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ON THE RAMANUJAN-NAGELL TYPE DIOPHANTINE EQUATION $x^2 + Ak^n = B$

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Abstract. In this paper, we prove that the Ramanujan-Nagell type Diophantine equation $x^2 + Ak^n = B$ has at most three nonnegative integer solutions (x, n) for $A = 1, 2, 4$, k an odd prime and B a positive integer. Therefore, we partially confirm two conjectures of Ulas from [23].

1. Introduction

It is well-known that the Diophantine equation

$$x^2 + 7 = 2^{n+2} \tag{1.1}$$

is called *the Ramanujan-Nagell equation.* In 1960, Nagell ([18]) proved that the only integer solutions to Diophantine equation (1.1) are

$$(x, n) = (1, 1),\ (3, 2),\ (5, 3),\ (11, 5),\ (181, 13).$$

A generalized Ramanujan-Nagell equation is the Diophantine equation

$$x^2 + D = k^n \ \text{ in integers } x \geq 1,\ \ n \geq 1. \tag{1.2}$$

This Diophantine equation has a very rich literature. For examples, one can see [1]–[18], [22]–[24]. One aspect of the study of equation (1.2) is to determine the integer solutions (x, k, n). Diophantine equation (1.2) was studied for fixed values of D or when $D = \prod_i p_i^{a_i}$ with fixed prime numbers p_i.

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Recently, many mathematicians have been interested in a more generalized Ramanujan-Nagell type equation of the form

$$x^2 = Ak^n + B, \quad k \in \mathbb{Z}_{\geq 2}, \quad A, B \in \mathbb{Z} \setminus \{0\}. \tag{1.3}$$

In 1996, Stiller ([20]) considered the equation

$$x^2 + 119 = 15 \cdot 2^n$$

and proved that this equation has exactly 6 solutions. His result motivated Ulas ([23]) to consider finding examples. He proved that for each $k \in \mathbb{Z} \setminus \{-1, 0, 1\}$ there are infinitely many pairs of integers A, B such that $\gcd(A, B)$ is square-free and Diophantine equation (1.3) has at least four solutions in non-negative integers. He was also able to solve some equations of the type (1.3) having five or more solutions. Besides proving many results, he set many conjectures. In [24], we completely proved his Conjectures 4.2 and 4.3.

In this paper, we consider following conjectures:

Conjecture 1.1 ([23, Conjecture 4.4]). *The Diophantine equation*

$$x^2 + k^n = B \tag{1.4}$$

has at most three nonnegative integers (x, n), for any given integers $k \geq 2$ and $B \geq 1$.

and

Conjecture 1.2 ([23, Conjecture 4.5]). *The Diophantine equation*

$$x^2 + Ak^n = B \tag{1.5}$$

has at most four nonnegative integers (x, n), for any given integers $k \geq 2$, $A \geq 1$, and $B \geq 1$.

Meng Bai and the first author in [4] confirmed Conjecture 1.1 for $k = 2$, i.e. they proved that for any positive integer B, the Diophantine equation

$$x^2 + 2^n = B$$

has at most 3 solutions (x, n) in nonnegative integers. The aim of this paper is not only to extend their result but also to partially confirm the above conjectures by proving the following result.

Theorem 1.3. *Let p be an odd prime, B a positive integer and $A = 1, 2, 4$. Then the Diophantine equation*

$$x^2 + Ap^n = B \tag{1.6}$$

has at most two nonnegative integer solutions (x, n) in the following three situations:

(i) $p \equiv 3 \pmod 4$, $A = 1, 4$;
(ii) $p \equiv 5, 7 \pmod 8$, $A = 2$;
(iii) $p^2 \nmid B$;

and at most three nonnegative integer solutions (x, n) *in the following two situations:*

(iv) $p \equiv 1 \pmod 4$, $A = 1, 4, p^2 | B$;
(v) $p \equiv 1, 3 \pmod 8$, $A = 2, p^2 | B$.

The organization of this paper is as follows. In Section 2, not only will we recall a result related to the solutions of a Pell equation, but also we will derive from it a lemma useful for the proof of Theorem 1.3 in Section 3.

2. Preliminaries

Let p be a prime and n an integer. We denote by $v_p(n)$ the p-adic valuation of n. First, we recall a well-known result on Pell equations. For example, one can refer to [17, Theorem 106 and Theorem 108a].

Lemma 2.1. *Let* $D \geq 3$ *be a nonsquare integer and suppose that the Pell equation*

$$x^2 - Dy^2 = -2^t, \ \ t = 0, 1, 2 \tag{2.1}$$

has an integer solution. Let ε_t *be the fundamental solution of equation (2.1), then all integer solutions of (2.1) can be written as*

$$x + y\sqrt{D} = \pm \frac{\varepsilon_t^{2k+1}}{2^{tk}}, \quad k \in \mathbb{Z}.$$

Now, we will prove the following result.

Lemma 2.2. *Let* D *be a positive integer and* p *an odd prime, then the Diophantine equation*

$$x^2 - Dp^{2n} = -2^t, \ \ t = 0, 1, 2 \tag{2.2}$$

has at most one positive integer solution (x, n).

Proof. It is obvious when D is a square. So, we assume that D is not a square. Moreover, we suppose that (a, n_1) and (b, n_2) are two distinct positive integer solutions of (2.2), with $n_2 > n_1 \geq 1$. Let $D_1 = Dp^{2n_1}$, $m = n_2 - n_1 \geq 1$, and let us consider the Pell equation

$$x^2 - D_1 y^2 = -2^t, \ \ t = 0, 1, 2. \tag{2.3}$$

It is easy to see that $a + \sqrt{D_1}$ is the fundamental solution of (2.3). By Lemma 2.1, there exists an integer $k \geq 1$ such that

$$b + p^m\sqrt{D_1} = \frac{(a + \sqrt{D_1})^{2k+1}}{2^{tk}}.$$

Then, we have

$$p^m = \frac{(a + \sqrt{D_1})^{2k+1} - (a - \sqrt{D_1})^{2k+1}}{2^{tk+1}\sqrt{D_1}} = 2^{-tk} \sum_{r=0}^{k} C_{2k+1}^{2r+1} a^{2k-2r} D_1^r.$$

This implies that

$$2^{tk}p^m = \sum_{r=0}^{k} C_{2k+1}^{2r+1} a^{2k-2r} D_1^r. \tag{2.4}$$

We will prove that

$$v_p(C_{2k+1}^{1} a^{2k}) < v_p(C_{2k+1}^{2r+1} a^{2k-2r} D_1^r), \quad r = 1, 2, ..., k. \tag{2.5}$$

As $a^2 = D_1 - 2^t$ and $D_1 = Dp^{2n_1}$, we obtain $p \nmid a$, then

$$v_p(C_{2k+1}^{1} a^{2k}) = v_p(2k+1)$$

and

$$v_p(C_{2k+1}^{2r+1} a^{2k-2r} D_1^r) = v_p(C_{2k+1}^{2r+1} D_1^r) = v_p(C_{2k+1}^{2r+1}) + rv_p(D_1).$$

Notice that

$$C_{2k+1}^{2r+1} = \frac{2k+1}{2r+1} C_{2k}^{2r}.$$

Thus, one has

$$v_p(C_{2k+1}^{2r+1}) \geq v_p(2k+1) - v_p(2r+1).$$

Then, we obtain the inequality (2.5) from

$$rv_p(D_1) - v_p(2r+1) \geq 2n_1 r - v_p(2r+1) \geq 2r - v_p(2r+1) > 0.$$

Therefore, using (2.4) and (2.5), we get $p^m | C_{2k+1}^{1} a^{2k}$, i.e. $p^m | 2k+1$. So, we have $p^m \leq 2k+1$. On the other hand, from (2.4) and $k \geq 1$, we deduce that

$$2^{tk} p^m > C_{2k+1}^{1} a^{2k} = (2k+1) a^{2k},$$

i.e. $2^{tk} > a^{2k}$, which yields $a^2 < 2^t \leq 4$. This contradicts the fact that

$$a^2 = D_1 - 2^t = p^{2n_1} D - 2^t \geq p^2 - 4 \geq 3^2 - 4 = 5.$$

Therefore, it completes the proof of Lemma 2.2. □

3. Proof of Theorem 1.3

If $B < Ap^2$, then $n \leq 1$ and therefore equation (1.6) has at most two nonnegative integer solutions (x, n). Thus, for the remainder of the proof, we assume that $B \geq Ap^2$ and we consider two cases: $p^2 \nmid B$ and $p^2 \mid B$.

Case 1: $p^2 \nmid B$. Here, we will study the following two claims. Notice that the restriction $p^2 \nmid B$ is necessary for the proof of Claim 2 but not for the proof of Claim 1. And from the following discussion, we also know that there is at most two nonnegative integer solutions for $A = 3$ in this case.

Claim 1: *There is at most one nonnegative integer solution (x, n) satisfying $Ap^n < 4\sqrt{B-A} + A - 4$.*

Assume that (x_1, n_1) and (x_2, n_2) are two distinct integer solutions of equation (1.6) satisfying

$$x_1 > x_2 \geq 0, \qquad Ap^{n_1} < Ap^{n_2} < 4\sqrt{B-A} + A - 4.$$

Then, we get

$$x_1^2 - x_2^2 = A(p^{n_2} - p^{n_1}).$$

Since p is an odd prime, we have $Ap^{n_1} \equiv Ap^{n_2} \pmod 2$, then $x_1^2 \equiv x_2^2 \pmod 2$, i.e. $2|(x_1 \pm x_2)$. As

$$p^{n_2} - p^{n_1} \leq p^{n_2} - 1$$

and

$$x_1^2 - x_2^2 = (x_1 + x_2)(x_1 - x_2) \geq 2(x_1 + x_2) \geq 2(x_2 + 2 + x_2) = 4x_2 + 4,$$

we get $Ap^{n_2} - (A + 4) \geq 4x_2$, i.e.

$$A^2p^{2n_2} - 2A(A + 4)p^{n_2} + (A + 4)^2 \geq 16x_2^2 = 16(B - Ap^{n_2}).$$

Therefore, we obtain

$$A^2p^{2n_2} - 2A(A - 4)p^{n_2} + (A - 4)^2 + 16A \geq 16B,$$

i.e.

$$(Ap^{n_2} - (A - 4))^2 \geq 16(B - A),$$

which yields $Ap^{n_2} \geq 4\sqrt{B - A} + A - 4$. This leads to a contradiction.

Claim 2: *There is at most one nonnegative integer solution (x, n) satisfying $Ap^n \geq 4\sqrt{B - A} + A - 4$.*

From $Ap^n \geq 4\sqrt{B - A} + A - 4$ and $B \geq Ap^2$, when $A = 1, 2, 4$ we have $n \geq 2$. Moreover, since $p^2 \nmid B$, we see that $p \nmid x$. Assume that (x_1, n_1) and (x_2, n_2) are two distinct integer solutions of equation (1.6) satisfying

$$x_1 > x_2 \geq 0, \qquad Ap^{n_2} > Ap^{n_1} \geq 4\sqrt{B - A} + A - 4.$$

Then, we get

$$x_1^2 - x_2^2 = Ap^{n_2} - Ap^{n_1} = Ap^{n_1}(p^{n_2 - n_1} - 1).$$

Similarly to Claim 1, we see that $2|(x_1 \pm x_2)$. Since p is an odd prime and $p \nmid x_1x_2$, we have $2p^{n_1}|x_1 + x_2$ or $2p^{n_1}|x_1 - x_2$, so we get

$$2x_1 - 2 \geq x_1 + x_2 \geq 2p^{n_1}.$$

This implies that

$$B - Ap^{n_1} = x_1^2 \geq (p^{n_1} + 1)^2 = p^{2n_1} + 2p^{n_1} + 1.$$

Thus, we deduce that

$$B + A + \frac{A^2}{4} \geq \left(p^{n_1} + 1 + \frac{A}{2}\right)^2,$$

which yields

$$Ap^{n_1} \leq A\sqrt{B + A + \frac{A^2}{4}} - A\left(1 + \frac{A}{2}\right).$$

Therefore, we have

$$A\sqrt{B+A+\frac{A^2}{4}}-A\left(1+\frac{A}{2}\right)\geq 4\sqrt{B-A}+A-4.$$

This gives

$$A\sqrt{B+A+\frac{A^2}{4}}-4\sqrt{B-A}\geq \frac{A^2}{2}+2A-4.$$

A direct calculation shows that this is impossible for $A=1,2,4$ and $B\geq Ap^2$. This justifies Claim 2 and also completes the proof of Theorem 1.3 (*iii*).

Case 2: $p^2|B$

In this case, we have $n=0$ or $n\geq 2$. We will prove Theorem 1.3 by induction on B.

• For Theorem 1.3 (*i*) and (*ii*), one can easily see that $\left(\frac{-A}{p}\right)=-1$ (the Legendre symbol) and then $n\neq 0$. This means that $n\geq 2$ and $p|x$. Let $x=pz$, $m=n-2$, $B_0=B/p^2$. Thus, we get the equation

$$z^2+Ap^m=B_0,$$

with $B_0<B$. By induction and Case 1, we see that the above equation has at most two nonnegative integer solutions (z,m). Therefore, equation (1.6) has at most two nonnegative integer solutions (x,n). This closes the case of Theorem 1.3 (*i*) and (*ii*).

• For Theorem 1.3 (*iv*) and (*v*), we will use Lemma 2.2 to prove that equation (1.6) has at most three nonnegative integer solutions (x,n).

Assume that $p^{2k}|B$ and $p^{2(k+1)}\nmid B$. Let $B=p^{2k}B_0$. We will prove that there is at most one nonnegative integer solution (x,n) satisfying $n<2k$ and at most two nonnegative integer solutions (x,n) satisfying $n\geq 2k$.

If (x,n) is a nonnegative integer solution of (1.6) with $n<2k$, then from $x^2+Ap^n=B=p^{2k}B_0$, we deduce that $2|n$. Put $n=2m$. Then, $p^m|x$. Put $x=p^mz$. Thus, we have

$$z^2+A=B_0p^{2(k-m)},$$

with $k-m=l\geq 1$, i.e.

$$z^2-B_0p^{2l}=-A.$$

As $A=1,2,4$, then by Lemma 2.2 the above equation has most one positive integer solution (z,l). This means that equation (1.6) has at most one nonnegative integer solution (x,n) satisfying $n<2k$.

If $n\geq 2k$, then $p^k|x$. Put $x=p^kz$, $u=n-2k$, $B=p^{2k}B_0$. Then, equation (1.6) becomes

$$z^2+Ap^u=B_0,$$

with $p^2\nmid B_0$. By Case 1, this equation has at most two nonnegative integer solution (z,u), i.e. equation (1.6) has at most two nonnegative integer solutions (x,n) satisfying $n\geq 2k$.

This completes the proof of Theorem 1.3.

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