1]. The q-Appell polynomials are defined by means of the following generating function

$$\frac{1}{g_q(t)}e_q(xt) = \sum_{n=0}^{\infty} A_{n,q}(x)\frac{t^n}{[n]_q!}, \qquad A_{n,q} := A_{n,q}(0).$$

Definition 1.5. *The continuous q-Hermite polynomials* $H_{n,q}^{(s)}(x)$ $(0 < q < 1,\ 0 \neq s \in \mathbb{R})$ *are defined by*

$$e_q\left(xt - \frac{st^2}{1+q}\right) = \sum_{n=0}^{\infty} H_{n,q}^{(s)}(x)\frac{t^n}{[n]_q!}, \tag{1.10}$$

where $H_{n,q}^{(s)} := H_{n,q}^{(s}(0)$ *are the q-Hermite numbers defined by*

$$e_q\left(-\frac{st^2}{1+q}\right) = \sum_{n=0}^{\infty} H_{n,q}^{(s)}(0)\frac{t^n}{[n]_q!}. \tag{1.11}$$

The continuous q-Hermite polynomials $H_{n,q}^{(s)}(x)$ *are q-Appell for* $g_q(t) = e_q\left(\frac{st^2}{1+q}\right)$.

To study hybrid forms of the q-polynomials by different means is a new approach. Very recently, Riyasat et al. [24] introduced and studied the composite $2D$ q-Appell polynomials. In order to extend this approach, in this article, a hybrid class of the q-Hermite based Apostol type Frobenius-Euler polynomials is introduced. The generating function, series representation and several important formulas and relations for these polynomials are derived. The $2D$ q-Hermite based Apostol-Bernoulli, Apostol-Euler and Apostol-Genocchi polynomials are also introduced and corresponding results are established. Finally, a new family of the $2D$ q-Hermite based Appell polynomials is introduced and studied from determinant point of view.

2. q-Hermite based Apostol type Frobenius-Euler polynomials

In this section, we introduce the q-Hermite based Apostol type Frobenius-Euler polynomials (qHbATFEP) by means of generating function and series representation. Certain relations for these polynomials are also derived by using various identities.

In order to establish the generating function for the qHbATFEP, the following result is proved.

Theorem 2.1. *Let* $q \in \mathbb{C}$, $\alpha \in \mathbb{N}$, $0 < |q| < 1$. *The following generating function for the q-Hermite based Apostol type Frobenius-Euler polynomials* ${}_H\mathcal{H}_{n,q}^{(\alpha,s)}(x,y;u;\lambda)$ *of order* α *holds true*

$$\left(\frac{1-u}{\lambda e_q(t)-u}\right)^{\alpha} e_q\left(xt - \frac{st^2}{1+q}\right)E_q(yt) = \sum_{n=0}^{\infty} {}_H\mathcal{H}_{n,q}^{(\alpha,s)}(x,y;u;\lambda)\frac{t^n}{[n]_q!}. \tag{2.1}$$

Proof. Expanding the exponential function $e_q(xt)$ and then replacing the powers of x, i.e. $x^0,\ x^1,\ x^2,\dots,x^n$ by the corresponding polynomials $H_{0,q}^{(s)}(x),\ H_{1,q}^{(s)}(x),\ \dots, H_{n,q}^{(s)}(x)$ in the l.h.s. and replacing x by the polynomial

$H_{1,q}^{(s)}(x)$ in the r.h.s. of equation (1.8) and after summing up the terms in the l.h.s. of the resultant equation, we find

$$\left(\frac{1-u}{\lambda e_q(t)-u}\right)^{\alpha} E_q(yt)\sum_{n=0}^{\infty} H_{n,q}^{(s)}(x)\frac{t^n}{[n]_q!}=\sum_{n=0}^{\infty}\mathcal{H}_{n,q}^{(\alpha)}(H_{1,q}^{(s)}(x),y;u;\lambda)\frac{t^n}{[n]_q!},$$

which on using equation (1.10) in the l.h.s. and denoting the resultant qHbATFEP in the r.h.s. by ${}_H\mathcal{H}_{n,q}^{(\alpha,s)}(x,y;u;\lambda)$ yields assertion (2.1). □

Remark 2.2. We note that ${}_H\mathcal{H}_{n,q}^{(\alpha,s)}(u;\lambda) := {}_H\mathcal{H}_{n,q}^{(\alpha,s)}(0,0;u;\lambda)$ are the q-Hermite based Apostol type Frobenius-Euler numbers defined by

$$\left(\frac{1-u}{\lambda e_q(t)-u}\right)^{\alpha} e_q\left(-\frac{st^2}{1+q}\right)=\sum_{n=0}^{\infty}{}_H\mathcal{H}_{n,q}^{(\alpha,s)}(u;\lambda)\frac{t^n}{[n]_q!}.$$

Theorem 2.3. *The following series representation for the q-Hermite based Apostol type Frobenius-Euler polynomials* ${}_H\mathcal{H}_{n,q}^{(\alpha,s)}(x,y;u;\lambda)$ *of order* α *hold true:*

$$(2.2)\qquad {}_H\mathcal{H}_{n,q}^{(\alpha,s)}(x,y;u;\lambda)=\sum_{k=0}^{n}\begin{bmatrix}n\\k\end{bmatrix}_q\mathcal{H}_{n-k,q}^{(\alpha)}(0,y;u;\lambda)H_{k,q}^{(s)}(x).$$

Proof. Using equations (1.8) and (1.10) in the l.h.s. of equation (2.1) and then applying the Cauchy product rule and equating the coefficients of same powers of t in both sides of resultant equation, we get representation (2.2). □

Theorem 2.4. *The following summation formulas for the q-Hermite based Apostol type Frobenius-Euler polynomials* ${}_H\mathcal{H}_{n,q}^{(\alpha,s)}(x,y;u;\lambda)$ *of order* α *holds true:*

$$(2.3)\qquad {}_H\mathcal{H}_{n,q}^{(\alpha,s)}(x,y;u;\lambda)=\sum_{k=0}^{n}\begin{bmatrix}n\\k\end{bmatrix}_q{}_H\mathcal{H}_{k,q}^{(\alpha)}(0,0;u;\lambda)(x+y)_q^{n-k},$$

$${}_H\mathcal{H}_{n,q}^{(\alpha,s)}(x,y;u;\lambda)=\sum_{k=0}^{n}\begin{bmatrix}n\\k\end{bmatrix}_q{}_H\mathcal{H}_{k,q}^{(\alpha)}(0,y;u;\lambda)x^{n-k},$$

$$(2.4)\qquad {}_H\mathcal{H}_{n,q}^{(\alpha,s)}(x,y;u;\lambda)=\sum_{k=0}^{n}\begin{bmatrix}n\\k\end{bmatrix}_q q^{(n-k)(n-k-1)/2}{}_H\mathcal{H}_{k,q}^{(\alpha)}(x,0;u;\lambda)y^{n-k}.$$

Proof. Suitably using equations (1.1)-(1.7) in generating function (2.20) to get three different forms. Further, making use of the Cauchy product rule in the resultant expressions and then comparing the like powers of t in both sides of resultant equation, we find formulas (2.3)-(2.4). □

THEOREM 2.5. *The following recursive formulas for the q-Hermite based Apostol type Frobenius-Euler polynomials* ${}_H\mathcal{H}_{n,q}^{(\alpha,s)}(x,y;u;\lambda)$ *of order* α *hold true:*

$$D_{q,x}\ {}_H\mathcal{H}_{n,q}^{(\alpha,s)}(x,y;u;\lambda)=[n]_q\ {}_H\mathcal{H}_{n-1,q}^{(\alpha,s)}(x,y;u;\lambda), \tag{2.5}$$

$$D_{q,y}\ {}_H\mathcal{H}_{n,q}^{(\alpha,s)}(x,y;u;\lambda)=[n]_q\ {}_H\mathcal{H}_{n-1,q}^{(\alpha,s)}(x,qy;u;\lambda). \tag{2.6}$$

PROOF. Differentiating generating function (2.1) with respect to x and y with the help of equation (1.2) and then simplifying with the help of the Cauchy product rule, formulas (2.5) and (2.6) are obtained. ∎

THEOREM 2.6. *The following recurrence relation for the q-Hermite based Apostol type Frobenius-Euler polynomials* ${}_H\mathcal{H}_{n,q}^{(s)}(x,y;u;\lambda)$ *holds true:*

$$\begin{aligned}
&{}_H\mathcal{H}_{n+1,q}^{(s)}(x,y;u;\lambda)\\
&\quad=-\left(\frac{2s}{1+q}\right)[n]_q\,{}_H\mathcal{H}_{n-1,q}^{(s)}(qx,qy;u;\lambda)\\
&\qquad+x\ {}_H\mathcal{H}_{n,q}^{(s)}(x,y;u;\lambda)+y\ {}_H\mathcal{H}_{n,q}^{(s)}(qx,qy;u;\lambda)\\
&\qquad-\frac{\lambda}{1-u}\sum_{k=0}^{n}\begin{bmatrix}n\\k\end{bmatrix}_q {}_H\mathcal{H}_{n-k,q}^{(s)}(x,y;u;\lambda)q^{n-k}\mathcal{H}_{k,q}(1,0;u;\lambda).
\end{aligned}\tag{2.7}$$

PROOF. Taking $\alpha=1$ and then applying q-derivative on both sides of generating function (2.1), it follows that

$$\sum_{n=0}^{\infty}{}_H\mathcal{H}_{n+1,q}^{(s)}(x,y;u;\lambda)\frac{t^n}{[n]_q!}=(1-u)D_{q,t}\left[\frac{e_q(xt)E_q(yt)e_q\left(-\frac{st^2}{1+q}\right)}{(\lambda e_q(t)-u)}\right],$$

which on performing differentiation in the l.h.s. using formula (1.4) yields

$$\begin{aligned}
&\sum_{n=0}^{\infty}{}_H\mathcal{H}_{n+1,q}^{(s)}(x,y;u;\lambda)\frac{t^n}{[n]_q!}\\
&\quad=(1-u)\left(\frac{(\lambda e_q(qt)-u)D_{q,t}\left(e_q(xt)E_q(yt)e_q\left(-\frac{st^2}{1+q}\right)\right)}{(\lambda e_q(t)-u)(\lambda e_q(qt)-u)}-\right.\\
&\qquad\left.\frac{e_q(xqt)E_q(yqt)e_q\left(-\frac{sq^2t^2}{1+q}\right)D_{q,t}(\lambda e_q(t)-u)}{(\lambda e_q(t)-u)(\lambda e_q(qt)-u)}\right).
\end{aligned}$$

Now, using product differentiation formula (1.3) in the above equation, it follows that

$$\begin{aligned}
&\sum_{n=0}^{\infty} {}_H\mathcal{H}^{(s)}_{n+1,q}(x,y;u;\lambda)\frac{t^n}{[n]_q!} \\
&\quad = -\left(\frac{2s}{1+q}\right)\left(\frac{1-u}{\lambda e_q(t)-u}\right)\ e_q\left(qxt-\frac{st^2}{1+q}\right)E_q(qyt)t \\
&\qquad + x\left(\frac{1-u}{\lambda e_q(t)-u}\right)\ e_q\left(xt-\frac{st^2}{1+q}\right)E_q(yt) \\
&\qquad + y\left(\frac{1-u}{\lambda e_q(t)-u}\right)\ e_q\left(qxt-\frac{st^2}{1+q}\right)E_q(qyt) \\
&\qquad - \frac{\lambda}{1-u}\left(\frac{1-u}{\lambda e_q(qt)-u}\right)\ e_q\left(qxt-\frac{sq^2t^2}{1+q}\right)E_q(qyt)\left(\frac{1-u}{\lambda e_q(t)-u}\right)e_q(t).
\end{aligned} \tag{2.8}$$

Further, using generating functions (1.8) and (2.1) (with $\alpha = 1$) in equation (2.8), we find

$$\begin{aligned}
&\sum_{n=0}^{\infty} {}_H\mathcal{H}^{(s)}_{n+1,q}(x,y;u;\lambda)\frac{t^n}{[n]_q!} \\
&\quad = -\left(\frac{2s}{1+q}\right)[n]_q\sum_{n=0}^{\infty} {}_H\mathcal{H}^{(s)}_{n-1,q}(qx,qy;u;\lambda)\frac{t^n}{[n]_q!} \\
&\qquad + x\sum_{n=0}^{\infty} {}_H\mathcal{H}^{(s)}_{n,q}(x,y;u;\lambda)\frac{t^n}{[n]_q!} + y\sum_{n=0}^{\infty} {}_H\mathcal{H}^{(s)}_{n,q}(qx,qy;u;\lambda)\frac{t^n}{[n]_q!} \\
&\qquad - \frac{\lambda}{1-u}\sum_{n=0}^{\infty} {}_H\mathcal{H}^{(s)}_{n,q}(x,y;u;\lambda)q^n\frac{t^n}{[n]_q!}\sum_{k=0}^{\infty}\mathcal{H}_{k,q}(1,0;u;\lambda)\frac{t^k}{[k]_q!},
\end{aligned}$$

which on making use of the Cauchy product rule in the r.h.s. and comparing the coefficients of $t^n/n!$ on both sides of the resultant equation gives recurrence relation (2.7). ∎

THEOREM 2.7. *The following relation for the q-Hermite based Apostol type Frobenius-Euler polynomials ${}_H\mathcal{H}^{(s)}_{n,q}(x,y;u;\lambda)$ holds true:*

$$\begin{aligned}
&(2u-1)\sum_{k=0}^{n}\begin{bmatrix} n \\ k \end{bmatrix}_q {}_H\mathcal{H}^{(s)}_{n-k,q}(x,y;1-u;\lambda)\mathcal{H}^{(s)}_{k,q}(0,0;u;\lambda) \\
&\qquad = u\,{}_H\mathcal{H}^{(s)}_{n,q}(x,y;u;\lambda) - (1-u)\,{}_H\mathcal{H}^{(s)}_{n,q}(x,y;1-u;\lambda).
\end{aligned} \tag{2.9}$$

PROOF. Making use of the identity

$$\frac{(2u-1)}{(\lambda e_q(t)-u)(\lambda e_q(t)-(1-u))} = \frac{1}{(\lambda e_q(t)-u)} - \frac{1}{(\lambda e_q(t)-(1-u))}$$

to evaluate the following fraction, so that we have

$$
\begin{aligned}
(2u-1)&\frac{(1-u)e_q\left(xt-\frac{st^2}{1+q}\right)(1-(1-u))E_q(yt)}{(\lambda e_q(t)-u)(\lambda e_q(t)-(1-u))}\\
&=\frac{(1-u)e_q\left(xt-\frac{st^2}{1+q}\right)uE_q(yt)}{(\lambda e_q(t)-u)}\\
&\quad-\frac{(1-u)e_q\left(xt-\frac{st^2}{1+q}\right)(1-(1-u))E_q(yt)}{(\lambda e_q(t)-(1-u))},
\end{aligned}
$$

which on using equations (2.1) and (1.9) in both sides gives

$$
\begin{aligned}
(2u-1)&\sum_{n=0}^{\infty}{}_H\mathcal{H}_{n,q}^{(s)}(x,y;1-u;\lambda)\frac{t^n}{[n]_q!}\sum_{k=0}^{\infty}\mathcal{H}_{k,q}^{(s)}(0,0;u;\lambda)\frac{t^k}{[k]_q!}\\
&=u\sum_{n=0}^{\infty}{}_H\mathcal{H}_{n,q}^{(s)}(x,y;u;\lambda)\frac{t^n}{[n]_q!}-(1-u)\sum_{n=0}^{\infty}{}_H\mathcal{H}_{n,q}^{(s)}(x,y;1-u;\lambda)\frac{t^n}{[n]_q!}.
\end{aligned}
$$

Applying the Cauchy product rule in the above equation and then equating the coefficients of like powers of t in both sides of the resultant equation, assertion (2.9) follows. ∎

THEOREM 2.8. *The following relation for the q-Hermite based Apostol type Frobenius-Euler polynomials* ${}_H\mathcal{H}_{n,q}^{(s)}(x,y;u;\lambda)$ *holds true:*

$$
\begin{aligned}
&u\,{}_H\mathcal{H}_{n,q}^{(s)}(x,y;u;\lambda)\\
&\quad=\sum_{k=0}^{n}\begin{bmatrix}n\\k\end{bmatrix}_q\left(\lambda\,{}_H\mathcal{H}_{n-k,q}^{(s)}(x,y;u;\lambda)-(1-u)H_{n-k,q}^{(s)}(0)(x+y)_q^k\right).
\end{aligned}
\tag{2.10}
$$

PROOF. By using $e_q(t)E_q(-t)=1$, we consider the following identity:

$$
\frac{u}{\lambda(\lambda e_q(t)-u)e_q(t)}=\frac{1}{(\lambda e_q(t)-u)}-\frac{1}{\lambda e_q(t)}.
$$

Evaluating the following fraction using above identity, we find

$$
\begin{aligned}
&\frac{u(1-u)e_q\left(xt-\frac{st^2}{1+q}\right)E_q(yt)}{\lambda(\lambda e_q(t)-u)e_q(t)}\\
&\quad=\frac{(1-u)e_q\left(xt-\frac{st^2}{1+q}\right)E_q(yt)}{(\lambda e_q(t)-u)}-\frac{(1-u)e_q\left(xt-\frac{st^2}{1+q}\right)E_q(yt)}{\lambda e_q(t)},
\end{aligned}
$$

which on using equations (2.1), (1.1) and (1.11) yields

$$u\sum_{n=0}^{\infty} {}_H\mathcal{H}_{n,q}^{(s)}(x,y;u;\lambda)\frac{t^n}{[n]_q!}$$
$$= \lambda\sum_{n=0}^{\infty} {}_H\mathcal{H}_{n,q}^{(s)}(x,y;u;\lambda)\frac{t^n}{[n]_q!}\sum_{k=0}^{\infty}\frac{t^k}{[k]_q!}$$
$$-(1-u)\sum_{n=0}^{\infty} H_{n,q}^{(s)}(0)\frac{t^n}{[n]_q!}\sum_{k=0}^{\infty}(x+y)_q^k\frac{t^k}{[k]_q!}.$$

Making use of the Cauchy product rule in the r.h.s. of above equation and then comparing the coefficients of $t^n/n!$ on both sides of the resultant equation, we get relation (2.10). ∎

Theorem 2.9. *The following relation for the q-Hermite based Apostol type Frobenius-Euler polynomials ${}_H\mathcal{H}_{n,q}^{(\alpha,s)}(x,y;u;\lambda)$ of order α holds true:*

(2.11)
$$ {}_H\mathcal{H}_{n,q}^{(\alpha,s)}(x,y;u;\lambda) = \frac{1}{1-u}\sum_{k=0}^{n}\begin{bmatrix} n \\ k \end{bmatrix}_q\Big(\lambda {}_H\mathcal{H}_{n-k,q}^{(\alpha,s)}(1,y;u;\lambda)\mathcal{H}_{k,q}(x,0;u;\lambda)$$
$$ - u {}_H\mathcal{H}_{n-k,q}^{(\alpha,s)}(x,0;u;\lambda)\mathcal{H}_{k,q}(0,y;u;\lambda)\Big).$$

Proof. Consider generating function (2.1) in the following form:

$$\sum_{n=0}^{\infty} {}_H\mathcal{H}_{n,q}^{(\alpha,s)}(x,y;u;\lambda)\frac{t^n}{[n]_q!}$$
$$=\left(\frac{1-u}{\lambda e_q(t)-u}\right)E_q(yt)\left(\frac{\lambda e_q(t)-u}{1-u}\right)\left(\frac{1-u}{\lambda e_q(t)-u}\right)^{\alpha}$$
$$e_q\left(xt-\frac{st^2}{1+q}\right).$$

Simplifying the above equation and using equations (2.1) and (1.8), we find

$$\sum_{n=0}^{\infty} {}_H\mathcal{H}_{n,q}^{(\alpha,s)}(x,y;u;\lambda)\frac{t^n}{[n]_q!}$$
$$=\frac{1}{1-u}\Big(\lambda\sum_{n=0}^{\infty} {}_H\mathcal{H}_{n,q}^{(\alpha,s)}(1,y;u;\lambda)\frac{t^n}{[n]_q!}\sum_{k=0}^{\infty}\mathcal{H}_{k,q}(x,0;u;\lambda)\frac{t^k}{[k]_q!}$$
$$-u\sum_{n=0}^{\infty} {}_H\mathcal{H}_{n,q}^{(\alpha,s)}(x,0;u;\lambda)\frac{t^n}{[n]_q!}\sum_{k=0}^{\infty}\mathcal{H}_{k,q}(0,y;u;\lambda)\frac{t^k}{[k]_q!}\Big).$$

Application of the Cauchy product rule in the r.h.s. and cancelation of the coefficients of same powers of t in both sides of the resultant equation yields relation (2.11). ∎

THEOREM 2.10. *The following relation between the q-Hermite based Apostol type Frobenius-Euler polynomials ${}_H\mathcal{H}_{n,q}^{(\alpha,s)}(x,y;u;\lambda)$ and the q-Apostol-Bernoulli polynomials $\mathfrak{B}_{n,q}(x,y;\lambda)$ holds true*

$$\begin{aligned} {}_H\mathcal{H}_{n,q}^{(\alpha,s)}(x,y;u;\lambda) = & \sum_{k=0}^{n+1} \frac{1}{[n+1]_q} \begin{bmatrix} n+1 \\ k \end{bmatrix}_q \Big(\lambda \sum_{r=0}^{k} \begin{bmatrix} k \\ r \end{bmatrix}_q \mathfrak{B}_{k-r,q}(0,y;\lambda) \\ & - \mathfrak{B}_{k,q}(0,y;\lambda)\Big)\, {}_H\mathcal{H}_{n-k+1,q}^{(\alpha,s)}(x,0;u;\lambda). \end{aligned} \tag{2.12}$$

PROOF. Consider generating function (2.1) in the following form

$$\begin{aligned} & \sum_{n=0}^{\infty} {}_H\mathcal{H}_{n,q}^{(\alpha,s)}(x,y;u;\lambda) \frac{t^n}{[n]_q!} \\ & \quad = \left(\frac{1-u}{\lambda e_q(t)-u}\right)^{\alpha} E_q(yt) \left(\frac{t}{\lambda e_q(t)-1}\right) \left(\frac{\lambda e_q(t)-1}{t}\right) \\ & \quad\quad e_q\left(xt - \frac{st^2}{1+q}\right), \end{aligned}$$

which on simplification and use of equations (2.1) and (1.5) (with $\alpha = 1$) gives

$$\begin{aligned} & \sum_{n=0}^{\infty} {}_H\mathcal{H}_{n,q}^{(\alpha,s)}(x,y;u;\lambda) \frac{t^n}{[n]_q!} \\ & \quad = \frac{1}{t}\Big(\lambda \sum_{n=0}^{\infty} {}_H\mathcal{H}_{n,q}^{(\alpha,s)}(x,0;u;\lambda) \frac{t^n}{[n]_q!} \sum_{k=0}^{\infty} \mathfrak{B}_{k,q}(0,y;\lambda) \frac{t^k}{[k]_q!} \sum_{r=0}^{\infty} \frac{t^r}{[r]_q!} \\ & \quad\quad - \sum_{n=0}^{\infty} {}_H\mathcal{H}_{n,q}^{(\alpha,s)}(x,0;u;\lambda) \frac{t^n}{[n]_q!} \sum_{k=0}^{\infty} \mathfrak{B}_{k,q}(0,y;\lambda) \frac{t^k}{[k]_q!}\Big). \end{aligned} \tag{2.13}$$

On equating the coefficients of same powers of t after using Cauchy product rule in equation (2.13), assertion (2.12) follows. ∎

THEOREM 2.11. *The following relation between the q-Hermite based Apostol type Frobenius-Euler polynomials ${}_H\mathcal{H}_{n,q}^{(\alpha,s)}(x,y;u;\lambda)$ and the q-Apostol-Euler polynomials $\mathfrak{E}_{n,q}(x,y;\lambda)$ holds true:*

$$\begin{aligned} {}_H\mathcal{H}_{n,q}^{(\alpha,s)}(x,y;u;\lambda) = & \frac{1}{2} \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q \Big(\lambda \sum_{r=0}^{k} \begin{bmatrix} k \\ r \end{bmatrix}_q \mathfrak{E}_{k-r,q}(x,0;\lambda) \\ & + \mathfrak{E}_{k,q}(x,0;\lambda)\Big)\, {}_H\mathcal{H}_{n-k,q}^{(\alpha,s)}(0,y;u;\lambda). \end{aligned} \tag{2.14}$$

PROOF. Consider generating function (2.1) in the following form:

$$\begin{aligned}\sum_{n=0}^{\infty} {}_H\mathcal{H}_{n,q}^{(\alpha,s)}(x,y;u;\lambda)\frac{t^n}{[n]_q!}&\\ =\left(\frac{1-u}{\lambda e_q(t)-u}\right)^{\alpha} E_q(yt)\left(\frac{2}{\lambda e_q(t)+1}\right)\left(\frac{\lambda e_q(t)+1}{2}\right)&\\ e_q\left(xt-\frac{st^2}{1+q}\right).&\end{aligned}$$

Simplifying the above equation and then making use of equations (2.1) and (1.13) (with $\alpha = 1$), it follows that

$$\begin{aligned}\sum_{n=0}^{\infty} {}_H\mathcal{H}_{n,q}^{(\alpha,s)}(x,y;u;\lambda)\frac{t^n}{[n]_q!}&\\ =\frac{1}{2}\Big(\lambda\sum_{n=0}^{\infty} {}_H\mathcal{H}_{n,q}^{(\alpha,s)}(0,y;u;\lambda)\frac{t^n}{[n]_q!}\sum_{k=0}^{\infty}\sum_{r=0}^{\infty}\mathfrak{E}_{k,q}(x,0;\lambda)\frac{t^k}{[k]_q!}\frac{t^r}{[r]_q!}&\\ +\sum_{n=0}^{\infty} {}_H\mathcal{H}_{n,q}^{(\alpha,s)}(0,y;u;\lambda)\frac{t^n}{[n]_q!}\sum_{k=0}^{\infty}\mathfrak{E}_{k,q}(x,0;\lambda)\frac{t^k}{[k]_q!}\Big),&\end{aligned}$$

which on using the Cauchy product rule and equating the coefficients of same powers of t in resultant equation yields relation (2.14). ∎

THEOREM 2.12. *The following relation between the q-Hermite based Apostol type Frobenius-Euler polynomials ${}_H\mathcal{H}_{n,q}^{(\alpha,s)}(x,y;u;\lambda)$ and the q-Apostol-Genocchi polynomials $\mathcal{G}_{n,q}(x,y;\lambda)$ holds true:*

$$\begin{aligned}{}_H\mathcal{H}_{n,q}^{(\alpha,s)}(x,y;u;\lambda) =&\frac{1}{2}\sum_{k=0}^{n+1}\frac{1}{[n+1]_q}\begin{bmatrix}n+1\\k\end{bmatrix}_q\Big(\lambda\sum_{r=0}^{k}\begin{bmatrix}k\\r\end{bmatrix}_q\mathcal{G}_{k-r,q}(0,y;\lambda)\\ &+\mathcal{G}_{k,q}(0,y;\lambda)\Big){}_H\mathcal{H}_{n-k+1,q}^{(\alpha,s)}(x,0;u;\lambda).\end{aligned}\tag{2.15}$$

PROOF. Consider generating function (2.1) in the following form:

$$\begin{aligned}\sum_{n=0}^{\infty} {}_H\mathcal{H}_{n,q}^{(\alpha,s)}(x,y;u;\lambda)\frac{t^n}{[n]_q!}&\\ =\left(\frac{1-u}{\lambda e_q(t)-u}\right)^{\alpha} E_q(yt)\left(\frac{2t}{\lambda e_q(t)+1}\right)\left(\frac{\lambda e_q(t)+1}{2t}\right)&\\ e_q\left(xt-\frac{st^2}{1+q}\right).&\end{aligned}\tag{2.16}$$

Simplifying equation (2.16) and using equations (2.1) and (1.7) (with $\alpha = 1$), we find

$$\sum_{n=0}^{\infty} {}_H\mathcal{H}_{n,q}^{(\alpha,s)}(x,y;u;\lambda)\frac{t^n}{[n]_q!}$$
$$= \frac{1}{2t}\Big(\lambda \sum_{n=0}^{\infty} {}_H\mathcal{H}_{n,q}^{(\alpha,s)}(x,0;u;\lambda)\frac{t^n}{[n]_q!}\sum_{k=0}^{\infty}\mathcal{G}_{k,q}(0,y;\lambda)\frac{t^k}{[k]_q!}\sum_{r=0}^{\infty}\frac{t^r}{[r]_q!}$$
$$+ \sum_{n=0}^{\infty} {}_H\mathcal{H}_{n,q}^{(\alpha,s)}(x,0;u;\lambda)\frac{t^n}{[n]_q!}\sum_{k=0}^{\infty}\mathcal{G}_{k,q}(0,y;\lambda)\frac{t^k}{[k]_q!}\Big).$$

Comparison of the like powers of $t^n/n!$ after using the Cauchy product rule in the above equation yields desired identity (2.15). □

In the next section, we introduce the $2D$ q-Hermite based Apostol-Bernoulli, Apostol-Euler and Apostol-Genocchi polynomials and establish some relations for these hybrid polynomials.

3. 2D q-Hermite based Apostol-Bernoulli, Apostol-Euler and Apostol-Genocchi polynomials

Keleshteri and Mahmudov in [10] introduced and studied the $2D$ q-Appell polynomials $A_{n,q}(x,y)$ which are the 2-variable generalizations of the q-Appell polynomials $A_{n,q}(x)$ [1]. The $2D$ q-Appell polynomials $A_{n,q}(x,y)$ are defined by means of the following generating function:

$$\frac{1}{g_q(t)}\ e_q(xt)E_q(yt) = \sum_{n=0}^{\infty} A_{n,q}(x,y)\frac{t^n}{[n]_q!}, \qquad A_{n,q} := A_{n,q}(0,0).$$

The $2D$ q-Bernoulli polynomials $B_{n,q}(x,y)$, q-Euler polynomials $E_{n,q}(x,y)$ and q-Genocchi polynomials $G_{n,q}(x,y)$ are the particular members of the $2D$ q-Appell family. Several important relations and formulas for these polynomials and for their generalizations are derived in [16, 18, 20, 5, 6].

The approach used in previous section is further exploited to introduce the $2D$ q-Hermite-based Apostol-Bernoulli, Apostol-Euler and Apostol-Genocchi polynomials. First, we give the following definitions.

Definition 3.1. *The $2D$ q-Hermite based Apostol-Bernoulli polynomials (2DqHbABP) of order α, ${}_H\mathfrak{B}_{n,q}^{(\alpha,s)}(x,y;u;\lambda)$ ($q \in \mathbb{C}$, $\alpha \in \mathbb{N}$, $0 < |q| < 1$) are defined by the following generating function*

$$\left(\frac{t}{\lambda e_q(t) - 1}\right)^{\alpha}\ e_q\left(xt - \frac{st^2}{1+q}\right)E_q(yt) = \sum_{n=0}^{\infty} {}_H\mathfrak{B}_{n,q}^{(\alpha,s)}(x,y;\lambda)\frac{t^n}{[n]_q!}, \tag{3.1}$$

where ${}_H\mathfrak{B}_{n,q}^{(\alpha,s)}(\lambda) := {}_H\mathfrak{B}_{n,q}^{(\alpha,s)}(0,0;\lambda)$ *are the* 2D *q-Hermite-based Apostol-Bernoulli numbers. The* 2*DqHbABP* ${}_H\mathfrak{B}_{n,q}^{(\alpha,s)}(x,y;\lambda)$ *are* 2D *q-Appell for* $g_q(t)=\left(\frac{\lambda e_q(t)-1}{t}\right)^\alpha\ e_q\left(\frac{st^2}{1+q}\right)$.

Definition 3.2. *The* 2D *q-Hermite based Apostol-Euler polynomials* ${}_H\mathfrak{E}_{n,q}^{(\alpha,s)}(x,y;u;\lambda)$ *(*2*DqHbAEP) of order* α *(*$q\in\mathbb{C}$, $\alpha\in\mathbb{N}$, $0<|q|<1$*) are defined by the following generating function*

$$(3.2)\qquad \left(\frac{2}{\lambda e_q(t)+1}\right)^\alpha\ e_q\left(xt-\frac{st^2}{1+q}\right)E_q(yt)=\sum_{n=0}^{\infty}{}_H\mathfrak{E}_{n,q}^{(\alpha,s)}(x,y;\lambda)\frac{t^n}{[n]_q!},$$

where ${}_H\mathfrak{E}_{n,q}^{(\alpha,s)}(\lambda) := {}_H\mathfrak{E}_{n,q}^{(\alpha,s)}(0,0;\lambda)$ *are the* 2D *q-Hermite-based Apostol-Euler numbers. The* 2*DqHbAEP* ${}_H\mathfrak{E}_{n,q}^{(\alpha,s)}(x,y;\lambda)$ *are* 2D *q-Appell for* $g_q(t)=\left(\frac{\lambda e_q(t)+1}{2}\right)^\alpha\ e_q\left(\frac{st^2}{1+q}\right)$.

Definition 3.3. *The* 2D *q-Hermite based Apostol-Genocchi polynomials* ${}_H\mathcal{G}_{n,q}^{(\alpha,s)}(x,y;u;\lambda)$ *(*2*DqHbAGP) of order* α *(*$q\in\mathbb{C}$, $\alpha\in\mathbb{N}$, $0<|q|<1$*) are defined by the following generating function*

$$(3.3)\qquad \left(\frac{2t}{\lambda e_q(t)+1}\right)^\alpha\ e_q\left(xt-\frac{st^2}{1+q}\right)E_q(yt)=\sum_{n=0}^{\infty}{}_H\mathcal{G}_{n,q}^{(\alpha,s)}(x,y;\lambda)\frac{t^n}{[n]_q!},$$

where ${}_H\mathcal{G}_{n,q}^{(\alpha,s)}(\lambda) := {}_H\mathcal{G}_{n,q}^{(\alpha,s)}(0,0;\lambda)$ *are the* 2D *q-Hermite-based Apostol-Genocchi numbers. The* 2*DqHbAGP* ${}_H\mathcal{G}_{n,q}^{(\alpha,s)}(x,y;\lambda)$ *are* 2D *q-Appell for* $g_q(t)=\left(\frac{\lambda e_q(t)+1}{2t}\right)^\alpha\ e_q\left(\frac{st^2}{1+q}\right)$.

Analogous to the results obtained for the qHbATFEP in Section 2, we obtain the series representations, summation formulas and recursive formulas for the 2DqHbABP, 2DqHbAEP and 2DqHbAGP. We present these results in TABLE 1 (I, II-IV, V-VI, respectively).

Next, we establish certain summation relations for the 2DqHbABP, 2DqHbAEP and 2DqHbAGP by proving the following results.

Theorem 3.4. *The following relations for the* 2D *q-Hermite based Apostol-Bernoulli polynomials* ${}_H\mathfrak{B}_{n,q}^{(\alpha,s)}(x,y;\lambda)$ *hold true:*

$$(a)\qquad \begin{aligned}{}_H\mathfrak{B}_{n,q}^{(\alpha,s)}(x,y;\lambda)&=\frac{1}{2m^n}\sum_{k=0}^{n}\begin{bmatrix}n\\k\end{bmatrix}_q m^k\Big({}_H\mathfrak{B}_{k,q}^{(\alpha,s)}(x,0;\lambda)\\&+m^{-k}\lambda\sum_{r=0}^{k}\begin{bmatrix}k\\r\end{bmatrix}_q m^r\,{}_H\mathfrak{B}_{r,q}^{(\alpha,s)}(x,0;\lambda)\Big)\mathfrak{E}_{n-k,q}(0,my;\lambda),\end{aligned}$$

$$(b)\quad {}_H\mathfrak{B}_{n,q}^{(\alpha,s)}(x,y;\lambda)=\frac{1}{2m^n}\sum_{k=0}^{n}\begin{bmatrix}n\\k\end{bmatrix}_q m^k\Big({}_H\mathfrak{B}_{k,q}^{(\alpha,s)}(0,y;\lambda)$$
$$+m^{-k}\lambda\sum_{r=0}^{k}\begin{bmatrix}k\\r\end{bmatrix}_q m^r\,{}_H\mathfrak{B}_{r,q}^{(\alpha,s)}(0,y;\lambda)\Big)\mathfrak{E}_{n-k,q}(mx,0;\lambda),$$

$$(c)\quad {}_H\mathfrak{B}_{n,q}^{(\alpha,s)}(x,y;\lambda)=\frac{1}{2m^n}\sum_{k=0}^{n+1}\frac{1}{[n+1]_q}\begin{bmatrix}n+1\\k\end{bmatrix}_q m^k\Big(m^{-k}\lambda\sum_{r=0}^{k}\begin{bmatrix}k\\r\end{bmatrix}_q m^r$$
$${}_H\mathfrak{B}_{r,q}^{(\alpha,s)}(0,y;\lambda)+{}_H\mathfrak{B}_{k,q}^{(\alpha,s)}(0,y;\lambda)\Big)\mathcal{G}_{n-k+1,q}(mx,0;\lambda),$$

$$(d)\quad {}_H\mathfrak{B}_{n,q}^{(\alpha,s)}(x,y;\lambda)=\frac{1}{2m^n}\sum_{k=0}^{n+1}\frac{1}{[n+1]_q}\begin{bmatrix}n+1\\k\end{bmatrix}_q m^k\Big(m^{-k}\lambda\sum_{r=0}^{k}\begin{bmatrix}k\\r\end{bmatrix}_q m^r$$
$${}_H\mathfrak{B}_{r,q}^{(\alpha,s)}(x,0;\lambda)+{}_H\mathfrak{B}_{k,q}^{(\alpha,s)}(x,0;\lambda)\Big)\mathcal{G}_{n-k+1,q}(0,my;\lambda).$$

PROOF. (a) Consider generating function (3.1) in the following form:

$$\sum_{n=0}^{\infty}{}_H\mathfrak{B}_{n,q}^{(\alpha,s)}(x,y;\lambda)\frac{t^n}{[n]_q!}=\left(\frac{t}{\lambda e_q(t)-1}\right)^{\alpha}e_q\left(xt-\frac{st^2}{1+q}\right)E_q\left(\frac{t}{m}.my\right)$$
$$\left(\frac{2}{\lambda e_q(t/m)+1}\right)\left(\frac{\lambda e_q(t/m)+1}{2}\right),$$

which on simplification becomes

(3.4)
$$\sum_{n=0}^{\infty}{}_H\mathfrak{B}_{n,q}^{(\alpha,s)}(x,y;\lambda)\frac{t^n}{[n]_q!}$$
$$=\frac{1}{2}\Bigg(\lambda\left(\frac{t}{\lambda e_q(t)-1}\right)^{\alpha}e_q\left(xt-\frac{st^2}{1+q}\right)E_q\left(\frac{t}{m}.my\right)\left(\frac{2}{\lambda e_q(t/m)+1}\right)e_q(\frac{t}{m})$$
$$+\left(\frac{t}{\lambda e_q(t)-1}\right)^{\alpha}e_q\left(xt-\frac{st^2}{1+q}\right)E_q\left(\frac{t}{m}.my\right)\left(\frac{2}{\lambda e_q(t/m)+1}\right)\Bigg).$$

Now, using equations (3.1) and (1.6) in equation (3.4), we find

$$\sum_{n=0}^{\infty}{}_H\mathfrak{B}_{n,q}^{(\alpha,s)}(x,y;\lambda)\frac{t^n}{[n]_q!}$$
$$=\frac{1}{2}\Bigg(\lambda\sum_{n=0}^{\infty}m^{-n}\mathfrak{E}_{n,q}(0,my;\lambda)\frac{t^n}{[n]_q!}\sum_{k=0}^{\infty}\sum_{r=0}^{\infty}m^{-k}\,{}_H\mathfrak{B}_{r,q}^{(\alpha,s)}(x,0;\lambda)\frac{t^k}{[k]_q!}\frac{t^r}{[r]_q!}$$
$$+\sum_{n=0}^{\infty}m^{-n}\mathfrak{E}_{n,q}(0,my;\lambda)\frac{t^n}{[n]_q!}\sum_{k=0}^{\infty}{}_H\mathfrak{B}_{k,q}^{(\alpha,s)}(x,0;\lambda)\frac{t^k}{[k]_q!}\Bigg).$$

TABLE 1. Results for ${}_H\mathfrak{B}^{(\alpha,s)}_{n,q}(x,y;\lambda)$, ${}_H\mathfrak{E}^{(\alpha,s)}_{n,q}(x,y;\lambda)$, ${}_H\mathcal{G}^{(\alpha,s)}_{n,q}(x,y;\lambda)$

S. No.	**Results for** ${}_H\mathfrak{B}^{(\alpha,s)}_{n,q}(x,y;\lambda)$	**Results for** ${}_H\mathfrak{E}^{(\alpha,s)}_{n,q}(x,y;\lambda)$	**Results for** ${}_H\mathcal{G}^{(\alpha,s)}_{n,q}(x,y;\lambda)$
I.	${}_H\mathfrak{B}^{(\alpha,s)}_{n,q}(x,y;\lambda) = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q \mathfrak{B}^{(\alpha)}_{n-k,q}(0,y;\lambda) \times H^{(s)}_{k,q}(x)$	${}_H\mathfrak{E}^{(\alpha,s)}_{n,q}(x,y;\lambda) = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q \mathfrak{E}^{(\alpha)}_{n-k,q}(0,y;\lambda) \times H^{(s)}_{k,q}(x)$	${}_H\mathcal{G}^{(\alpha,s)}_{n,q}(x,y;\lambda) = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q \mathcal{G}^{(\alpha)}_{n-k,q}(0,y;\lambda) \times H^{(s)}_{k,q}(x)$
II.	${}_H\mathfrak{B}^{(\alpha,s)}_{n,q}(x,y;\lambda) = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q {}_H\mathfrak{B}^{(\alpha)}_{k,q}(\lambda) \times (x+y)^{n-k}_q$	${}_H\mathfrak{E}^{(\alpha,s)}_{n,q}(x,y;\lambda) = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q {}_H\mathfrak{E}^{(\alpha)}_{k,q}(\lambda) \times (x+y)^{n-k}_q$	${}_H\mathcal{G}^{(\alpha,s)}_{n,q}(x,y;\lambda) = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q {}_H\mathcal{G}^{(\alpha)}_{k,q}(\lambda) \times (x+y)^{n-k}_q$
III.	${}_H\mathfrak{B}^{(\alpha,s)}_{n,q}(x,y;\lambda) = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q q^{\frac{(n-k)(n-k-1)}{2}} \times {}_H\mathfrak{B}^{(\alpha)}_{k,q}(x,0;\lambda) y^{n-k}$	${}_H\mathfrak{E}^{(\alpha,s)}_{n,q}(x,y;\lambda) = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q q^{\frac{(n-k)(n-k-1)}{2}} \times {}_H\mathfrak{E}^{(\alpha)}_{k,q}(x,0;\lambda) y^{n-k}$	${}_H\mathcal{G}^{(\alpha,s)}_{n,q}(x,y;\lambda) = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q q^{\frac{(n-k)(n-k-1)}{2}} \times {}_H\mathcal{G}^{(\alpha)}_{k,q}(x,0;\lambda) y^{n-k}$
IV.	${}_H\mathfrak{B}^{(\alpha,s)}_{n,q}(x,y;\lambda) = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q {}_H\mathfrak{B}^{(\alpha)}_{k,q}(0,y;\lambda) \times x^{n-k}$	${}_H\mathfrak{E}^{(\alpha,s)}_{n,q}(x,y;\lambda) = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q {}_H\mathfrak{E}^{(\alpha)}_{k,q}(0,y;\lambda) \times x^{n-k}$	${}_H\mathcal{G}^{(\alpha,s)}_{n,q}(x,y;\lambda) = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q {}_H\mathcal{G}^{(\alpha)}_{k,q}(0,y;\lambda) \times x^{n-k}$
V.	$D_{q,x}\ {}_H\mathfrak{B}^{(\alpha,s)}_{n,q}(x,y;\lambda) = [n]_q\ {}_H\mathfrak{B}^{(\alpha,s)}_{n-1,q}(x,y;\lambda)$	$D_{q,x}\ {}_H\mathfrak{E}^{(\alpha,s)}_{n,q}(x,y;\lambda) = [n]_q\ {}_H\mathfrak{E}^{(\alpha,s)}_{n-1,q}(x,y;\lambda)$	$D_{q,x}\ {}_H\mathcal{G}^{(\alpha,s)}_{n,q}(x,y;\lambda) = [n]_q\ {}_H\mathcal{G}^{(\alpha,s)}_{n-1,q}(x,y;\lambda)$
VI.	$D_{q,y}\ {}_H\mathfrak{B}^{(\alpha,s)}_{n,q}(x,y;\lambda) = [n]_q\ {}_H\mathfrak{B}^{(\alpha,s)}_{n-1,q}(x,qy;\lambda)$	$D_{q,y}\ {}_H\mathfrak{E}^{(\alpha,s)}_{n,q}(x,y;\lambda) = [n]_q\ {}_H\mathfrak{E}^{(\alpha,s)}_{n-1,q}(x,qy;\lambda)$	$D_{q,y}\ {}_H\mathcal{G}^{(\alpha,s)}_{n,q}(x,y;\lambda) = [n]_q\ {}_H\mathcal{G}^{(\alpha,s)}_{n-1,q}(x,qy;\lambda)$

Further, making use of the Cauchy product rule in the r.h.s. of above equation and comparing the coefficients of like powers of t, assertion (a) follows.

(b) Consider generating function (3.1) in the following form:

$$\sum_{n=0}^{\infty} {}_H\mathfrak{B}_{n,q}^{(\alpha,s)}(x,y;\lambda)\frac{t^n}{[n]_q!} = \left(\frac{t}{\lambda e_q(t)-1}\right)^{\alpha} e_q\left(\frac{t}{m}.mx\right) e_q\left(-\frac{st^2}{1+q}\right) E_q(yt) \left(\frac{2}{\lambda e_q(t/m)+1}\right)\left(\frac{\lambda e_q(t/m)+1}{2}\right).$$

Following the same lines of proof as in (a), we are led to assertion (b).

(c) Consider generating function (3.1) in the following form:

$$\sum_{n=0}^{\infty} {}_H\mathfrak{B}_{n,q}^{(\alpha,s)}(x,y;\lambda)\frac{t^n}{[n]_q!} = \left(\frac{t}{\lambda e_q(t)-1}\right)^{\alpha} e_q\left(\frac{t}{m}.mx\right) e_q\left(-\frac{st^2}{1+q}\right) E_q(yt) \left(\frac{2t}{\lambda e_q(t/m)+1}\right)\left(\frac{\lambda e_q(t/m)+1}{2t}\right).$$

Simplifying the above equation and using equations (3.1) and (1.7), we find

$$\begin{aligned}
&\sum_{n=0}^{\infty} {}_H\mathfrak{B}_{n,q}^{(\alpha,s)}(x,y;\lambda)\frac{t^n}{[n]_q!} \\
&= \frac{1}{2t}\Big(m^{1-n}\lambda\sum_{n=0}^{\infty}\mathcal{G}_{n,q}(mx,0;\lambda)\frac{t^n}{[n]_q!}\sum_{k=0}^{\infty}\sum_{r=0}^{\infty} m^{-k}\,{}_H\mathfrak{B}_{r,q}^{(\alpha,s)}(0,y;\lambda)\frac{t^k}{[k]_q!}\frac{t^r}{[r]_q!} \\
&\quad + m^{1-n}\sum_{n=0}^{\infty}\mathcal{G}_{n,q}(mx,0;\lambda)\frac{t^n}{[n]_q!}\sum_{k=0}^{\infty} {}_H\mathfrak{B}_{k,q}^{(\alpha,s)}(0,y;\lambda)\frac{t^k}{[k]_q!}\Big).
\end{aligned}$$

Use of the Cauchy product rule in the r.h.s. of the simplified form of the above equation and finally equating the coefficients of same powers of t gives assertion (c).

(d) Consider generating function (3.1) in the following form:

$$\sum_{n=0}^{\infty} {}_H\mathfrak{B}_{n,q}^{(\alpha,s)}(x,y;\lambda)\frac{t^n}{[n]_q!} = \left(\frac{t}{\lambda e_q(t)-1}\right)^{\alpha} E_q\left(\frac{t}{m}.my\right) e_q\left(xt-\frac{st^2}{1+q}\right) \left(\frac{2t}{\lambda e_q(t/m)+1}\right)\left(\frac{\lambda e_q(t/m)+1}{2t}\right).$$

Proceeding on the same lines of proof as in (c), we are led to relation (d). □

THEOREM 3.5. *The following relations for the 2D q-Hermite based Apostol-Euler polynomials* ${}_H\mathfrak{E}_{n,q}^{(\alpha,s)}(x,y;\lambda)$ *hold true:*

$$(a)\quad {}_H\mathfrak{E}_{n,q}^{(\alpha,s)}(x,y;\lambda) = \frac{1}{m^n}\sum_{k=0}^{n+1}\frac{1}{[n+1]_q}\begin{bmatrix} n+1 \\ k \end{bmatrix}_q m^k\Big(m^{-k}\lambda\sum_{r=0}^{k}\begin{bmatrix} k \\ r \end{bmatrix}_q m^r \\ {}_H\mathfrak{E}_{r,q}^{(\alpha,s)}(0,y;\lambda) - {}_H\mathfrak{E}_{k,q}^{(\alpha,s)}(0,y;\lambda)\Big)\mathfrak{B}_{n-k+1,q}(mx,0;\lambda),$$

$$(b)\quad {}_H\mathfrak{E}_{n,q}^{(\alpha,s)}(x,y;\lambda) = \frac{1}{m^n}\sum_{k=0}^{n+1}\frac{1}{[n+1]_q}\begin{bmatrix} n+1 \\ k \end{bmatrix}_q m^k\Big(m^{-k}\lambda\sum_{r=0}^{k}\begin{bmatrix} k \\ r \end{bmatrix}_q m^r \\ {}_H\mathfrak{E}_{r,q}^{(\alpha,s)}(x,0;\lambda) - {}_H\mathfrak{E}_{k,q}^{(\alpha,s)}(x,0;\lambda)\Big)\mathfrak{B}_{n-k+1,q}(0,my;\lambda),$$

$$(c)\quad {}_H\mathfrak{E}_{n,q}^{(\alpha,s)}(x,y;\lambda) = \frac{1}{2m^n}\sum_{k=0}^{n+1}\frac{1}{[n+1]_q}\begin{bmatrix} n+1 \\ k \end{bmatrix}_q m^k\Big(m^{-k}\lambda\sum_{r=0}^{k}\begin{bmatrix} k \\ r \end{bmatrix}_q m^r \\ {}_H\mathfrak{E}_{r,q}^{(\alpha,s)}(0,y;\lambda) + {}_H\mathfrak{E}_{k,q}^{(\alpha,s)}(0,y;\lambda)\Big)\mathcal{G}_{n-k+1,q}(mx,0;\lambda),$$

$$(d)\quad {}_H\mathfrak{E}_{n,q}^{(\alpha,s)}(x,y;\lambda) = \frac{1}{2m^n}\sum_{k=0}^{n+1}\frac{1}{[n+1]_q}\begin{bmatrix} n+1 \\ k \end{bmatrix}_q m^k\Big(m^{-k}\lambda\sum_{r=0}^{k}\begin{bmatrix} k \\ r \end{bmatrix}_q m^r \\ {}_H\mathfrak{E}_{r,q}^{(\alpha,s)}(x,0;\lambda) + {}_H\mathfrak{E}_{k,q}^{(\alpha,s)}(x,0;\lambda)\Big)\mathcal{G}_{n-k+1,q}(0,my;\lambda).$$

PROOF. Taking suitable arrangements of generating function (3.2) and proceeding on the same lines of proof as in Theorem 3.4, assertions (a)–(d) can be proved. Thus, we omit it. ∎

THEOREM 3.6. *The following relations for the 2D q-Hermite based Apostol-Genocchi polynomials* ${}_H\mathcal{G}_{n,q}^{(\alpha,s)}(x,y;\lambda)$ *hold true:*

$$(a)\quad {}_H\mathcal{G}_{n,q}^{(\alpha,s)}(x,y;\lambda) = \frac{1}{m^n}\sum_{k=0}^{n+1}\frac{1}{[n+1]_q}\begin{bmatrix} n+1 \\ k \end{bmatrix}_q m^k\Big(m^{-k}\lambda\sum_{r=0}^{k}\begin{bmatrix} k \\ r \end{bmatrix}_q m^r \\ {}_H\mathcal{G}_{r,q}^{(\alpha,s)}(0,y;\lambda) - {}_H\mathcal{G}_{k,q}^{(\alpha,s)}(0,y;\lambda)\Big)\mathfrak{B}_{n-k+1,q}(mx,0;\lambda),$$

$$(b)\quad {}_H\mathcal{G}_{n,q}^{(\alpha,s)}(x,y;\lambda) = \frac{1}{m^n}\sum_{k=0}^{n+1}\frac{1}{[n+1]_q}\begin{bmatrix} n+1 \\ k \end{bmatrix}_q m^k\Big(m^{-k}\lambda\sum_{r=0}^{k}\begin{bmatrix} k \\ r \end{bmatrix}_q m^r \\ {}_H\mathcal{G}_{r,q}^{(\alpha,s)}(x,0;\lambda) - {}_H\mathcal{G}_{k,q}^{(\alpha,s)}(x,0;\lambda)\Big)\mathfrak{B}_{n-k+1,q}(0,my;\lambda),$$

$$(c)\quad {}_H\mathcal{G}^{(\alpha,s)}_{n,q}(x,y;\lambda)=\frac{1}{2m^n}\sum_{k=0}^{n}\begin{bmatrix}n\\k\end{bmatrix}_q m^k\Big({}_H\mathcal{G}^{(\alpha,s)}_{k,q}(x,0;\lambda)+m^{-k}\lambda \sum_{r=0}^{k}\begin{bmatrix}k\\r\end{bmatrix}_q m^r\,{}_H\mathcal{G}^{(\alpha,s)}_{r,q}(x,0;\lambda)\Big)\mathfrak{E}_{n-k,q}(0,my;\lambda),$$

$$(d)\quad {}_H\mathcal{G}^{(\alpha,s)}_{n,q}(x,y;\lambda)=\frac{1}{2m^n}\sum_{k=0}^{n}\begin{bmatrix}n\\k\end{bmatrix}_q m^k\Big({}_H\mathcal{G}^{(\alpha,s)}_{k,q}(0,y;\lambda)+m^{-k}\lambda \sum_{r=0}^{k}\begin{bmatrix}k\\r\end{bmatrix}_q m^r\,{}_H\mathcal{G}^{(\alpha,s)}_{r,q}(0,y;\lambda)\Big)\mathfrak{E}_{n-k,q}(mx,0;\lambda).$$

Proof. Considering appropriate arrangements of generating function (3.3) and following the same lines of proof as in Theorem 3.4, we get assertions (a)-(d). Thus, we omit it. ◻

In the next section, we introduce a new class of the $2D$ q-Hermite based Appell polynomials (2DqHbAP) by means of generating function and series representation.

4. 2D q-Hermite based Appell polynomials

First, we establish the generating function for the 2DqHbAP by making use of replacement technique. For, this we consider the following definitions.

Definition 4.1. *The $2D$ q-Hermite based Appell polynomials ${}_HA^{(s)}_{n,q}(x,y)$ ($q\in\mathbb{C}$, $0<|q|<1$) are defined by means of the following generating function:*

$$\frac{1}{g_q(t)}e_q\left(xt-\frac{st^2}{1+q}\right)E_q(yt)=\sum_{n=0}^{\infty}{}_HA^{(s)}_{n,q}(x,y)\frac{t^n}{[n]_q!},\tag{4.1}$$
$${}_HA^{(s)}_{n,q}:={}_HA^{(s)}_{n,q}(0,0).$$

Definition 4.2. *The $2D$ q-Hermite based Appell polynomials ${}_HA^{(s)}_{n,q}(x,y)$ are defined by the following series representation:*

$${}_HA^{(s)}_{n,q}(x,y)=\sum_{k=0}^{n}\begin{bmatrix}n\\k\end{bmatrix}_q A_{k,q}(x,y)H^{(s)}_{n-k,q}(0).$$

Note. We note that by taking $g_q(t)=1$ in equation (4.1), the 2DqHbAP ${}_HA^{(s)}_{n,q}(x,y)$ reduce to $2D$ q-Hermite polynomials $H^{(s)}_{n,q}(x,y)$. Thus, we have the following definition.

Definition 4.3. *The $2D$ q-Hermite polynomials $H^{(s)}_{n,q}(x,y)$ are defined by means of the following generating function:*

$$e_q\left(xt-\frac{st^2}{1+q}\right)E_q(yt)=\sum_{n=0}^{\infty}H^{(s)}_{n,q}(x,y)\frac{t^n}{[n]_q!}.\tag{4.2}$$

REMARK 4.4. From definitions 4.1 and 4.3, it is clear that

$$ {}_HA^{(s)}_{n,q}(x,0) = {}_HA^{(s)}_{n,q}(x), \qquad H^{(s)}_{n,q}(x,0) = H^{(s)}_{n,q}(x), $$

where ${}_HA^{(s)}_{n,q}(x)$ and $H^{(s)}_{n,q}(x)$ are the q-Hermite based Appell polynomials and the q-Hermite polynomials, respectively.

To study the two-variable forms of the q-polynomials from determinant point of view is a new investigation. In particular, the determinant definition for the $2D$ q-Appell polynomials is considered in [10]. Motivated by this, we find the determinant definition for the 2DqHbAP ${}_HA^{(s)}_{n,q}(x,y)$.

- By using a similar approach as in [10, p.359 Theorem 7] and taking help of equations (4.1) and (4.2), we find the following determinant definition for the 2DqHbAP ${}_HA^{(s)}_{n,q}(x,y)$.

DEFINITION 4.5. *The $2D$ q-Hermite based Appell polynomials ${}_HA^{(s)}_{n,q}(x,y)$ of degree n are defined by*

(4.3)

$$ {}_HA^{(s)}_{0,q}(x,y) = \frac{1}{\beta_{0,q}}, $$

$$ {}_HA^{(s)}_{n,q}(x,y) = \frac{(-1)^n}{(\beta_{0,q})^{n+1}} \begin{vmatrix} 1 & H^{(s)}_{1,q}(x,y) & H^{(s)}_{2,q}(x,y) & \cdots & H^{(s)}_{n-1,q}(x,y) & H^{(s)}_{n,q}(x,y) \\ \beta_{0,q} & \beta_{1,q} & \beta_{2,q} & \cdots & \beta_{n-1,q} & \beta_{n,q} \\ 0 & \beta_{0,q} & \begin{bmatrix} 2 \\ 1 \end{bmatrix}_q \beta_{1,q} & \cdots & \begin{bmatrix} n-1 \\ 1 \end{bmatrix}_q \beta_{n-2,q} & \begin{bmatrix} n \\ 1 \end{bmatrix}_q \beta_{n-1,q} \\ 0 & 0 & \beta_{0,q} & \cdots & \begin{bmatrix} n-1 \\ 2 \end{bmatrix}_q \beta_{n-3,q} & \begin{bmatrix} n \\ 2 \end{bmatrix}_q \beta_{n-2,q} \\ . & . & . & \cdots & . & . \\ . & . & . & \cdots & . & . \\ 0 & 0 & 0 & \cdots & \beta_{0,q} & \begin{bmatrix} n \\ n-1 \end{bmatrix}_q \beta_{1,q} \end{vmatrix}, $$

where $n = 1, 2, \dots$, $\beta_{0,q} \neq 0$; $\beta_{0,q}, \ \beta_{1,q}, \ \beta_{2,q}, \dots, \beta_{n,q} \in \mathbb{R}$.

For suitable choices of $g_q(t)$, different members belonging to the family of 2DqHbAP can be obtained. Particularly, we note that the 2DqHbABP, 2DqHbAEP and 2DqHbAGP are the special members of the 2DqHbAP ${}_HA^{(s)}_{n,q}(x,y)$.

- We conclude that for $\lambda = 1$, the polynomials ${}_H\mathfrak{B}^{(\alpha,s)}_{n,q}(x,y;\lambda)$, ${}_H\mathfrak{E}^{(\alpha,s)}_{n,q}(x,y;\lambda)$ and ${}_H\mathcal{G}^{(\alpha,s)}_{n,q}(x,y;\lambda)$ reduce to the $2D$ q-Hermite based Bernoulli polynomials ${}_HB^{(\alpha,s)}_{n,q}(x,y)$, $2D$ q-Hermite based Euler polynomials ${}_HE^{(\alpha,s)}_{n,q}(x,y)$ and $2D$ q-Hermite based Genocchi polynomials ${}_HG^{(\alpha,s)}_{n,q}(x,y)$, each of order α. For $\alpha = 1$, these polynomials reduce to the $2D$ q-Hermite based Bernoulli polynomials ${}_HB^{(s)}_{n,q}(x,y)$, $2D$ q-Hermite based Euler polynomials ${}_HE^{(s)}_{n,q}(x,y)$ and $2D$ q-Hermite based Genocchi polynomials ${}_HG^{(s)}_{n,q}(x,y)$.

Recently, Riyasat et al. in [25] gave the determinant definitions of the q-Bernoulli, q-Euler and q-Genocchi polynomials. First, we slightly focus on the determinant definitions of the $2D$ q-Bernoulli, q-Euler and q-Genocchi polynomials.

- By taking

$$\Big(\beta_{0,q}=1, \beta_{i,q}=\frac{1}{[i+1]_q}\Big); \qquad \Big(\beta_{0,q}=1, \beta_{i,q}=\frac{1}{2}\Big)$$

and

$$\Big(\beta_{0,q}=1, \beta_{i,q}=\frac{1}{2[i+1]_q}\Big)$$

$i=1,2,\cdots,n$, respectively in determinant definition of the $2D$ q-Appell polynomials [10, p.359 Theorem 7], we can obtain the determinant definitions of the $2D$ q-Bernoulli, q-Euler and q-Genocchi polynomials $B_{n,q}(x,y)$, $E_{n,q}(x,y)$ and $G_{n,q}(x,y)$, respectively.

- Again, by taking $\big(\beta_{0,q}=1,\ \beta_{i,q}=\frac{1}{[i+1]_q}\big)$; $\big(\beta_{0,q}=1,\ \beta_{i,q}=\frac{1}{2}\big)$ and $\big(\beta_{0,q}=1,\ \beta_{i,q}=\frac{1}{2[i+1]_q}\big)$ $\big(i=1,2,\cdots,n\big)$, respectively in determinant definition (4.3) of the $2D$ q-Hermite based Appell polynomials, we find the following determinant definitions for the polynomials ${}_HB^{(s)}_{n,q}(x,y)$, ${}_HE^{(s)}_{n,q}(x,y)$ and ${}_HG^{(s)}_{n,q}(x,y)$, respectively.

Definition 4.6. *The $2D$ q-Hermite based Bernoulli polynomials of degree n ${}_HB^{(s)}_{n,q}(x,y)$ are defined by*

$$\begin{aligned}
&{}_HB^{(s)}_{0,q}(x,y)=1,\\
&{}_HB^{(s)}_{n,q}(x,y)\\
&=(-1)^n\begin{vmatrix}
1 & H^{(s)}_{1,q}(x,y) & H^{(s)}_{2,q}(x,y) & \cdots & H^{(s)}_{n-1,q}(x,y) & H^{(s)}_{n,q}(x,y)\\
1 & \frac{1}{[2]_q} & \frac{1}{[3]_q} & \cdots & \frac{1}{[n]_q} & \frac{1}{[n+1]_q}\\
0 & 1 & \begin{bmatrix}2\\1\end{bmatrix}_q\frac{1}{[2]_q} & \cdots & \begin{bmatrix}n-1\\1\end{bmatrix}_q\frac{1}{[n-1]_q} & \begin{bmatrix}n\\1\end{bmatrix}_q\frac{1}{[n]_q}\\
0 & 0 & 1 & \cdots & \begin{bmatrix}n-1\\2\end{bmatrix}_q\frac{1}{[n-2]_q} & \begin{bmatrix}n\\2\end{bmatrix}_q\frac{1}{[n-1]_q}\\
. & . & . & \cdots & . & .\\
. & . & . & \cdots & . & .\\
0 & 0 & 0 & \cdots & 1 & \begin{bmatrix}n\\n-1\end{bmatrix}_q\frac{1}{[2]_q}
\end{vmatrix},
\end{aligned}$$

$n=1,2,\cdots$.

Definition 4.7. *The $2D$ q-Hermite based Euler polynomials ${}_HE^{(s)}_{n,q}(x,y)$ of degree n are defined by*

$$
\begin{aligned}
&{}_HE^{(s)}_{0,q}(x,y)=1,\\
&{}_HE^{(s)}_{n,q}(x,y)\\
&=(-1)^n\begin{vmatrix}
1 & H^{(s)}_{1,q}(x,y) & H^{(s)}_{2,q}(x,y) & \cdots & H^{(s)}_{n-1,q}(x,y) & H^{(s)}_{n,q}(x,y)\\
1 & \frac{1}{2} & \frac{1}{2} & \cdots & \frac{1}{2} & \frac{1}{2}\\
0 & 1 & \frac{1}{2}\begin{bmatrix}2\\1\end{bmatrix}_q & \cdots & \frac{1}{2}\begin{bmatrix}n-1\\1\end{bmatrix}_q & \frac{1}{2}\begin{bmatrix}n\\1\end{bmatrix}_q\\
0 & 0 & 1 & \cdots & \frac{1}{2}\begin{bmatrix}n-1\\2\end{bmatrix}_q & \frac{1}{2}\begin{bmatrix}n\\2\end{bmatrix}_q\\
. & . & . & \cdots & . & .\\
. & . & . & \cdots & . & .\\
0 & 0 & 0 & \cdots & 1 & \frac{1}{2}\begin{bmatrix}n\\n-1\end{bmatrix}_q
\end{vmatrix},
\end{aligned}
$$

$n=1,2,\cdots$.

Definition 4.8. *The $2D$ q-Hermite based Genocchi polynomials of degree n ${}_HG^{(s)}_{n,q}(x,y)$ are defined by*

$$
\begin{aligned}
&{}_HG^{(s)}_{0,q}(x,y)=1,\\
&{}_HG^{(s)}_{n,q}(x,y)\\
&=(-1)^n\begin{vmatrix}
1 & H^{(s)}_{1,q}(x,y) & H^{(s)}_{2,q}(x,y) & \cdots & H^{(s)}_{n-1,q}(x,y) & H^{(s)}_{n,q}(x,y)\\
1 & \frac{1}{2[2]_q} & \frac{1}{2[3]_q} & \cdots & \frac{1}{2[n]_q} & \frac{1}{2[n+1]_q}\\
0 & 1 & \begin{bmatrix}2\\1\end{bmatrix}_q\frac{1}{2[2]_q} & \cdots & \begin{bmatrix}n-1\\1\end{bmatrix}_q\frac{1}{2[n-1]_q} & \begin{bmatrix}n\\1\end{bmatrix}_q\frac{1}{2[n]_q}\\
0 & 0 & 1 & \cdots & \begin{bmatrix}n-1\\2\end{bmatrix}_q\frac{1}{2[n-2]_q} & \begin{bmatrix}n\\2\end{bmatrix}_q\frac{1}{2[n-1]_q}\\
. & . & . & \cdots & . & .\\
. & . & . & \cdots & . & .\\
0 & 0 & 0 & \cdots & 1 & \begin{bmatrix}n\\n-1\end{bmatrix}_q\frac{1}{2[2]_q}
\end{vmatrix},
\end{aligned}
$$

$n=1,2,\cdots$.

The q-difference equations for the $2D$ q-Appell and composite 2D q-Appell polynomials are established in [11, 24]. This provides motivation to establish q-difference equations for the $2D$ q-Hermite based Appell polynomials and also for their composite forms. This aspect will be taken in next investigation.

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