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ON APPROXIMATE LEFT ϕ-BIPROJECTIVE BANACH ALGEBRAS

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Abstract. Let A be a Banach algebra. We introduce the notions of approximate left ϕ-biprojective and approximate left character biprojective Banach algebras, where ϕ is a non-zero multiplicative linear functional on A. We show that for a SIN group G, the Segal algebra $S(G)$ is approximate left ϕ_1-biprojective if and only if G is amenable, where ϕ_1 is the augmentation character on $S(G)$. Also we show that the measure algebra $M(G)$ is approximate left character biprojective if and only if G is discrete and amenable. For a Clifford semigroup S, we show that $\ell^1(S)$ is approximate left character biprojective if and only if $\ell^1(S)$ is pseudo-amenable. We study the hereditary property of these notions. Finally we give some examples to show the differences of these notions and the classical ones.

1. Introduction

A Banach algebra A is called amenable if for every Banach A-bimodule X, every continuous derivation D from A into X^* is inner, that is, there exists $x_0 \in X^*$ such that

$$D(a) = a \cdot x_0 - x_0 \cdot a \quad (a \in A).$$

An equivalent notion to amenability is the existence of a bounded net (m_α) in $A \otimes_p A$, where $\otimes_p$ denotes the projective tensor product, such that

$$a \cdot m_\alpha - m_\alpha \cdot a \to 0, \quad \pi_A(m_\alpha)a \to a \qquad (a \in A),$$

here $\pi_A : A \otimes_p A \to A$ is given by $\pi_A(a \otimes b) = ab$ for every $a, b \in A$ ([13]). In the homological theory, two important notions, biflatness and biprojectivity for Banach algebras, have a key role. In fact a Banach algebra A is called

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biflat (biprojective), if there exists a bounded A-bimodule morphism $\rho : A \to (A\otimes_p A)^{**}$ ($\rho : A \to A\otimes_p A$) such that $\pi_A^{**}\circ\rho$ is the canonical embedding of A into A^{**} (ρ is a right inverse for π_A), respectively. Note that a Banach algebra A is amenable if and only if A is biflat and it has a bounded approximate identity. It is well known that for a locally compact group G, $L^1(G)$ is biflat (biprojective) if and only if G is amenable (compact), respectively, see [10].

Let A be a Banach algebra. Throughout, the character space of A is denoted by $\Delta(A)$, that is, all non-zero multiplicative linear functionals on A. Recently a notion of amenability related to a character has been introduced in [14]. Indeed a Banach algebra A is called left ϕ-amenable, if there exists a bounded net (a_α) in A such that $aa_\alpha - \phi(a)a_\alpha \to 0$ and $\phi(a_\alpha) \to 1$ for all $a \in A$, where $\phi \in \Delta(A)$. For a locally compact group G, the Fourier algebra $A(G)$ is always left ϕ-amenable. Also the group algebra $L^1(G)$ is left ϕ-amenable if and only if G is amenable, for further information see [25] and [2].

In [21] the authors introduced the character version of homological properties of Banach algebras like ϕ-biflat and ϕ-biprojective. A Banach algebra A is called ϕ-biflat (ϕ-biprojective) if there exists a bounded A-bimodule morphism

$$\rho : A \to (A\otimes_p A)^{**}, \quad (\rho : A \to A\otimes_p A)$$

such that

$$\tilde{\phi}\circ\pi_A^{**}\circ\rho(a) = \phi(a), \quad (\phi\circ\pi_A\circ\rho(a) = \phi(a)) \quad (a\in A),$$

respectively, where $\tilde{\phi}(F) = F(\phi)$ for all $F \in A^{**}$. For a locally compact group G, they showed that the Segal algebra $S(G)$ is ϕ-biflat (ϕ-biprojective) if and only if G is amenable (compact). Also $A(G)$ is ϕ-biprojective if and only if G is discrete, see [17] and [21]. In [5] another definition of ϕ-biflatness has been given. A Banach algebra A with a character $\phi \in \Delta(A)$ has condition W (according to our approach we say right ϕ-biflat), if there exists a bounded linear map $\rho : A \to (A\otimes_p A)^{**}$ that satisfies

(i) $\rho(ab) = \phi(b)\rho(a) = \rho(a)\cdot b$,
(ii) $\tilde{\phi}\circ\pi_A^{**}\circ\rho(a) = \phi(a)$,

$a, b \in A$. Also the characterization of the right ϕ-biflatness for symmetric Segal algebras has been given in [5]. The definition of the left ϕ-biflatness is similar.

Recently approximate versions of the amenability and the homological properties of Banach algebras have been under more observations. In [26] Zhang introduced the notion of approximate biprojective Banach algebras. A Banach algebra A is *approximate biprojective* if there exists a net of A-bimodule morphisms $\rho_\alpha : A \to A\otimes_p A$ such that

$$\pi_A\circ\rho_\alpha(a) \to a \quad (a\in A).$$

The authors investigated approximate biprojectivity of some semigroup algebras and some related Triangular Banach algebras, see [22] and [23]. Approximate amenable Banach algebras have been introduced by Ghahramani and Loy. Indeed a Banach algebra A is *approximate amenable* if for every Banach A-bimodule X and every continuous derivation $D : A \to X^*$, there exists a net (x_α) in X^* such that

$$D(a) = \lim_\alpha a \cdot x_\alpha - x_\alpha \cdot a \quad (a \in A).$$

Other extended notions are pseudo-amenability and pseudo-contractibility. A Banach algebra A is *pseudo-amenable (pseudo-contractible)* if there exists a not necessarily bounded net (m_α) in $A \otimes_p A$ such that

$$a \cdot m_\alpha - m_\alpha \cdot a \to 0, \quad (a \cdot m_\alpha = m_\alpha \cdot a), \qquad \pi_A(m_\alpha)a \to a \quad (a \in A),$$

respectively, for more information the reader is referred to [9], [7] and [8]. The character version of approximate notions of amenability have been introduced and studied in [1]. A Banach algebra A is called *approximate left ϕ-amenable* if there exists a (not necessarily bounded) net (a_α) in A such that $aa_\alpha - \phi(a)a_\alpha \to 0$ and $\phi(a_\alpha) \to 1$ for all $a \in A$. Also A is *approximate character amenable*, if A is approximate left ϕ-amenable for all $\phi \in \Delta(A) \cup \{0\}$. Note that $L^1(G)^{**}$ is character amenable if and only if G is discrete and amenable. Also $M(G)$ is character amenable if and only if G is discrete and amenable ([1]).

In this paper we give a new approximate homological notion with respect to a character which is weaker than ϕ-biflatness and also right ϕ-biflatness.

Definition 1.1. *Let A be a Banach algebra and $\phi \in \Delta(A)$. A is called approximate left ϕ-biprojective if there exists a net of bounded linear maps from A into $A \otimes_p A$, say $(\rho_\alpha)_{\alpha \in I}$, such that*

(i) $a \cdot \rho_\alpha(b) - \rho_\alpha(ab) \xrightarrow{\|\cdot\|} 0$,
(ii) $\rho_\alpha(ba) - \phi(a)\rho_\alpha(b) \xrightarrow{\|\cdot\|} 0$,
(iii) $\phi \circ \pi_A \circ \rho_\alpha(a) - \phi(a) \to 0$,

for every $a, b \in A$. We say that A is approximate left character biprojective if A is approximate left ϕ-biprojective for all $\phi \in \Delta(A)$.

In this paper, first we show that the approximate left ϕ-amenability is a stronger notion than the approximate left ϕ-biprojectivity. While we study the hereditary properties of this notion, we show that for a SIN group G, the Segal algebra $S(G)$ is approximate left ϕ_1-biprojective if and only if G is amenable, where ϕ_1 is the augmentation character on $S(G)$ and the measure algebra $M(G)$ is approximate left character biprojective if and only if G is discrete and amenable. Finally we give some examples of Banach algebras among Triangular Banach algebras which are never approximate left ϕ-biprojective and some examples which reveal the differences of our new notion and the classical ones.

2. Approximate left ϕ-biprojectivity

In this section we study the general properties of approximate left ϕ-biprojective Banach algebras.

Proposition 2.1. *Let A be a Banach algebra and $\phi \in \Delta(A)$. Suppose that A is approximate left ϕ-biprojective and A has an element a_0 such that $aa_0 = a_0a$ for all $a \in A$ and $\phi(a_0) = 1$. Then A is approximate left ϕ-amenable.*

Proof. Let $(\rho_\alpha)_{\alpha \in I}$ be as in Definition 1.1. Let a_0 be an element in A such that $aa_0 = a_0a$ and $\phi(a_0) = 1$ for every $a \in A$. Set $n_\alpha = \rho_\alpha(a_0)$. It is clear that (n_α) is a net in $A \otimes_p A$ such that

$$\begin{aligned} a \cdot n_\alpha - \phi(a)n_\alpha &= a \cdot \rho_\alpha(a_0) - \phi(a)\rho_\alpha(a_0) \\ &= a \cdot \rho_\alpha(a_0) - \rho_\alpha(aa_0) + \rho_\alpha(aa_0) - \rho_\alpha(a_0a) \\ &\quad + \rho_\alpha(a_0a) - \phi(a)\rho_\alpha(a_0) \to 0 \end{aligned}$$

for every $a \in A$. Also we have

$$\phi \circ \pi_A(n_\alpha) - 1 = \phi \circ \pi_A \circ \rho_\alpha(a_0) - \phi(a_0) \to 0.$$

Define $T : A \otimes_p A \to A$ by $T(a \otimes b) = \phi(b)a$ for each $a, b \in A$. It is clear that T is a bounded linear map which satisfies

$$T(a \cdot x) = aT(x), \quad T(x \cdot a) = \phi(a)T(x), \quad \phi \circ T = \phi \circ \pi_A, \quad (a \in A, x \in A \otimes_p A).$$

Set $m_\alpha = T(n_\alpha)$. One can show that

$$aT(n_\alpha) - \phi(a)T(n_\alpha) = T(a \cdot n_\alpha - \phi(a)n_\alpha) \to 0, \quad (a \in A)$$

and

$$\phi(m_\alpha) = \phi \circ T(n_\alpha) = \phi \circ \pi_A(n_\alpha) \to 1.$$

Thus A is approximate left ϕ-amenable. □

Proposition 2.2. *Let A be a Banach algebra and $\phi \in \Delta(A)$. If A is approximate biprojective, then A is approximate left ϕ-biprojective.*

Proof. Since A is approximate biprojective, there exists a net of A-bimodule morphisms $\rho_\alpha : A \to A \otimes_p A$ such that

$$\pi_A \circ \rho_\alpha(a) \to a \quad (a \in A).$$

Pick $a_0 \in A$ such that $\phi(a_0) = 1$. Let $T : A \otimes_p A \to A \otimes_p A$ be defined by $T(a \otimes b) = \phi(b)a \otimes a_0$ for each $a, b \in A$. Clearly T is a bounded linear map. It is easy to see that

$$x \cdot T(a \otimes b) = \phi(b)x \cdot a \otimes a_0 = T(x \cdot (a \otimes b)), \tag{2.1}$$

$$T(a\otimes b)\phi(x)=\phi(bx)a\otimes a_0=T((a\otimes b)\cdot x) \tag{2.2}$$

and

$$\phi\circ\pi_A\circ T(a\otimes b)=\phi(\phi(a)a_0b)=\phi(ab)=\phi\circ\pi_A(a\otimes b), \tag{2.3}$$

for each $a,b,x\in A$. We claim that $(T\circ\rho_\alpha)_\alpha$ satisfies the conditions of Definition 1.1. To see this, using (2.1) we have

$$a\cdot T\circ\rho_\alpha(b)=T(a\cdot\rho_\alpha(b))=T(\rho_\alpha(ab)),$$

also by (2.2) we have

$$\begin{aligned}T(\rho_\alpha(ba))-\phi(a)T(\rho_\alpha(b))&=T(\rho_\alpha(ba))-T(\rho_\alpha(b)\cdot a)\\&=T(\rho_\alpha(ba))-T(\rho_\alpha(ba))=0,\end{aligned}$$

and also (2.3) implies that

$$\phi\circ\pi_A\circ T(\rho_\alpha(a))=\phi\circ\pi_A(\rho_\alpha(a))\to a,$$

for each $a,b\in A$. Thus A is approximate left ϕ-biprojective. $\square$

Let A and B be Banach algebras and let X be a Banach A,B-module, that is, X is a Banach space, a left A-module and a right B-module with the compatible module action that satisfies $(a\cdot x)\cdot b=a\cdot(x\cdot b)$ and $\|a\cdot x\cdot b\|\leq\|a\|\|x\|\|b\|$ for every $a\in A, x\in X, b\in B$. With the usual matrix operation and

$$\left\|\begin{pmatrix}a & x\\ 0 & b\end{pmatrix}\right\|=\|a\|+\|x\|+\|b\|,\qquad T=\begin{pmatrix}A & X\\ 0 & B\end{pmatrix}$$

becomes a Banach algebra which is called Triangular Banach algebra. Take $\phi\in\Delta(B)$. We define a character $\psi_\phi\in\Delta(T)$ via $\psi_\phi\begin{pmatrix}a & x\\ 0 & b\end{pmatrix}=\phi(b)$ for every $a\in A$, $b\in B$ and $x\in X$. In the following example we present a non-approximate left ϕ-biprojective Banach algebra.

EXAMPLE 2.3. Consider the triangular Banach algebra $T=\begin{pmatrix}\mathbb{C} & \mathbb{C}\\ 0 & \mathbb{C}\end{pmatrix}$. Define $\phi\in\Delta(T)$ by $\phi(\begin{pmatrix}a & b\\ 0 & c\end{pmatrix})=c$ for all $a,b,c\in\mathbb{C}$. We claim that T is not approximate left ϕ-biprojective. To see this we go toward a contradiction and assume that T is approximate left ϕ-biprojective. Since T is unital, by Proposition 2.1 T is approximate left ϕ-amenable. Set $I=\begin{pmatrix}0 & \mathbb{C}\\ 0 & \mathbb{C}\end{pmatrix}$. It is easy to see that $\phi|_I\neq 0$ and I is approximate left ϕ-amenable. Thus there exists a net (i_α) in I such that

$$ii_\alpha-\phi(i)i_\alpha\to 0,\quad \phi(i_\alpha)\to 1,\quad (i\in I).$$

Hence there exist nets (a_α) and (b_α) in $\mathbb{C}$ such that $i_\alpha = \begin{pmatrix} 0 & a_\alpha \\ 0 & b_\alpha \end{pmatrix}$. So for each $i = \begin{pmatrix} 0 & a \\ 0 & b \end{pmatrix}$ in I, we have

$$\begin{pmatrix} 0 & a \\ 0 & b \end{pmatrix}\begin{pmatrix} 0 & a_\alpha \\ 0 & b_\alpha \end{pmatrix} - b\begin{pmatrix} 0 & a_\alpha \\ 0 & b_\alpha \end{pmatrix} \to 0,$$

which implies that $ab_\alpha - ba_\alpha \to 0$, for each $a, b \in \mathbb{C}$. Since $b_\alpha \to 1$, taking $a = 1$ and $b = 0$, gives a contradiction.

We remind that by [1, Proposition 2.7], A is approximate left ϕ-amenable if and only if there exists a net (m_α) in $(A \otimes_p A)^{**}$ such that $a \cdot m_\alpha - \phi(a)m_\alpha \to 0$ and $\tilde{\phi} \circ \pi_A^{**}(m_\alpha) \to 1$ for all $a \in A$.

For each $\phi \in \Delta(A)$ there exists a unique extension $\tilde{\phi}$ to A^{**} which is defined by $\tilde{\phi}(F) = F(\phi)$. It is easy to see that $\tilde{\phi} \in \Delta(A^{**})$.

Proposition 2.4. *Let A be a Banach algebra and $\phi \in \Delta(A)$. If A is approximate left ϕ-amenable, then A is approximate left ϕ-biprojective.*

Proof. Let A be approximate left ϕ-amenable. Then there exists a net m_α in $(A \otimes_p A)^{**}$ such that $a \cdot m_\alpha - \phi(a)m_\alpha \to 0$ and $\tilde{\phi} \circ \pi^{**}(m_\alpha) = 1$, for each $a \in A$, see [1, Proposition 2.7]. Take $\epsilon > 0$ and arbitrary finite subsets $F \subseteq A$ and $\Lambda \subseteq (A \otimes_p A)^*$. Then we have

$$\|a \cdot m_\alpha - \phi(a)m_\alpha\| < \epsilon, \quad |\tilde{\phi} \circ \pi_A^{**}(m_\alpha) - 1| < \epsilon, \quad (a \in F).$$

It is well-known that for each α, there exists a net $(n_\beta^\alpha)_\beta$ in $A \otimes_p A$ such that $n_\beta^\alpha \xrightarrow{w^*} m_\alpha$. Since π_A^{**} is a w^*-continuous map, we have

$$\pi_A(n_\beta^\alpha) = \pi_A^{**}(n_\beta^\alpha) \xrightarrow{w^*} \pi_A^{**}(m_\alpha).$$

Thus we have

$$|a \cdot n_\beta^\alpha(f) - am_\alpha(f)| < \frac{\epsilon}{K_0}, \quad |\phi(a)n_\beta^\alpha(f) - \phi(a)m_\alpha(f)| < \frac{\epsilon}{K_0}$$

and

$$|\phi \circ \pi_A(n_\beta^\alpha) - \tilde{\phi} \circ \pi^{**}(m_\alpha)| < \epsilon,$$

for each $a \in F$ and $f \in \Lambda$, where $K_0 = \sup\{\|f\| : f \in \Lambda\}$. Since $a \cdot m_\alpha - \phi(a)m_\alpha \to 0$ and $\tilde{\phi} \circ \pi^{**}(m_\alpha) = 1$, we can find $\beta = \beta(F, \Lambda, \epsilon)$ such that

$$|a \cdot n_\beta^\alpha(f) - \phi(a)n_\beta^\alpha(f)| < c\frac{\epsilon}{K_0}, \quad |\phi \circ \pi_A(n_\beta^\alpha) - 1| < \epsilon, \quad (a \in F, f \in \Lambda)$$

for some $c \in \mathbb{R}^+$. Using Mazur's lemma, we have a net $(n_{(F,\Lambda,\epsilon)})$ in $A \otimes_p A$ such that

$$\|a \cdot n_{(F,\Lambda,\epsilon)} - \phi(a)n_{(F,\Lambda,\epsilon)}\| \to 0, \quad |\phi \circ \pi_A(n_{(F,\Lambda,\epsilon)}) - 1| \to 0, \quad (a \in F).$$

Define $\rho_{(F,\Lambda,\epsilon)} : A \to A \otimes_p A$ by $\rho_{(F,\Lambda,\epsilon)}(a) = a \cdot n_{(F,\Lambda,\epsilon)}$ for each $a \in A$. It is clear that $\rho_{(F,\Lambda,\epsilon)}(ab) = a \cdot \rho_{(F,\Lambda,\epsilon)}(b)$ for each $a, b \in A$. So we have

$$\begin{aligned} \|\rho_{(F,\Lambda,\epsilon)}(ab) - \phi(b)\rho_{(F,\Lambda,\epsilon)}(a)\| &= \|ab \cdot n_{(F,\Lambda,\epsilon)} - \phi(b)a \cdot n_{(F,\Lambda,\epsilon)}\| \\ &\leq \|a\|\|b \cdot n_{(F,\Lambda,\epsilon)} - \phi(b)n_{(F,\Lambda,\epsilon)}\| \to 0, \end{aligned} \tag{2.4}$$

for each $a, b \in A$. Also

$$\begin{aligned} |\phi \circ \pi_A \circ \rho_{(F,\Lambda,\epsilon)}(a) - \phi(a)| &= |\phi \circ \pi_A(a \cdot n_{(F,\Lambda,\epsilon)}) - \phi(a)| \\ &= |\phi(a)||\phi \circ \pi_A(n_{(F,\Lambda,\epsilon)}) - 1| \to 0, \end{aligned} \tag{2.5}$$

for each $a \in A - \ker\phi$. It is easy to see that $\phi \circ \pi_A \circ \rho_{(F,\Lambda,\epsilon)}(a) = \phi(a)$ for each $a \in \ker\phi$. ∎

REMARK 2.5. Let A be a Banach algebra and $\phi \in \Delta(A)$. Using the arguments of the previous proposition one can see that if A is either pseudo-amenable or approximate amenable, then A is approximate left ϕ-biprojective.

THEOREM 2.6. *Let A be a Banach algebra and $\phi \in \Delta(A)$. If A is ϕ-biflat, then A is approximate left ϕ-biprojective.*

PROOF. Since A is ϕ-biflat, there exists a bounded A-bimodule morphism $\rho : A \to (A \otimes_p A)^{**}$ such that $\tilde{\phi} \circ \pi_A^{**} \circ \rho(a) = \phi(a)$ for each $a \in A$. There exists a net (ρ_α) in $B(A, A \otimes_p A)$, the set of all bounded linear maps from A into $A \otimes_p A$, such that ρ_α converges to ρ in the weak-star operator topology. Since π_A^{**} is a w^*-continuous map, for each $a \in A$ we have

$$\pi_A \circ \rho_\alpha(a) = \pi_A^{**} \circ \rho_\alpha(a) \xrightarrow{w^*} \pi_A^{**} \circ \rho(a),$$

so

$$\phi \circ \pi_A \circ \rho_\alpha(a) \to \tilde{\phi} \circ \pi_A^{**} \circ \rho(a).$$

Let $\epsilon > 0$ and take arbitrary finite subsets $F = \{a_1, a_2, \dots, a_r\}$ and $G = \{x_1, x_2, \dots, x_r\}$ of A. Set

$$\begin{aligned} M = \{(&a_1 \cdot T(x_1) - T(a_1x_1), a_2 \cdot T(x_2) - T(a_2x_2), \dots, a_r \cdot T(x_r) - T(a_rx_r), \\ &\phi \circ \pi_A \circ T(x_1) - \phi(x_1), \phi \circ \pi_A \circ T(x_2) - \phi(x_2), \dots, \\ &\phi \circ \pi_A \circ T(x_r) - \phi(x_r))| T \in B(A, A \otimes_p A), a_i \in F, x_i \in G\} \end{aligned}$$

as a subset of $\prod_{i=1}^r (A \otimes_p A) \oplus_1 \prod_{i=1}^r \mathbb{C}$. It is clear that M is a convex set and $(0, 0, \dots, 0)$ belongs to $\overline{M}^w$ and by Mazur's Lemma $(0, 0, \dots, 0) \in \overline{M}^w = \overline{M}^{\|\cdot\|}$. Then we can find an element $\theta_{(F,G,\epsilon)}$ in $B(A, A \otimes_p A)$ such that

$$\|a_i \cdot \theta_{(F,G,\epsilon)}(b_i) - \theta_{(F,G,\epsilon)}(a_ib_i)\| < \epsilon, \quad \|\theta_{(F,G,\epsilon)}(a_ib_i) - \theta_{(F,G,\epsilon)}(a_i) \cdot b_i\| < \epsilon$$

and

$$|\phi \circ \pi_A \circ \theta_{(F,G,\epsilon)}(a_i) - \phi(a_i)| < \epsilon,$$

for each $i \in \{1, 2, \dots, r\}$. Hence the net $(\theta_{(F,G,\epsilon)})_{(F,G,\epsilon)}$ satisfies

$$a \cdot \theta_{(F,G,\epsilon)}(b) - \theta_{(F,G,\epsilon)}(ab) \to 0, \quad \theta_{(F,G,\epsilon)}(ab) - \theta_{(F,G,\epsilon)}(a) \cdot b \to 0$$

and $\phi\circ\pi_A\circ\theta_{(F,G,\epsilon)}(a)-\phi(a)\to 0$ for each $a,b\in A$. Consider the map T as in the proof of Proposition 2.2. It is easy to see that $(T\circ\theta_{(F,G,\epsilon)})_{(F,G,\epsilon)}$ satisfies the conditions of Definition 1.1. So A is approximate left ϕ-biprojective. ∎

We have to remind that every biflat Banach algebra A with $\phi\in\Delta(A)$ is ϕ-biflat. Then using the previous theorem, we have the following corollary.

Corollary 2.7. *Suppose that A is a biflat Banach algebra and $\phi\in\Delta(A)$. Then A is approximate left $\phi-$biprojective.*

Proposition 2.8. *Let A be a Banach algebra and $\phi\in\Delta(A)$. Suppose that I is a closed ideal of A such that $\phi|_I\neq 0$. If A is approximate left ϕ-biprojective, then I is approximate left ϕ-biprojective.*

Proof. Let $(\rho_\alpha)_\alpha$ be a net of maps which satisfies Definition 1.1. Take i_0 in I such that $\phi(i_0)=1$. Define $T:A\otimes_p A\to I\otimes_p I$ by $T(a\otimes b)=ai_0\otimes i_0b$ for every $a,b\in A$. It is easy to see that T is a bounded linear map. Set $\eta_\alpha=T\circ\rho_\alpha|_I:I\to I\otimes_p I$. Then we have

$$i\cdot\eta_\alpha(j)-\eta_\alpha(ij)=T(i\cdot\rho_\alpha(j)-\rho_\alpha(ij))\to 0$$

and

$$\eta_\alpha(ij)-\phi(j)\eta_\alpha(i)=T(\rho_\alpha(ij)-\phi(j)\rho_\alpha(i))\to 0,$$

also

$$\phi\circ\pi_I\circ\eta_\alpha(i)-\phi(i)=\phi\circ\pi_I\circ T\circ\rho_\alpha(i)-\phi(i)=\phi\circ\pi_A\circ\rho_\alpha(i)-\phi(i)\to 0$$

for each $i,j\in I$. ∎

Let A and B be Banach algebras, $\phi\in\Delta(A)$ and $\psi\in\Delta(B)$. We denote by $\phi\otimes\psi$ a map defined by $\phi\otimes\psi(a\otimes b)=\phi(a)\psi(b)$ for all $a\in A$ and $b\in B$. It is easy to see that $\phi\otimes\psi\in\Delta(A\otimes_p B)$. Also note that $A\otimes_p B$ with the following actions becomes a Banach A-bimodule:

$$a_1\cdot(a_2\otimes b)=a_1a_2\otimes b,\quad (a_2\otimes b)\cdot a_1=a_2a_1\otimes b,\quad (a_1,a_2\in A, b\in B).$$

Theorem 2.9. *Let A and B be Banach algebras, $\phi\in\Delta(A)$ and $\psi\in\Delta(B)$. Suppose that A is unital and B has an idempotent x_0 with $x_0\notin\ker\psi$. If $A\otimes_p B$ is approximate left $\phi\otimes\psi$-biprojective, then A is approximate left ϕ-biprojective.*

Proof. Let $(\rho_\alpha):A\otimes_p B\to(A\otimes_p B)\otimes_p(A\otimes_p B)$ be a net of continuous linear maps such that

$$x\cdot\rho_\alpha(y)-\rho_\alpha(xy)\to 0,\quad \rho_\alpha(yx)-\phi(x)\rho_\alpha(y)\to 0 \tag{2.6}$$

and

$$\phi\otimes\psi\circ\pi_{A\otimes_p B}\circ\rho_\alpha(x)-\phi\otimes\psi(x)\to 0 \tag{2.7}$$

for each $x, y \in A \otimes_p B$. Since x_0 is an idempotent, for each a_1 and a_2 in A we have

$$a_1 a_2 \otimes x_0 = (a_1 \otimes x_0)(a_2 \otimes x_0). \tag{2.8}$$

Using also the unit element e of A, we have

$$\begin{aligned}
\rho_\alpha(a_1 a_2 \otimes x_0) - a_1 \cdot \rho_\alpha(a_2 \otimes x_0) &= \rho_\alpha((a_1 \otimes x_0)(a_2 \otimes x_0)) - a_1 \cdot \rho_\alpha(a_2 \otimes x_0) \\
&= \rho_\alpha((a_1 \otimes x_0)(a_2 \otimes x_0)) - (a_1 \otimes x_0) \cdot \rho_\alpha(a_2 \otimes x_0) \\
&\quad + (a_1 \otimes x_0) \cdot \rho_\alpha(a_2 \otimes x_0) - a_1 \cdot \rho_\alpha(a_2 \otimes x_0) \\
&= \rho_\alpha((a_1 \otimes x_0)(a_2 \otimes x_0)) - (a_1 \otimes x_0) \cdot \rho_\alpha(a_2 \otimes x_0) \\
&\quad + (a_1 \cdot (e \otimes x_0)) \cdot \rho_\alpha(a_2 \otimes x_0) - a_1 \cdot \rho_\alpha(a_2 \otimes x_0) \\
&= \rho_\alpha((a_1 \otimes x_0)(a_2 \otimes x_0)) - (a_1 \otimes x_0) \cdot \rho_\alpha(a_2 \otimes x_0) \\
&\quad + (a_1 \cdot (e \otimes x_0)) \cdot \rho_\alpha(a_2 \otimes x_0) - a_1 \cdot \rho_\alpha(e a_2 \otimes x_0 x_0) \\
&\quad + a_1 \cdot \rho_\alpha(e a_2 \otimes x_0 x_0) - a_1 \cdot \rho_\alpha(a_2 \otimes x_0) \to 0.
\end{aligned}$$

Also using (2.7) and (2.6) we have

$$\begin{aligned}
\rho_\alpha(a_1 a_2 \otimes x_0) - \phi(a_2)\rho_\alpha(a_1 \otimes x_0) &= \rho_\alpha((a_1 \otimes x_0)(a_2 \otimes x_0)) - \phi(a_2)\rho_\alpha(a_1 \otimes x_0) \\
&= \rho_\alpha((a_1 \otimes x_0)(a_2 \otimes x_0)) - \phi \otimes \psi(a_2 \otimes x_0)\rho_\alpha(a_1 \otimes x_0) \\
&\quad + \phi \otimes \psi(a_2 \otimes x_0)\rho_\alpha(a_1 \otimes x_0) - \phi(a_2)\rho_\alpha(a_1 \otimes x_0) \to 0,
\end{aligned}$$

for each $a_1, a_2 \in A$. Define

$$T : (A \otimes_p B) \otimes_p (A \otimes_p B) \to A \otimes_p A$$

by

$$T((a \otimes b) \otimes (c \otimes d)) = \psi(bd) a \otimes c,$$

for each $a, c \in A, b, d \in B$. One can see that T is a bounded linear operator and $\pi_A \circ T = (id \otimes \psi) \circ \pi_{A \otimes_p B}$, where $id \otimes \psi(a \otimes b) = \psi(b)a$ for all $a \in A, b \in B$. Set $\eta_\alpha(a) = T \circ \rho_\alpha(a \otimes x_0)$. It is easy to see that for each α, the map $\eta_\alpha : A \to A \otimes_p A$ is linear, continuous and satisfies

$$a \cdot \eta_\alpha(b) - \eta_\alpha(ab) \to 0, \quad \eta_\alpha(ba) - \phi(a)\eta_\alpha(b) \to 0, \quad (a, b \in A).$$

Also we have

$$\phi \circ \pi_A \circ \eta_\alpha(a) = \phi \circ \pi_A \circ T \circ \rho_\alpha(a \otimes x_0) = \phi \circ (id \otimes \psi) \circ \pi_{A \otimes_p B} \circ \rho_\alpha(a \otimes x_0) \to \phi(a),$$

for each $a \in A$. Hence A is approximate left ϕ-biprojective. □

3. Application to Banach algebras related to a locally compact group

Let G be a locally compact group. A linear subspace $S(G)$ of $L^1(G)$ is said to be a Segal algebra, if it satisfies the following conditions:

(i) $S(G)$ is dense in $L^1(G)$;
(ii) $S(G)$ with a norm $\|\cdot\|_S$ is a Banach space and $\|f\|_1 \leq \|f\|_S$ for every $f \in S(G)$;
(iii) For $f \in S(G)$ and $y \in G$, we have $L_y f \in S(G)$ the map $y \mapsto L_y(f)$ from G into $S(G)$ is continuous, where $L_y(f)(x) = f(y^{-1}x)$;
(iv) $\|L_y(f)\|_S = \|f\|_S$ for every $f \in S(G)$ and $y \in G$.

For more information see [18].

Let G be a locally compact group and let $\widehat{G}$ be its dual group, which consists of all non-zero continuous homomorphism ζ from G into the circle group $\mathbb{T}$. It is well-known that $\Delta(L^1(G)) = \{\phi_\zeta : \zeta \in \widehat{G}\}$, where $\phi_\zeta(f) = \int_G \overline{\zeta(x)} f(x)dx$ and dx is a left Haar measure on G, for more details, see [11, Theorem 23.7].

The map $\phi_1 : L^1(G) \to \mathbb{C}$ which is specified by

$$\phi_1(f) = \int_G f(x)dx$$

is called augmentation character. It is well known that the augmentation character induces a character on $S(G)$ is still denoted by ϕ_1, see [2].

A locally compact group G is called SIN group if it contains a fundamental family of compact invariant neighborhoods of the identity, see [4, p. 86].

Theorem 3.1. *Let G be a locally compact SIN-group. Then $S(G)$ is approximate ϕ_1-biprojective if and only if G is amenable.*

Proof. Since G is a SIN group, $S(G)$ has a central approximate identity [15]. We have an element $f \in S(G)$ such that $gf = fg$ and $\phi_1(f) = 1$ for each $g \in S(G)$. Applying Proposition 2.1, approximate left ϕ_1-biprojectivity of $S(G)$ implies that $S(G)$ is approximate left ϕ_1-amenable. So there exists a net (m_α) in $S(G)$ such that

$$\|gm_\alpha - \phi_1(g)m_\alpha\|_S \to 0, \quad \phi_1(m_\alpha) \to 1 \quad (g \in S(G)).$$

Since $\|\cdot\|_1 \leq \|\cdot\|_S$, we have

$$\|gm_\alpha - \phi_1(g)m_\alpha\|_1 \to 0, \quad \phi_1(m_\alpha) \to 1 \quad (g \in S(G)).$$

Define $f_\alpha = fm_\alpha$, where $\phi_1(f) = 1$. For each $y \in G$ and $g \in S(G)$, we have

$$\phi_1(\delta_y g) = \int_G \delta_y g(x)dx = \int_G g(y^{-1}x)dx = \int_G g(x)dx = \phi_1(g),$$

where δ_y denotes the point mass at $\{y\}$. By the condition (iii) in the definition of $S(G)$, we have

$$\begin{aligned}\|\delta_y f_\alpha - f_\alpha\|_1 &= \|(\delta_y f)m_\alpha - fm_\alpha\|_1 \\ &\leq \|(\delta_y f)m_\alpha - m_\alpha\|_1 + \|m_\alpha - fm_\alpha\|_1 \\ &\leq \|(\delta_y f)m_\alpha - \phi_1(\delta_y f)m_\alpha\|_1 + \|\phi_1(\delta_y f)m_\alpha - m_\alpha\|_1 \\ &\quad + \|m_\alpha - \phi_1(f)m_\alpha\|_1 + \|\phi_1(f)m_\alpha - fm_\alpha\|_1 \to 0.\end{aligned} \tag{3.1}$$

On the other hand

$$\phi_1(f_\alpha) = \phi_1(fm_\alpha) = \phi_1(f)\phi_1(m_\alpha) \to 1.$$

Since ϕ_1 is a bounded linear functional, that is, $|f_\alpha| \leq \|f_\alpha\|_1$, f_α stays away from 0. Without loss of generality we may assume that $\|f_\alpha\|_1 \geq \frac{1}{2}$. Define $g_\alpha = \frac{|f_\alpha|}{\|f_\alpha\|_1}$. It is clear that (g_α) is a bounded net in $L^1(G)$. Consider

$$\|\delta_y g_\alpha - g_\alpha\|_1 \leq 2\|\delta_y|f_\alpha| - |f_\alpha|\|_1 \leq 2\|\delta_y f_\alpha - f_\alpha\|_1 \to 0.$$

Now by [20, Exercise 1.1.6], G is amenable.

Conversely, suppose that G is an amenable group. Since G is a SIN group, by [24, Corollary 3.2] amenability of G implies that $S(G)$ is pseudo-amenable. Now using Remark (2.5) $S(G)$ is approximate ϕ_1-biprojective.

□

We give a non-approximate left ϕ-biprojective Banach algebra defined on locally compact groups.

Example 3.2. Let $T = \begin{pmatrix} A(G) & A(G) \\ 0 & A(G) \end{pmatrix}$, where $A(G)$ is the Fourier algebra with respect to a locally compact group G. Suppose that $\phi \in \Delta(A(G))$. Define $\psi_\phi(\begin{pmatrix} a & b \\ 0 & c \end{pmatrix}) = \phi(c)$ for every $a, b, c \in A(G)$. It is easy to see that $\psi_\phi \in \Delta(T)$. Note that $A(G)$ is a commutative Banach algebra, hence there exists $a_0 \in A(G)$ such that $aa_0 = a_0a$ for every $a \in A(G)$ and $\phi(a_0) = 1$. Set $t_0 = \begin{pmatrix} a_0 & 0 \\ 0 & a_0 \end{pmatrix}$, clearly for every $t \in T$ we have $tt_0 = t_0t$ and $\psi_\phi(t_0) = 1$. Suppose conversely that T is ψ_ϕ-approximate left biprojective. By Proposition 2.1, T is approximate left ψ_ϕ-amenable. Using a similar argument as in the Example 2.3, we have a net (a_α) in $A(G)$ such that $a - ba_\alpha \to 0$ for every $a, b \in A(G)$. By taking $a \in A(G)$ such that $\phi(a) = 1$ and $b \in \ker\phi$, we have $\phi(a) = \phi(a) - \phi(b)\phi(a_\alpha) = \phi(a - ba_\alpha) \to 0$ which is a contradiction. Then T is not ψ_ϕ-approximate left biprojective.

Lemma 3.3. *Let G be a locally compact group. Then $A(G)$ is approximate left ϕ-biprojective.*

PROOF. By [14, Example 2.6] $A(G)$ is left ϕ-amenable for each $\phi \in \Delta(A(G))$. So it is approximate left ϕ-amenable. Proposition 2.4 implies that $A(G)$ is approximate left ϕ-biprojective, for each $\phi \in \Delta(A(G))$. □

Let G be a locally compact group and let $M(G)$ be the measure algebra with respect to G. It is well-known that $L^1(G)$ is a closed ideal of $M(G)$. So every character of $L^1(G)$ has an extension to $M(G)$, particularly the augmentation character ϕ_1. We again denote this extension by ϕ_1.

THEOREM 3.4. *Let G be a locally compact group. Then $M(G)$ is approximate left ϕ_1-biprojective if and only if G is amenable.*

PROOF. Suppose that $M(G)$ is approximate left ϕ_1-biprojective. Since $M(G)$ is unital, by Proposition 2.1 $M(G)$ is approximate left ϕ_1-amenable. Since $L^1(G)$ is a closed ideal of $M(G)$ and $\phi_1|_{L^1(G)} \neq 0$, by [14, Lemma 3.1] $L^1(G)$ is approximate left ϕ_1-amenable. Using similar method as in the proof of Theorem 3.1, one can show that G is amenable.

Conversely, let G be an amenable group. Then $L^1(G)$ is amenable. Hence $L^1(G)$ is left ϕ_1-amenable. So there exists a bounded net (a_α) in $L^1(G)$ such that

$$aa_\alpha - \phi_1(a)a_\alpha \to 0, \quad \phi_1(a_\alpha) = 1 \qquad (a \in L^1(G)).$$

Pick $i_0 \in L^1(G)$ such that $\phi_1(i_0) = 1$. Set $m_\alpha = i_0 a_\alpha$. Thus

$$\begin{aligned} am_\alpha - \phi_1(a)m_\alpha &= ai_0a_\alpha - \phi_1(a)i_0a_\alpha \\ &= ai_0a_\alpha - \phi_1(ai_0)a_\alpha + \phi_1(ai_0)a_\alpha - \phi_1(a)i_0a_\alpha \to 0 \end{aligned}$$

and $\phi_1(m_\alpha) = \phi_1(i_0 a_\alpha) = 1$, for each $a \in M(G)$. It follows that $M(G)$ is left ϕ_1-amenable and by [1, p. 1332] we have $M(G)$ is approximate left ϕ_1-amenable. Proposition 2.4 implies that $M(G)$ is approximate left ϕ_1-biprojective. □

COROLLARY 3.5. *Let G be a locally compact group. Then $M(G)$ is approximate left character biprojective if and only if G is discrete and amenable.*

PROOF. Suppose that $M(G)$ is approximate left character biprojective. Since $M(G)$ is unital, by Proposition 2.1 approximate character biprojectivity implies that $M(G)$ is approximate character amenable. Applying [1, Theorem 7.2] G is discrete and amenable.

Conversely, let G be amenable and discrete. Then by [9, Proposition 4.2] $M(G)$ is pseudo-amenable. Hence by Remark 2.5, $M(G)$ is approximate character left biprojective. □

Now we give a Banach algebra which is not pseudo-amenable but it is approximate left ϕ-biprojective.

EXAMPLE 3.6. Let G be an infinite compact group. Since G is compact $\widehat{G} \subseteq L^\infty(G) \subseteq L^1(G)$. It is easy to see that for every $\rho \in \widehat{G}$ we have

$$\begin{aligned} f\rho(x) &= \int f(y)\rho(y^{-1}x)dy = \rho(x)\int f(y)\rho(y^{-1})dy \\ &= \rho(x)\int f(y)\overline{\rho(y)}dy = \phi_\rho(f)\rho(x) \quad (x \in G) \end{aligned}$$

and

$$\phi_\rho(\rho) = \int_G \rho(x)\overline{\rho(x)}dx = \int_G 1dx = 1, \quad (f \in L^1(G)),$$

where we considered the normalized left Haar measure on G. Since $\rho \in \mathrm{L}^1(G)$, the map $f \mapsto f\rho$ is w^*-continuous on $L^1(G)^{**}$. Hence for $\tilde{\phi}_\rho \in \Delta(L^1(G)^{**})$ we have

$$f\rho = \tilde{\phi}_\rho(f)\rho, \quad \phi_\rho(\rho) = \tilde{\phi}_\rho(\rho) = 1, \quad (f \in L^1(G)^{**}).$$

It means that $L^1(G)^{**}$ is left $\tilde{\phi}_\rho$-amenable, so $L^1(G)^{**}$ is approximate left $\tilde{\phi}_\rho$-amenable. Therefore, by Proposition 2.4, $L^1(G)^{**}$ is approximate left $\tilde{\phi}_\rho$-biprojective. But if $L^1(G)^{**}$ is pseudo-amenable, then by [9, Proposition 4.2] G is discrete and amenable. Since G is compact, then G must be finite which is a contradiction.

Theorem 3.7. *Let G be a locally compact SIN group. Then $L^1(G)^{**}$ is approximate left character biprojective if and only if G is amenable.*

Proof. Suppose that $L^1(G)^{**}$ is approximate left character biprojective. Since G is a SIN group, $L^1(G)$ has a central approximate identity. Then for every $\phi \in \Delta(L^1(G))$ there exists an element $a_0 \in L^1(G)$ such that $aa_0 = a_0a$ and $\phi(a_0) = 1$ every $a \in L^1(G)$. Since the maps $b \mapsto ab$ and $b \mapsto ba$ are w^*-continuous on $L^1(G)^{**}$, we have

$$aa_0 = a_0a, \quad \phi(a_0) = \widetilde{\phi}(a_0) = 1 \quad (a \in L^1(G)^{**}).$$

Proposition 2.4, implies that $L^1(G)^{**}$ is approximate left ϕ-amenable for all $\phi \in \Delta(L^1(G)^{**})$. By [1, Proposition 3.9] $L^1(G)$ is approximate left ϕ-amenable. Hence [1, Theorem 7.1] implies that G is amenable.

Conversely, suppose that G is amenable. Then $L^1(G)$ is amenable, hence $L^1(G)$ is left $\phi-$amenable. By [14, Proposition 3.4] we have $L^1(G)^{**}$ is left $\tilde{\phi}-$amenable for all $\phi \in \Delta(L^1(G))$. Hence $L^1(G)^{**}$ is approximate left $\tilde{\phi}-$amenable for all $\phi \in \Delta(L^1(G))$. Now by Theorem 2.4 $L^1(G)^{**}$ is approximate left character biprojective, see also Definition 1.1. □

The semigroup S is called *inverse semigroup*, if for each $s \in S$ there exists $s^* \in S$ such that $ss^*s = s^*$ and $s^*ss^* = s$. An inverse semigroup S is called *Clifford semigroup* if for each $s \in S$ there exists $s^* \in S$ such that $ss^* = s^*s$. There exists a partial order on each inverse semigroup S, that is,

$$s \leq t \Leftrightarrow s = ss^*t \quad (s, t \in S).$$

Let $(S,\leq)$ be an inverse semigroup. For each $s\in S$, set $(x]=\{y\in S|\, y\leq x\}$. S is called *uniformly locally finite* if $\sup\{|(x]| : x\in S\}<\infty$. Suppose that S is an inverse semigroup and $e\in E(S)$, where $E(S)$ is the set of all idempotents of S. Then $G_e=\{s\in S|ss^*=s^*s=e\}$ is a maximal subgroup of S with respect to e. See [12] as a main reference of semigroup theory.

THEOREM 3.8. *Let $S=\cup_{e\in E(S)}G_e$ be a Clifford semigroup such that $E(S)$ is uniformly locally finite. Then $\ell^1(S)$ is approximate left character biprojective if and only if $\ell^1(S)$ pseudo-amenable.*

PROOF. Suppose that $\ell^1(S)$ is approximate left character biprojective. By [19, Theorem 2.16], $\ell^1(S)\cong \ell^1-\oplus_{e\in E(S)}\ell^1(G_e)$. Since $\ell^1(G_e)$ has a character ϕ_1(at least augmentation character), then this character extends to $\ell^1(S)$ which still is denoted by ϕ_1. So $\ell^1(S)$ is approximate left ϕ_1-biprojective. Since $\phi_1|_{\ell^1(G_e)}\neq 0$ and $\ell^1(G_e)$ is a closed ideal of $\ell^1(S)$, by Proposition 2.8, $\ell^1(G_e)$ is approximate left ϕ_1-biprojective. On the other hand, since $\ell^1(G_e)$ is unital, by Proposition 2.1, $\ell^1(G_e)$ is approximate left ϕ_1-amenable. So by [1, Theorem 7.1], G_e is amenable for all $e\in E(S)$. Thus by [6, Corollary 3.9] $\ell^1(S)$ is pseudo-amenable.

The converse is true by Remark 2.5. □

4. EXAMPLES

EXAMPLE 4.1. We give a Banach algebra which is approximate left ϕ-biprojective but it is not ϕ-biprojective. Also this Banach algebra is character left biprojective but it is not character biprojective. Consider the semigroup $\mathbb{N}_\vee$, with semigroup operation $m\vee n=\max\{m,n\}$, where m and n are in $\mathbb{N}$. The character space $\Delta(\ell^1(\mathbb{N}_\vee))$ precisely consists of all functions $\phi_n:\ell^1(\mathbb{N}_\vee)\to\mathbb{C}$ defined by $\phi_n(\sum_{i=1}^{\infty}\alpha_i\delta_i)=\sum_{i=1}^{n}\alpha_i$ for every $n\in\mathbb{N}\cup\{\infty\}$, for more information see [3]. In [21], authors showed that $\ell^1(\mathbb{N}_\vee)$ is ϕ_n-biflat for each $n\in\mathbb{N}\cup\{\infty\}$. Since this algebra is commutative, by [21, Proposition 3.3] $\ell^1(\mathbb{N}_\vee)$ is left ϕ_n-amenable. Thus $\ell^1(\mathbb{N}_\vee)$ is approximate left ϕ_n-amenable. By Proposition 2.4, $\ell^1(\mathbb{N}_\vee)$ is approximate character left biprojective. Hence $\ell^1(\mathbb{N}_\vee)$ is approximate left $\phi_\infty-$biprojective. Moreover we showed that $\ell^1(\mathbb{N}_\vee)$ is ϕ_n-biprojective for each $n\in\mathbb{N}$. But if $\ell^1(\mathbb{N}_\vee)$ is ϕ_∞-biprojective, then $\ell^1(\mathbb{N}_\vee)$ is character biprojective. So by [17, Remark 3.6] and [17, Lemma 3.7], the maximal ideal space of $\ell^1(\mathbb{N}_\vee)$ is finite which is impossible, because the maximal ideal space of $\ell^1(\mathbb{N}_\vee)$ is $\mathbb{N}\cup\{\infty\}$.

EXAMPLE 4.2. We give a Banach algebra which is neither left ϕ-amenable nor ϕ-biflat but it is approximate left ϕ-biprojective. Hence the converse of Theorem 2.6 does not always hold. We denote by ℓ^1 the set of all sequences $a=((a_n))$ of complex numbers with $\|a\|=\sum_{n=1}^{\infty}|a_n|<\infty$. With the following product:

$$(a*b)(n) = \begin{cases} a(1)b(1), & \text{if } n=1 \\ a(1)b(n)+b(1)a(n)+a(n)b(n), & \text{if } n>1 \end{cases},$$

$A = (\ell^1, \|\cdot\|)$ becomes a Banach algebra. It is easy to see that $\Delta(\ell^1) = \{\phi_1, \phi_1 + \phi_n\}$, where $\phi_n(a) = a(n)$ for each $a \in \ell^1$. By [16, Example 2.9] ℓ^1 is not left ϕ_1-amenable. Suppose that ℓ^1 is ϕ-biflat. Since ℓ^1 is commutative, by [21, Proposition 3.3] ϕ-biflatness follows that ℓ^1 is left ϕ_1-amenable, which is a contradiction. Moreover by [5, Corollary 2.6], ℓ^1 is not right ϕ_1-biflat (it does not have condition W).

Using [16, Example 2.9], ℓ^1 is approximate left ϕ_1-amenable. Then Proposition 2.4 implies that ℓ^1 is approximate left ϕ_1-biprojective. Moreover [16, Example 2.9] showed that ℓ^1 is left $\phi_1 + \phi_n$-amenable so ℓ^1 is approximate left $\phi_1 + \phi_n$-biprojective. Hence ℓ^1 is approximate left character biprojective.

Example 4.3. We give a Banach algebra which is approximate left ϕ-biprojective but it is not approximate left ϕ-amenable. Then the converse of Proposition 2.4 does not always hold. Let S be a left zero semigroup with $|S| \geq 2$, that is, a semigroup with product $st = s$ for all $s, t \in S$. For the semigroup algebra $\ell^1(S)$, we have $fg = \phi_S(g)f$, where ϕ_S is the augmentation character on $\ell^1(S)$. We claim that $\ell^1(S)$ is approximate left ϕ_S-biprojective. To see this, let $f_0 \in \ell^1(S)$ be an element such that $\phi_S(f_0) = 1$. Define $\rho : \ell^1(S) \to \ell^1(S) \otimes_p \ell^1(S)$ by $\rho(f) = f \otimes f_0$ for all $f \in \ell^1(S)$. It is easy to see that

$$f \cdot \rho(g) = \rho(fg), \quad \rho(fg) = \phi_S(g)\rho(f)$$

and

$$\phi_S \circ \pi_A \circ \rho(f) = \phi_S(f_0 f) = \phi(f)$$

for each $f, g \in \ell^1(S)$. We show that $\ell^1(S)$ is not approximate left ϕ-amenable, provided that $|S| \geq 2$. We go toward a contradiction and suppose that $\ell^1(S)$ is approximate left ϕ-amenable. Then there exists a net (f_α) in $\ell^1(S)$ such that $\phi_S(f_\alpha) = 1$ and

$$\phi_S(f_\alpha)f - \phi_S(f)f_\alpha = ff_\alpha - \phi_S(f)f_\alpha \to 0 \quad (f \in \ell^1(S)).$$

It follows that $f - \phi_S(f)f_\alpha \to 0$ for each $f \in \ell^1(S)$. Since S has at least two elements s_1 and s_2, consider δ_{s_1} and δ_{s_2} and replace them in $f - \phi_S(f)f_\alpha \to 0$. It follows that $\delta_{s_1} = \delta_{s_2}$, so $s_1 = s_2$ which is impossible.

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