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PROPER INCLUSIONS OF MORREY SPACES

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Abstract. In this paper, we prove that the inclusions between Morrey spaces, between weak Morrey spaces, and between a Morrey space and a weak Morrey space are all proper. The proper inclusion between a Morrey space and a weak Morrey space is established via the unboundedness of the Hardy-Littlewood maximal operator on Morrey spaces of exponent 1. In addition, we also give a necessary condition for each inclusion. Our results refine previous inclusion properties studied in [4].

1. Introduction

Morrey spaces were first introduced by C. B. Morrey in [7] in relation to the study of the solution of certain elliptic partial differential equations. For $1 \le p \le q < \infty$, the *Morrey space* $\mathcal{M}^p_q = \mathcal{M}^p_q(\mathbb{R}^d)$ is defined to be the set of all $f \in L^p_{\rm loc}(\mathbb{R}^d)$ such that

$$\|f\|_{\mathcal{M}^p_q} := \sup_{a\in\mathbb{R}^d,\ r>0} |B(a,r)|^{\frac{1}{q}} \left(\frac{1}{|B(a,r)|}\int_{B(a,r)} |f(y)|^p \, dy\right)^{\frac{1}{p}} < \infty.$$

Here, $B(a,r)$ is an open ball centered at a with radius r, and $|B(a,r)|$ denotes its Lebesgue measure. Notice that, when $p = q$, one can recover the Lebesgue space $L^p = L^p(\mathbb{R}^d)$ as the special case of $\mathcal{M}^p_q$. See [9] for various spaces related to Morrey spaces. Many researchers have proved the boundedness of classical integral operators on Morrey spaces and their generalizations. See, for instance, [1, 2] and the references therein.

Concerning the Hardy-Littlewood maximal operator (defined in Section 3), one may prove its boundedness on Morrey spaces using the inclusion $\mathcal{M}^p_q \subseteq$

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$\mathcal{M}^1_q$. In general, we have the following inclusions of Morrey spaces

$$L^q = \mathcal{M}^q_q \subseteq \mathcal{M}^{p_2}_q \subseteq \mathcal{M}^{p_1}_q \subseteq \mathcal{M}^1_q$$

provided that $1 \le p_1 \le p_2 \le q < \infty$. These inclusions may be obtained by applying Hölder's inequality. Note that, for $1 \le p_2 < q < \infty$, we have $f(x) := |x|^{-\frac{d}{q}} \in \mathcal{M}^{p_2}_q \setminus \mathcal{M}^q_q$. This tells us that the inclusion $\mathcal{M}^q_q \subseteq \mathcal{M}^{p_2}_q$ is proper for $1 \le p_2 < q < \infty$.

Besides the 'strong' Morrey spaces, we also have weak Morrey spaces whose definitions are given as follows.

DEFINITION 1.1. *Let $1 \le p \le q < \infty$. A measurable functions f on $\mathbb{R}^d$ is said to belong to the weak Morrey space $w\mathcal{M}^p_q = w\mathcal{M}^p_q(\mathbb{R}^d)$ if the quasi-norm*

$$\|f\|_{w\mathcal{M}^p_q} := \sup_{\gamma>0} \|\gamma\chi_{\{|f|>\gamma\}}\|_{\mathcal{M}^p_q}$$

is finite.

Note that, by using the inequality $\gamma\chi_{\{|f|>\gamma\}} \le |f|$ for every $\gamma > 0$, we have $\mathcal{M}^p_q \subseteq w\mathcal{M}^p_q$. The inclusion properties of weak Morrey spaces, generalized Morrey spaces, generalized weak Morrey spaces, and their necessary conditions were discussed in [4]. In particular, for the case of Morrey spaces and weak Morrey spaces, the results can be stated as follows.

THEOREM 1.2 ([4]). *For $1 \le p_1 \le p_2 \le q < \infty$, the following inclusion holds*

$$w\mathcal{M}^{p_2}_q \subseteq w\mathcal{M}^{p_1}_q.$$

Further, if $p_1 < p_2$, then

$$w\mathcal{M}^{p_2}_q \subseteq \mathcal{M}^{p_1}_q.$$

In addition to the above inclusion relations of Morrey spaces, we have the following theorems.

THEOREM 1.3. *Let $1 \le p_1 < p_2 < q < \infty$. Then each of the following inclusions is proper:*

(i) $\mathcal{M}^{p_2}_q \subseteq \mathcal{M}^{p_1}_q$;
(ii) $w\mathcal{M}^{p_2}_q \subseteq \mathcal{M}^{p_1}_q$;
(iii) $w\mathcal{M}^{p_2}_q \subseteq w\mathcal{M}^{p_1}_q$.

THEOREM 1.4. *Let $1 \le p \le q < \infty$. Then the inclusion $\mathcal{M}^p_q \subseteq w\mathcal{M}^p_q$ is proper.*

REMARK 1.5. The claim about the proper inclusion $\mathcal{M}^{p_2}_q \subseteq \mathcal{M}^{p_1}_q$ is stated in [4, p. 2] without proof. We shall see the detailed explanation of this claim in the proof of Theorem 1.3 (i). In [4, Remark 2.4], the authors refer to [3] for the proper inclusion between the generalized Morrey space $L^{1,\phi}$ and the corresponding weak type space $wL^{1,\phi}$, where $\phi(t) = t^{-1}\log(3+t)$. Since

$L^{1,\phi} \neq \mathcal{M}^1_q$ for this choice of ϕ, Theorem 1.4 can be seen as a complement of the result in [3].

We also obtain the following necessary conditions for inclusion of Morrey spaces and weak Morrey spaces which can be seen as a refinement of some necessary conditions given in [4].

THEOREM 1.6. *Let $1 \le p_i \le q_i < \infty$ for $i = 1, 2$. Then the following implications hold:*

(i) $\mathcal{M}^{p_2}_{q_2} \subseteq \mathcal{M}^{p_1}_{q_1}$ *implies* $q_1 = q_2$ *and* $p_1 \le p_2$*;*
(ii) $w\mathcal{M}^{p_2}_{q_2} \subseteq w\mathcal{M}^{p_1}_{q_1}$ *implies* $q_1 = q_2$ *and* $p_1 \le p_2$*;*
(iii) $w\mathcal{M}^{p_2}_{q_2} \subseteq \mathcal{M}^{p_1}_{q_1}$ *implies* $q_1 = q_2$ *and* $p_1 < p_2$*.*

REMARK 1.7. A necessary and sufficient condition for inclusion of Morrey spaces on a bounded domain can be found in [10, Theorem 2.1] and [11]. The case of Morrey spaces on $\mathbb{R}^d$ is mentioned in [6, Eq. (3.9)] and the authors refer to [12, Satz 1.6]. However, we do not have the access to the paper, so that we do not know how the proof goes. See also [6, Corollary 3.14] for weighted version of Theorem 1.6. Here we present a proof of the necessary and sufficient condition for the inclusion property, which is different from and simpler than that in [10].

The organization of this paper is as follows. In the next section, we prove that for $1 \le p_1 < p_2 < q < \infty$ the set $\mathcal{M}^{p_1}_q \setminus \mathcal{M}^{p_2}_q$ is not empty. By the same example, we also show that for $1 \le p_1 < p_2 < q$ the inclusion $w\mathcal{M}^{p_2}_q \subseteq w\mathcal{M}^{p_1}_q$ is proper. In Section 3, we give the proof of Theorem 1.4 using the unboundedness of the Hardy-Littlewood maximal operator on Morrey spaces of exponent 1. The proof of Theorem 1.6 is given in the last section. Throughout this paper, we denote by C a positive constant which is independent of the function f and its value may be different from line to line.

2. THE PROOF OF THEOREM 1.3

We shall first prove Theorem 1.3 (i) by constructing a function which belongs to $\mathcal{M}^{p_1}_q$ but not to $\mathcal{M}^{p_2}_q$, for $1 \le p_1 < p_2 < q < \infty$.

PROOF OF THEOREM 1.3 (i). Let $1 \le p_1 < p_2 < q < \infty$ and $\beta := \frac{d(p_1+p_2)}{2q}$. Then we have

$$\frac{dp_1}{q} < \beta < \frac{dp_2}{q} \tag{2.1}$$

and

$$d - \beta = \frac{d(q-p_1) + d(q-p_2)}{2q} > 0.$$

Define $g(x) := \chi_{B(0,1)}(x) + \chi_{\mathbb{R}^n\setminus B(0,1)}(x)|x|^{-\beta}$. Then, for each $k \in \mathbb{N}$, we choose $r_k \in (k, k+1)$ such that

$$\int_{B(0,k+1)\setminus B(0,k)} g(x)\ dx = |B(0,r_k) \setminus B(0,k)|.$$

Next define

$$f(x) := \chi_{B(0,1)}(x) + \sum_{k=1}^{\infty} \chi_{B(0,r_k)\setminus B(0,k)}(x). \tag{2.2}$$

We shall show that $f \in \mathcal{M}_q^{p_1} \setminus \mathcal{M}_q^{p_2}$. First observe that

$$\int_{B(a,r)} |f(x)|^p dx \leq \int_{B(0,r)} |f(x)|^p\ dx$$

for every $1 \leq p < \infty,\ a \in \mathbb{R}^d,\ r > 0$. Now, for $1 \leq p < \infty$ and $r > 2$, we have

$$\int_{B(0,r)} |f(x)|^p\ dx = \int_{B(0,r)} |f(x)|\ dx \leq \int_{B(0,2r)} g(x)\ dx,$$

so

$$\int_{B(0,r)} |f(x)|^p\ dx \leq \int_{B(0,2r)} |x|^{-\beta}\ dx = Cr^{d-\beta} \tag{2.3}$$

and

$$\begin{aligned}\int_{B(0,r)} |f(x)|^p\ dx &\geq \int_{B(0,r)\setminus B(0,1)} |x|^{-\beta}\ dx \\ &= C(r^{d-\beta} - 1) \geq C\left(1 - \frac{1}{2^{d-\beta}}\right) r^{d-\beta}.\end{aligned} \tag{2.4}$$

Therefore, by substituting $p = p_1$ into (2.3) and recalling (2.1), we have

$$|B(0,r)|^{\frac{1}{q}-\frac{1}{p_1}} \left(\int_{B(0,r)} |f(x)|^{p_1}\ dx\right)^{\frac{1}{p_1}} \leq Cr^{\frac{d}{q}-\frac{d}{p_1}} r^{\frac{d}{p_1}-\frac{\beta}{p_1}} = Cr^{\frac{d}{q}-\frac{\beta}{p_1}} \leq C. \tag{2.5}$$

On the other hand, for each $r \leq 2$, we have

$$\begin{aligned}|B(0,r)|^{\frac{1}{q}-\frac{1}{p_1}} \left(\int_{B(0,r)} |f(x)|^{p_1}\ dx\right)^{\frac{1}{p_1}} &\leq Cr^{\frac{d}{q}-\frac{d}{p_1}} \left(\int_{B(0,r)} |f(x)|^{p_1} dx\right)^{\frac{1}{p_1}} \\ &\leq Cr^{\frac{d}{q}} \leq C.\end{aligned} \tag{2.6}$$

By combining (2.5) and (2.6) we conclude that $f \in \mathcal{M}_q^{p_1}$.

Meanwhile, by substituting $p = p_2$ into (2.4), we have

$$|B(0,r)|^{\frac{1}{q}-\frac{1}{p_2}} \left(\int_{B(0,r)} |f(x)|^{p_2}\ dx\right)^{\frac{1}{p_2}} \geq Cr^{\frac{d}{q}-\frac{d}{p_2}} r^{\frac{d-\beta}{p_2}} = Cr^{\frac{d}{q}-\frac{\beta}{p_2}}.$$

Since $\frac{d}{q} - \frac{\beta}{p_2} > 0$, we have

$$\sup_{a\in\mathbb{R}^d, r>0} |B(a,r)|^{\frac{1}{q}-\frac{1}{p_2}} \left(\int_{B(a,r)} |f(x)|^{p_2}\ dx \right)^{\frac{1}{p_2}}$$
$$\geq C \sup_{r>2} |B(0,r)|^{\frac{1}{q}-\frac{1}{p_2}} \left(\int_{B(0,r)} |f(x)|^{p_2}\ dx \right)^{\frac{1}{p_2}}$$
$$\geq C \sup_{r>2} r^{\frac{d}{q}-\frac{\beta}{p_2}} = \infty.$$

Thus $f \notin \mathcal{M}^{p_2}_q$, and we are done. $\square$

Theorem 1.3 (ii) and (iii) are proved by using the function f from the proof of Theorem 1.3 (i) and its relation with the characteristic function of its level set. The detailed proof goes as follows.

Proof of Theorem 1.3 (ii)-(iii). For $1 \leq p_1 < p_2 < q < \infty$, let f be defined by (2.2). Observe that

$$\chi_{\{|f|>\gamma\}} = \begin{cases} 0, & \gamma \geq 1, \\ f, & \gamma \in (0,1). \end{cases}$$

This together with the fact that $f \notin \mathcal{M}^{p_2}_q$ gives

$$\|f\|_{w\mathcal{M}^{p_2}_q} = \sup_{\gamma\in(0,1)} \gamma \|\chi_{\{|f|>\gamma\}}\|_{\mathcal{M}^{p_2}_q} = \sup_{\gamma\in(0,1)} \gamma \|f\|_{\mathcal{M}^{p_2}_q} = \|f\|_{\mathcal{M}^{p_2}_q} = \infty,$$

and hence $f \in \mathcal{M}^{p_1}_q \setminus w\mathcal{M}^{p_2}_q$. Thus we have shown that $w\mathcal{M}^{p_2}_q \subseteq \mathcal{M}^{p_1}_q$ is a proper inclusion. Since $\mathcal{M}^{p_1}_q \subseteq w\mathcal{M}^{p_1}_q$, we also have $f \in w\mathcal{M}^{p_1}_q \setminus w\mathcal{M}^{p_2}_q$, so the inclusion (iii) is proper. $\square$

3. The proof of Theorem 1.4

In order to prove Theorem 1.4, we need the following lemma.

Lemma 3.1. *Let $1 \leq p \leq q < \infty$. Then*

$$\|f\|_{\mathcal{M}^p_q} = \||f|^p\|^{\frac{1}{p}}_{\mathcal{M}^1_{\frac{q}{p}}}$$

for every $f \in \mathcal{M}^p_q$ and

$$\|f\|_{w\mathcal{M}^p_q} = \||f|^p\|^{\frac{1}{p}}_{w\mathcal{M}^1_{\frac{q}{p}}}$$

for every $f \in w\mathcal{M}^p_q$.

PROOF. We calculate

$$\|f\|_{\mathcal{M}^p_q} = \sup_B \left(|B|^{\frac{p}{q}-1} \int_B |f(x)|^p \, dx \right)^{\frac{1}{p}} = \||f|^p\|^{\frac{1}{p}}_{\mathcal{M}^1_{\frac{q}{p}}}.$$

By applying the first identity for $\chi_{\{|f|>\gamma^{\frac{1}{p}}\}}$, we have

$$\||f|^p\|^{\frac{1}{p}}_{w\mathcal{M}^1_{\frac{q}{p}}} = \sup_{\gamma>0} \gamma^{\frac{1}{p}} \|\chi_{\{|f|^p>\gamma\}}\|^{\frac{1}{p}}_{\mathcal{M}^1_{\frac{q}{p}}} = \sup_{\gamma>0} \gamma^{\frac{1}{p}} \|\chi_{\{|f|>\gamma^{\frac{1}{p}}\}}\|_{\mathcal{M}^p_q} = \|f\|_{w\mathcal{M}^p_q},$$

as desired. □

We also use the following fact about the unboundedness of the Hardy-Littlewood maximal operator M on Morrey spaces of exponent 1. The operator M maps a locally integrable function f to Mf which is given by

$$Mf(x) := \sup_{r>0} \frac{1}{|B(x,r)|} \int_{B(x,r)} |f(y)| \, dy, \quad x \in \mathbb{R}^d.$$

LEMMA 3.2. *The Hardy-Littlewood maximal operator M is not bounded on the Morrey space $\mathcal{M}^1_q$ for $1 < q < \infty$.*

REMARK 3.3. Lemma 3.2 is a consequence of a necessary condition of the boundedness of M on generalized Orlicz-Morrey spaces given in [8, Corollary 5.3]. The Morrey space $\mathcal{M}^p_q$ in this paper is recognized as the Orlicz-Morrey space $L^{(\Phi,\phi)}$ with $\Phi(t) = t^p$ and $\phi(t) = t^{-\frac{1}{q}}$. Based on [8, Corollary 5.3], the maximal operator M is bounded on $L^{(\Phi,\phi)}$ if and only if $\Phi \in \nabla_2$ (that is, $\Phi(r) \le \frac{1}{2k}\Phi(kr)$ for some $k \ge 1$). Clearly $\Phi(t) = t \notin \nabla_2$.

Now, we are ready to prove Theorem 1.4.

PROOF OF THEOREM 1.4. Let $1 \le p \le q$. If $p = q$, then $f(x) := |x|^{-\frac{d}{q}} \in w\mathcal{M}^p_q \setminus \mathcal{M}^p_q$. So assume that $p < q$ and write $r = \frac{q}{p}$. In view of Lemma 3.1, it suffices for us to prove that $\mathcal{M}^1_r \subset w\mathcal{M}^1_r$ properly. Suppose to the contrary that $\mathcal{M}^1_r = w\mathcal{M}^1_r$. Since the Hardy-Littlewood maximal operator M is bounded from $\mathcal{M}^1_r$ to $w\mathcal{M}^1_r$, we obtain

$$\|Mg\|_{w\mathcal{M}^1_r} \le C\,\|g\|_{\mathcal{M}^1_r},$$

for every $g \in \mathcal{M}^1_r$. Meanwhile, by the Closed Graph Theorem, there must exist a constant $C' > 0$ such that

$$\|Mg\|_{\mathcal{M}^1_r} \le C'\|Mg\|_{w\mathcal{M}^1_r}$$

for every $g \in \mathcal{M}^1_r$. Combining the two inequalities, we obtain

$$\|Mg\|_{\mathcal{M}^1_r} \le C\|g\|_{\mathcal{M}^1_r}$$

for every $g \in \mathcal{M}^1_r$. This tells us that M is bounded on $\mathcal{M}^1_r$, which contradicts Lemma 3.2. Therefore, $w\mathcal{M}^1_r \setminus \mathcal{M}^1_r \neq \emptyset$, as desired. □

To conclude this section, we write a proposition which gives us a subset of weak Morrey spaces with norm equivalence between the Morrey norm $\|\cdot\|_{\mathcal{M}^p_q}$ and the weak Morrey quasi-norm $\|\cdot\|_{w\mathcal{M}^p_q}$.

PROPOSITION 3.4. *Let $1 \le p < q < \infty$. Suppose that f is a positive radial decreasing function in $w\mathcal{M}^p_q(\mathbb{R}^d)$. Then $f \in \mathcal{M}^p_q(\mathbb{R}^d)$ with*

$$\|f\|_{w\mathcal{M}^p_q} \le \|f\|_{\mathcal{M}^p_q} \le \left(\frac{q\omega_{d-1}}{d(q-p)|B(0,1)|}\right)^{\frac{1}{p}} \|f\|_{w\mathcal{M}^p_q},$$

that is, $\|f\|_{w\mathcal{M}^p_q} \sim \|f\|_{\mathcal{M}^p_q}$.

PROOF. Recall that, since $\gamma\chi_{\{|f|>\gamma\}} \le |f|$ for every $\gamma > 0$, we have $\|f\|_{w\mathcal{M}^p_q} \le \|f\|_{\mathcal{M}^p_q}$. Next, let $x \in \mathbb{R}^d$. Since $\{y \in B(0,|x|) : f(y) > f(x)\} = B(0,|x|)$, we have

$$\begin{aligned} f(x) &= \frac{f(x)|\{y \in B(0,|x|) : f(y) > f(x)\}|^{\frac{1}{p}}}{|B(0,|x|)|^{\frac{1}{p}}} \\ &\le \frac{|B(0,|x|)|^{\frac{1}{p}-\frac{1}{q}}\|f\|_{w\mathcal{M}^p_q}}{|B(0,|x|)|^{\frac{1}{p}}} = |B(0,1)|^{-\frac{1}{q}}\|f\|_{w\mathcal{M}^p_q}|x|^{-\frac{d}{q}}. \end{aligned}$$

By combining the last estimate and

$$\||x|^{-\frac{d}{q}}\|_{\mathcal{M}^p_q} = |B(0,1)|^{\frac{1}{q}} \left(\frac{q\omega_{d-1}}{d(q-p)|B(0,1)|}\right)^{\frac{1}{p}},$$

where ω_{d-1} is the surface area of the unit sphere $\mathbb{S}^{d-1}$, we get

$$\|f\|_{\mathcal{M}^p_q} \le (|B(0,1)|^{-\frac{1}{q}}\||x|^{-\frac{d}{q}}\|_{\mathcal{M}^p_q})\|f\|_{wM^p_q} = \left(\frac{q\omega_{d-1}}{d(q-p)|B(0,1)|}\right)^{\frac{1}{p}} \|f\|_{wM^p_q}.$$

Hence $\|f\|_{w\mathcal{M}^p_q} \sim \|f\|_{\mathcal{M}^p_q}$. ∎

4. The proof of Theorem 1.6

PROOF OF THEOREM 1.6 (i). It follows from the inclusion $\mathcal{M}^{p_2}_{q_2} \subseteq \mathcal{M}^{p_1}_{q_1}$ that

$$\|\chi_{B(0,r)}\|_{\mathcal{M}^{p_1}_{q_1}} \le C\|\chi_{B(0,r)}\|_{\mathcal{M}^{p_2}_{q_2}},$$

for every $r > 0$. Therefore

$$r^{\frac{d}{q_1}-\frac{d}{q_2}} \le C$$

for every $r > 0$, which implies that $q_1 = q_2$. Now choose $\epsilon \in \left(0, \min\{\frac{dp_1}{q_1}, \frac{dp_2}{q_2}\}\right)$. For $j \in \mathbb{N}$, define $h_j(x) := \chi_{\{j \le |x| \le j+j^{-\epsilon}\}}(x)$, and for $K \in \mathbb{N}$ write

$f(x) := \chi_{\{0\le|x|<1\}}(x) + \sum_{j=1}^{K} h_j(x)$. Then

$$\|f\|_{\mathcal{M}^{p_1}_{q_1}} \ge |B(0,K+K^{-\epsilon})|^{\frac{1}{q_1}-\frac{1}{p_1}} \left(\int_{B(0,K+K^{-\epsilon})} |f(x)|^{p_1}\, dx\right)^{\frac{1}{p_1}}$$

$$\ge C(K+K^{-\epsilon})^{\frac{d}{q_1}-\frac{d}{p_1}}(K+K^{-\epsilon})^{\frac{d}{p_1}-\frac{\epsilon}{p_1}} = C(K+K^{-\epsilon})^{\frac{d}{q_1}-\frac{\epsilon}{p_1}}. \tag{4.1}$$

Meanwhile, for each $L \in \mathbb{N},\ L \le K$, we observe that

$$|B(0,L+L^{-\epsilon})|^{\frac{1}{q_2}-\frac{1}{p_2}} \left(\int_{B(0,L+L^{-\epsilon})} |f(x)|^{p_2} dx\right)^{\frac{1}{p_2}} \le C(L+L^{-\epsilon})^{\frac{d}{q_2}-\frac{\epsilon}{p_2}}.$$

Hence,

$$\|f\|_{\mathcal{M}^{p_2}_{q_2}} \le C(K+K^{-\epsilon})^{\frac{d}{q_2}-\frac{\epsilon}{p_2}}. \tag{4.2}$$

By combining (4.1), (4.2), $q_1 = q_2$, and $\|f\|_{\mathcal{M}^{p_1}_{q_1}} \le C\|f\|_{\mathcal{M}^{p_2}_{q_2}}$, we get

$$(K+K^{-\epsilon})^{\frac{\epsilon}{p_2}-\frac{\epsilon}{p_1}} \le C.$$

As this holds for every $K \in \mathbb{N}$, we conclude that $p_1 \le p_2$. ◻

Remark 4.1. Note that the difference between the proof of Theorem 1.6 (i) and [4, Remark 3.4] is that we do not assume $p_1 \le p_2$.

Proof of Theorem 1.6 (ii). By arguing as in the proof of Theorem 1.6 (i) and using the identities

$$\|\chi_{B(0,r)}\|_{w\mathcal{M}^{p_1}_{q_1}} = |B(0,r)|^{\frac{1}{q_1}}$$

and

$$\|\chi_{B(0,r)}\|_{w\mathcal{M}^{p_2}_{q_2}} = |B(0,r)|^{\frac{1}{q_2}},$$

we have $q_1 = q_2$. Assume to the contrary that $p_1 > p_2$. Define f by (2.2). By a similar argument as in the proof of Theorem 1.3 (ii)-(iii), we have $f \in w\mathcal{M}^{p_2}_{q_2}$ but $f \notin w\mathcal{M}^{p_1}_{q_1}$, which contradicts $w\mathcal{M}^{p_2}_{q_2} \subseteq w\mathcal{M}^{p_1}_{q_1}$. Hence $p_1 \le p_2$. ◻

Remark 4.2. Observe that unlike [4, Theorem 4.4 and Remark 4.5], the condition $p_1 \le p_2$ is not assumed in Theorem 1.6 (ii).

Proof of Theorem 1.6 (iii). Since $\mathcal{M}^{p_2}_{q_2} \subseteq w\mathcal{M}^{p_2}_{q_2}$, we have $\mathcal{M}^{p_2}_{q_2} \subseteq \mathcal{M}^{p_1}_{q_1}$. Therefore, by virtue of Theorem 1.6 (ii), we have $q_1 = q_2$ and $p_1 \le p_2$. Now, assume to the contrary that $p_1 = p_2$. According to Theorem 1.4, there exists $f_0 \in w\mathcal{M}^{p_2}_{q_2}$ such that $f_0 \notin \mathcal{M}^{p_1}_{q_1}$. This contradicts $w\mathcal{M}^{p_2}_{q_2} \subseteq \mathcal{M}^{p_1}_{q_1}$. Thus $p_1 < p_2$, as desired. ◻

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