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TWO DIVISORS OF $(n^2+1)/2$ SUMMING UP TO $\delta n+\delta\pm 2$, δ EVEN

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Abstract. We prove there exist infinitely many odd integers n for which there exists a pair of positive divisors d_1, d_2 of $(n^2+1)/2$ such that

$$d_1+d_2=\delta n+\varepsilon \quad \text{for} \quad \varepsilon=\delta+2,$$

where δ is an even positive integer. Furthermore, we deal with the same problem where $\varepsilon=\delta-2$ and $\delta\equiv 4,6 \pmod 8$. Using different approaches and methods we obtain similar but conditional results since the proofs rely on Schinzel's Hypothesis H.

1. Introduction

Ayad [1] conjectured that there do not exist two divisors d_1, d_2 of $(p^2+1)/2$ such that

$$d_1+d_2=p+1,$$

where p is an odd prime number.

Ayad and Luca [2] dealt with a similar, but more general problem. Namely, they proved that there does not exist an odd integer $n>1$ and two positive divisors d_1, d_2 of $(n^2+1)/2$ such that

$$d_1+d_2=n+1. \tag{1.1}$$

Dujella and Luca [4] replaced the linear polynomial $n+1$ in (1.1) by an arbitrary linear polynomial $\delta n+\varepsilon$ where $\delta>0$ and ε are given integers and tried to answer whether there exist infinitely many odd positive integers n for which there are two divisors d_1, d_2 of $(n^2+1)/2$ such that $d_1+d_2=\delta n+\varepsilon$.

Since $d_1+d_2\equiv 2 \pmod 4$, then either $\delta\equiv\varepsilon\equiv 1 \pmod 2$, or $\delta\equiv\varepsilon+2\equiv 0,2 \pmod 4$. In [4] the authors dealt with the case $\delta\equiv\varepsilon\equiv 1 \pmod 2$.

Bujačić Babić [3] dealt with the case $\delta\equiv\varepsilon+2\equiv 0,2 \pmod 4$, for some fixed δ or ε.

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In this paper, we discuss one-parametric families of even coefficients δ and ε of the linear polynomial $\delta n+\varepsilon$ where $\varepsilon=\delta\pm 2$. We prove the existence of infinitely many odd integers n for which there exists a pair of positive divisors d_1, d_2 of $(n^2+1)/2$ such that $d_1+d_2=\delta n+\delta+2$.

Furthermore, we deal with the same problem where $\varepsilon=\delta-2$ and $\delta\equiv 4,6 \pmod 8$ using different approaches and methods and give conditional proofs relying on Schinzel's Hypothesis H. The same problem for $\delta\equiv 0,2 \pmod 8$ still remains open. Our conditional and unconditional proofs rely on known facts from the theory of Pellain equations.

2. The case $d_1+d_2=\delta n+\varepsilon$ for $\varepsilon=\delta+2$

In this section, we consider one-parametric family of linear polynomials $\delta n+\varepsilon$, where δ is an even positive integer and $\varepsilon=\delta+2$.

Theorem 2.1. *For every even positive integer δ there are infinitely many odd positive integers n for which there exist divisors d_1, d_2 of $(n^2+1)/2$ such that*

$$d_1+d_2=\delta(n+1)+2.$$

Proof. Let δ be an even positive integer, n an odd positive integer and d_1, d_2 positive divisors of $(n^2+1)/2$ such that

$$d_1+d_2=\delta(n+1)+2.$$

We follow the idea from [4] (see also [3]). Let $g=\gcd(d_1,d_2)$. There exists a positive integer d such that

$$d_1d_2=\frac{g(n^2+1)}{2d}.$$

From the identity

$$(d_2-d_1)^2=(d_1+d_2)^2-4d_1d_2,$$

we easily get

$$d(d_2-d_1)^2=(\delta^2d-2g)n^2+2d\delta(\delta+2)n+\delta^2d+4d\delta+4d-2g. \tag{2.1}$$

Multiplying (2.1) by δ^2d-2g, we obtain

$$\begin{aligned} &d(\delta^2d-2g)(d_2-d_1)^2\\ &=(\delta^2d-2g)^2n^2+2d\delta(\delta^2d-2g)(\delta+2)n+\delta^4d^2+4\delta^3d^2\\ &\quad+4d^2\delta^2-4\delta^2dg-8d\delta g-8dg+4g^2.\end{aligned}$$

After introducing the substitutions $X=(\delta^2d-2g)n+d\delta(\delta+2)$ and $Y=d_2-d_1$, the previous equation becomes

$$X^2-d(\delta^2d-2g)Y^2=4\delta^2dg+8d\delta g+8dg-4g^2. \tag{2.2}$$

For $d=g$ the right-hand side of (2.2) becomes a perfect square. Finally, the equation of the form

$$X^2-d(\delta^2 d-2d)Y^2=(2d(\delta+1))^2$$

is obtained. Since X is divisible by d, we denote $X=dX'$ and we get

$$X'^2-(\delta^2-2)Y^2=4(\delta+1)^2. \tag{2.3}$$

For δ even, $\delta^2-2\equiv 2 \pmod 4$ is never a perfect square, so (2.3) is a Pellian equation. If we denote $X'=2(\delta+1)U$ and $Y=2(\delta+1)V$ and divide (2.3) by $(2(\delta+1))^2$, we get

$$U^2-(\delta^2-2)V^2=1, \tag{2.4}$$

which is a Pell equation that has infinitely many solutions (U,V). Consequently, the Pellian equation (2.3) has infinitely many solutions (X',Y). Since the continuous fraction expansion of $\sqrt{\delta^2-2}$ is

$$\sqrt{\delta^2-2}=[\delta-1;\overline{1,\delta-2,1,2\delta-2}],$$

the fundamental solution of (2.4) is $(U_1,V_1)=(\delta^2-1,\delta)$. All solutions (U,V) of equation (2.4) in nonnegative integers are given by $(U,V)=(U_m,V_m)$ for some $m\geq 0$, where

$$U_0=1,\ \ U_1=\delta^2-1,\ \ U_{m+2}=2(\delta^2-1)U_{m+1}-U_m, \tag{2.5}$$

$$V_0=0,\ \ V_1=\delta,\ \ V_{m+2}=2(\delta^2-1)V_{m+1}-U_m,\ \ m\in\mathbb{N}_0. \tag{2.6}$$

From $X=2d(\delta+1)U$ and $X=(\delta^2 d-2d)n+d\delta(\delta+2)$, it is easily obtained that

$$n=\frac{2(\delta+1)U-\delta(\delta+2)}{\delta^2-2}. \tag{2.7}$$

Now, we will show that numbers n of the form (2.7) with $U=U_m$ are odd positive integers for all $m\geq 1$. Indeed, by induction on m, using recurrence (2.5), we get that $U_m\equiv 1 \pmod{(\delta^2-2)}$ for every $m\geq 0$. Hence,

$$2(\delta+1)U_m-\delta(\delta+2)\equiv 2\delta+2-\delta^2-2\delta\equiv-(\delta^2-2)\equiv 0 \pmod{(\delta^2-2)},$$

which implies that numbers n are integers. Note that n is a positive integer if $m\geq 1$. Furthermore, since δ is even, numbers U_m are odd for all $m\geq 0$. Therefore, we have

$$2(\delta+1)U_m-\delta(\delta+2)\equiv 2U\equiv 2 \pmod 4 \quad\text{and}\quad \delta^2-2\equiv 2 \pmod 4,$$

which implies that numbers n are odd. This completes the proof of the theorem. ∎

EXAMPLE 2.2. According to the proof of Theorem 2.1, we can generate integers n, d_1 and d_2 from each solution $(U,V) = (U_m, V_m),\ m \geq 1$, of the equation (2.4). Since

$$d_1 + d_2 = \delta n + \delta + 2,\ \ d_1 d_2 = \frac{n^2+1}{2}, \tag{2.8}$$

d_1 and d_2 can be interpreted as solutions of the quadratic equation. Using Vieta's formulas we are able to determine expressions for d_1, d_2 for each odd positive integer n. Namely, d_1 and d_2 are roots of the quadratic polynomial of the form $t^2 - (d_1+d_2)t + d_1 d_2$. We obtain the polynomial

$$t^2 - (\delta n + \delta + 2)t + \frac{n^2+1}{2}. \tag{2.9}$$

The roots of (2.9) are given by

$$t_{1,2} = \frac{2\delta n + 2\delta + 4 \pm \sqrt{4(\delta n + \delta + 2)^2 - 8(n^2+1)}}{4}. \tag{2.10}$$

For $U = U_1 = \delta^2 - 1$, we get

$$n = \frac{2(\delta+1)(\delta^2-1) - \delta(\delta+2)}{\delta^2 - 2} = 2\delta + 1.$$

Inserting $n = 2\delta + 1$ into (2.10), we obtain

$$t_{1,2} = \frac{2\delta(2\delta+1) + 2\delta + 4 \pm 4\delta(1+\delta)}{4} = \frac{4\delta^2 + 4\delta + 4 \pm (4\delta^2 + 4\delta)}{4},$$

so

$$d_1 = t_1 = 1,\ \ d_2 = t_2 = 2\delta^2 + 2\delta + 1.$$

For $U = U_2 = 2\delta^4 - 4\delta^2 + 1$, we have

$$n = \frac{2(\delta+1)(2\delta^4 - 4\delta^2 + 1) - \delta(\delta+2)}{\delta^2 - 2} = 4\delta^3 + 4\delta^2 - 1.$$

Inserting $n = 4\delta^3 + 4\delta^2 - 1$ into (2.10), we get

$$t_{1,2} = \frac{8\delta^4 + 8\delta^3 + 4 \pm 8\delta(\delta-1)(\delta+1)^2}{4},$$

hence

$$d_1 = 2\delta^2 + 2\delta + 1,\ \ d_2 = 4\delta^4 + 4\delta^3 - 2\delta^2 - 2\delta + 1.$$

Analogously, for $U = U_3 = 4\delta^6 - 12\delta^4 + 9\delta^2 - 1$, we obtain

$$n = \frac{2(\delta+1)(4\delta^6 - 12\delta^4 + 9\delta^2 - 1) - \delta(\delta+2)}{\delta^2 - 2} = 8\delta^5 + 8\delta^4 - 8\delta^3 - 8\delta^2 + 2\delta + 1,$$

and

$$d_1 = 4\delta^4 + 4\delta^3 - 2\delta^2 - 2\delta + 1,\ \ d_2 = 8\delta^6 + 8\delta^5 - 12\delta^4 - 12\delta^3 + 4\delta^2 + 4\delta + 1.$$

Consequently, we generate infinitely many triples (n, d_1, d_2) of the form

$$\begin{cases} n = 2\delta + 1, \\ d_1 = 1, \\ d_2 = 2\delta^2 + 2\delta + 1, \end{cases}$$

$$\begin{cases} n = 4\delta^3 + 4\delta^2 - 1, \\ d_1 = 2\delta^2 + 2\delta + 1, \\ d_2 = 4\delta^4 + 4\delta^3 - 2\delta^2 - 2\delta + 1, \end{cases}$$

$$\begin{cases} n = 8\delta^5 + 8\delta^4 - 8\delta^3 - 8\delta^2 + \delta + 1, \\ d_1 = 4\delta^4 + 4\delta^3 - 2\delta^2 - 2\delta + 1, \\ d_2 = 8\delta^6 - 12\delta^4 - 12\delta^3 + 4\delta^2 + 4\delta + 1. \end{cases}$$

It is easy to notice that the divisor d_2 of $\frac{n^2+1}{2}$ for $n = 2\delta+1$ is the divisor d_1 of $\frac{n^2+1}{2}$ for $n = 4\delta^3+4\delta^2-1$. The divisor d_2 of $\frac{n^2+1}{2}$ for $n = 4\delta^3+4\delta^2-1$ is the divisor d_1 of $\frac{n^2+1}{2}$ for $n = 8\delta^5 + 8\delta^4 - 8\delta^3 - 8\delta^2 + \delta + 1$, etc.

Namely, the quadratic equations of the form (2.9) that are obtained using two integers n generated by two consecutive terms of recurrence sequence (U_m), $m \geq 1$ have a common root. In what follows, we prove that claim.

Definition 2.3. *Let $f(x) = a_0x^l + \cdots + a_l$ and $g(x) = b_0x^m + \cdots + b_m$ be two polynomials of the degrees $l \geq 1$ and $m \geq 1$, respectively, with coefficients in an arbitrary field K. The* resultant *of f and g, denoted by* $\mathrm{Res}(f, g)$*, is the determinant of order $l + m$ of the form*

$$\mathrm{Res}(f,g) = \det \begin{bmatrix} a_0 & & & & b_0 & & & \\ a_1 & a_0 & & & b_1 & b_0 & & \\ a_2 & a_1 & \ddots & & b_2 & b_1 & \ddots & \\ \vdots & a_2 & \ddots & a_0 & \vdots & b_2 & \ddots & b_0 \\ a_l & \vdots & \ddots & a_1 & b_m & \vdots & \ddots & b_1 \\ & a_l & & a_2 & & b_m & & b_2 \\ & & \ddots & \vdots & & & \ddots & \vdots \\ & & & a_l & & & & b_m \end{bmatrix},$$

where empty spaces stand for zeros.

Two polynomials f, g have a common root if and only if their resultant satisfies $\mathrm{Res}(f, g) = 0$. In our case, for the first two values of n, $n = 2\delta + 1$ and $n = 4\delta^3 + 4\delta^2 - 1$ the corresponding quadratic polynomials (2.9) are of the form

$$f_1(t) = 2t^2 - 2(2\delta^2 + 2\delta + 2)t + 4\delta^2 + 4\delta + 2, \tag{2.11}$$

$$f_2(t) = 2t^2 - 2(4\delta^4 + 4\delta^3 + 2)t + 16\delta^6 + 32\delta^5 + 16\delta^4 - 8\delta^3 - 8\delta^2 + 2, \tag{2.12}$$

respectively. The polynomials (2.11) and (2.12) have one common root which implies $\mathrm{Res}(f_1, f_2) = 0$. This property holds in general.

Proposition 2.4. *Two quadratic polynomials of the form (2.9) generated by two integers n determined by two consecutive terms of the recurrence sequence (U_m), $m \geq 1$, have a common root.*

Proof. Let U_{m-1}, U_m, $m \geq 2$ be two consecutive terms of the recurrence sequence given by (2.5), and let f_{m-1}, f_m be polynomials of the form (2.9) generated by the integers n of the form (2.7) with $U = U_{m-1}$ and $U = U_m$, respectively. Then polynomials f_{m-1} and f_m are of the form

$$f_{m-1}(t) = 2t^2 - 2\left(\delta\frac{2(\delta+1)U_{m-1} - \delta(\delta+2)}{\delta^2-2} + \delta + 2\right)t + \left(\frac{2(\delta+1)U_{m-1} - \delta(\delta+2)}{\delta^2-2}\right)^2 + 1, \tag{2.13}$$

$$f_m(t) = 2t^2 - 2\left(\delta\frac{2(\delta+1)U_m - \delta(\delta+2)}{\delta^2-2} + \delta + 2\right)t + \left(\frac{2(\delta+1)U_m - \delta(\delta+2)}{\delta^2-2}\right)^2 + 1. \tag{2.14}$$

We get

$$\mathrm{Res}(f_{m-1}, f_m) = \frac{64(1+\delta^4)(U_m - U_{m-1})^2(\delta^4 + (U_m + U_{m-1})^2 - 2\delta^2(1 + U_mU_{m-1}))}{(\delta^2-2)^4}.$$

Hence, $\mathrm{Res}(f_{m-1}, f_m) = 0$ if and only if

$$\delta^4 - 2\delta^2(U_mU_{m-1} + 1) + (U_m + U_{m-1})^2 = 0. \tag{2.15}$$

Therefore, in order to prove the proposition, it suffices to show that (2.15) is valid for all $m \geq 1$. Since, $U_0 = 1$ and $U_1 = \delta^2 - 1$, the relation (2.15) is obviously true for $m = 1$. Assume that (2.15) is valid for m. By (2.5) we have

$$U_{m+1} = 2(\delta^2 - 1)U_m - U_{m-1},\ m \geq 1.$$

Then

$$\begin{aligned}
&\delta^4 - 2\delta^2(U_{m+1}U_m + 1) + (U_{m+1} + U_m)^2 \\
&= \delta^4 - 2\delta^2U_{m+1}U_m - 2\delta^2 + U_{m+1}^2 + 2U_{m+1}U_m + U_m^2 \\
&= \delta^4 - 2\delta^2(2(\delta^2-1)U_m - U_{m-1})U_m - 2\delta^2 + (2(\delta^2-1)U_m - U_{m-1})^2 \\
&\quad + 2(2(\delta^2-1)U_m - U_{m-1})U_m + U_m^2 \\
&= \delta^4 - 2\delta^2(U_mU_{m-1} + 1) + (U_m + U_{m-1})^2 = 0,
\end{aligned}$$

by the inductive hypothesis. □

EXAMPLE 2.5. Let $\delta = 8$. We get

$$(n, \frac{n^2+1}{2}, d_1, d_2, \delta, \varepsilon) = (17, 145, 1, 145, 8, 10),\ (2303, 2651905, 145, 18289, 8, 10), \dots$$

Let $\delta = 10$. We get

$$(n, \frac{n^2+1}{2}, d_1, d_2, \delta, \varepsilon) = (21, 221, 1, 221, 10, 12),\ (4399, 9675601, 221, 43781, 10, 12), \dots$$

3. The case $d_1 + d_2 = \delta n + \varepsilon$ for $\varepsilon = \delta - 2$

In this section, we assume that coefficients δ and ε of the linear polynomial $\delta n + \varepsilon$ are even and $\varepsilon = \delta - 2$. Our goal is to show that there exist infinitely many odd positive integers n such that two divisors d_1, d_2 of $(n^2+1)/2$ satisfy

$$d_1 + d_2 = \delta n + \delta - 2. \tag{3.1}$$

Like in the previous section, we set $g = \gcd(d_1, d_2)$. Then, there exists $d \in \mathbb{N}$ such that

$$d_1 d_2 = \frac{g(n^2+1)}{2d}.$$

It can be easily concluded that $g \equiv d \equiv d_1 \equiv d_2 \equiv 1 \pmod 4$. From the identity

$$(d_2 - d_1)^2 = (d_1 + d_2)^2 - 4d_1 d_2,$$

we get the equation

$$X^2 - d(d\delta^2 - 2g)Y^2 = 2dg(\delta^2 + \varepsilon^2) - 4g^2, \tag{3.2}$$

where $X = n(d\delta^2 - 2g) + d\delta\varepsilon$ and $Y = d_2 - d_1$.

Since $g \mid (\delta^2 n^2 - \varepsilon^2)$ and $g \mid \delta^2(n^2+1)$, we conclude that

$$g \mid (\delta^2 + \varepsilon^2).$$

For even integers δ, ε we get $\delta^2 + \varepsilon^2 \equiv 0 \pmod 4$ and since $g \equiv 1 \pmod 4$, we conclude that $g \mid \frac{\delta^2+\varepsilon^2}{4}$. In particular, for $\varepsilon = \delta - 2$, we get

$$g \mid \frac{\delta^2 - 2\delta + 2}{2}.$$

Taking $g = \frac{\delta^2+\varepsilon^2}{4} = \frac{\delta^2-2\delta+2}{2}$ equation (3.2) becomes

$$X^2 - d(d\delta^2 - 2g)Y^2 = 4g^2(2d-1). \tag{3.3}$$

For $d = 2k^2 - 2k + 1,\ k \in \mathbb{N}$, the right-hand side of (3.3) is a perfect square and equation (3.3) takes the form

$$X^2 - 2(2k^2 - 2k + 1)(\delta k - 1)(\delta k - \delta + 1)Y^2 = (2g(2k-1))^2. \tag{3.4}$$

The corresponding Pell equation is

$$U^2 - 2(2k^2 - 2k + 1)(\delta k - 1)(\delta k - \delta + 1)V^2 = 1. \tag{3.5}$$

Since the period length of the continued fraction expansion of

$$\sqrt{2(2k^2 - 2k + 1)(\delta k - 1)(\delta k - \delta + 1)}$$

depends on $k \in \mathbb{N}$, the approach we have used in the previous section cannot be used here. In this case, we are looking for the positive integer solutions of (3.4) of the form $(X, Y) = (2g(2k-1)U, 2g(2k-1)V)$, where (U, V) are solutions of the equation (3.5). Those solutions have to satisfy the additional condition

$$X \equiv d\delta\varepsilon \equiv d\delta(\delta - 2) \pmod{2(\delta k - 1)(\delta k - \delta + 1)}, \tag{3.6}$$

in order that n be an integer.

If we set

$$a = 2k^2 - 2k + 1, \;\; b = \delta k - 1, \;\; c = \delta k - \delta + 1,$$

then the equation (3.5) becomes

$$U^2 - 2abcV^2 = 1. \tag{3.7}$$

The fundamental solution (U_0, V_0) of that equation satisfies

$$(U_0 - 1)(U_0 + 1) = 2abcV_0^2.$$

It is easy to conclude that $4 | (U_0 - 1)(U_0 + 1)$ and V_0 is even. So, we set $V_0 = 2st, \;\; s, t \in \mathbb{N}$. The previous equation becomes

$$(U_0 - 1)(U_0 + 1) = 8abcs^2t^2.$$

If we assume that a, b, c are primes, number of factorizations of the equation (3.7) is the smallest possible. Since a, b, c are odd primes, all possible factorizations are:

$$\begin{array}{ll}
1^{\pm}) & U_0 \pm 1 = 2abcs^2, \;\; U_0 \mp 1 = 2^2t^2, \\
2^{\pm}) & U_0 \pm 1 = 2^2abcs^2, \;\; U_0 \mp 1 = 2t^2, \\
3^{\pm}) & U_0 \pm 1 = 2abs^2, \;\; U_0 \mp 1 = 2^2ct^2, \\
4^{\pm}) & U_0 \pm 1 = 2acs^2, \;\; U_0 \mp 1 = 2^2bt^2, \\
5^{\pm}) & U_0 \pm 1 = 2bcs^2, \;\; U_0 \mp 1 = 2^2at^2, \\
6^{\pm}) & U_0 \pm 1 = 2as^2, \;\; U_0 \mp 1 = 2^2bct^2, \\
7^{\pm}) & U_0 \pm 1 = 2bs^2, \;\; U_0 \mp 1 = 2^2act^2, \\
8^{\pm}) & U_0 \pm 1 = 2cs^2, \;\; U_0 \mp 1 = 2^2abt^2.
\end{array}$$

From (3.5) we get $U_0^2 \equiv 1 \pmod{(\delta k - 1)}$ and $U_0^2 \equiv 1 \pmod{(\delta k - \delta + 1)}$, so we assume

$$U_0 \equiv -1 \pmod{(\delta k - 1)}, \;\; U_0 \equiv 1 \pmod{(\delta k - \delta + 1)}. \tag{3.8}$$

Then, for corresponding $X = X_0$, we have

$$\begin{aligned} X_0 &= 2g(2k-1)U_0 \equiv -2g(2k-1) \equiv -d\delta^2(2k-1) \\ &\equiv -d(2\delta(\delta k-1) - \delta(\delta-2)) \equiv d\delta(\delta-2) \pmod{(\delta k-1)}, \end{aligned}$$

$$\begin{aligned} X_0 &= 2g(2k-1)U_0 \equiv 2g(2k-1) \equiv d\delta^2(2k-1) \\ &\equiv 2d\delta(\delta k-\delta+1) + d\delta(\delta-2) \equiv d\delta(\delta-2) \pmod{(\delta k-\delta+1)}. \end{aligned}$$

Since $\delta k-1$ and $\delta k-\delta+1$ are coprime, we obtain $X_0 \equiv d\delta(\delta-2) \pmod{(\delta k-1)(\delta k-\delta+1)}$. Furthermore, we have $X_0 \equiv d\delta(\delta-2) \equiv 0 \pmod 2$, which implies

$$X_0 \equiv d\delta(\delta-2) \pmod{2(\delta k-1)(\delta k-\delta+1)}. \tag{3.9}$$

Methods that we use in this section depend on which residue class modulo 8 the even number δ belongs.

3.1. *Case* $\delta \equiv 4 \pmod 8$. We set $\delta \equiv 4 \pmod 8$ and $k \equiv 3 \pmod 8$, so we obtain

$$\begin{aligned} a &= 2k^2-2k+1 \equiv 5 \pmod 8, \\ b &= \delta k-1 \equiv 3 \pmod 8, \\ c &= \delta k-\delta+1 \equiv 1 \pmod 8. \end{aligned}$$

We want to show that there exist infinitely many integers k such that only factorizations 4^-) and 7^+) are possible, which implies that (3.8) holds and, consequently, that corresponding (X_0, Y_0) are integer solutions of (3.4).

1^+) $U_0+1 = 2abcs^2$, $U_0-1 = 2^2t^2$.
This factorization gives $abcs^2-2t^2 = 1$, which implies $7s^2-2t^2 \equiv 1 \pmod 8$ and this is impossible.

1^-) $U_0+1 = 2^2t^2$, $U_0-1 = 2abcs^2$.
In this case we get $2t^2-abcs^2 = 1$, which implies $(2t)^2 = 2abcs^2+2$. Since

$$\left(\frac{2}{a}\right) = \left(\frac{2}{b}\right) = -1,$$

this factorization is impossible.

2^+) $U_0+1 = 2^2abcs^2$, $U_0-1 = 2t^2$.
We get $2abcs^2-t^2 = 1$, which implies $7t^2-2s^2 \equiv 1 \pmod 8$, and this is not satisfied for any $s, t \in \mathbb{Z}$.

2^-) $U_0+1 = 2t^2$, $U_0-1 = 2^2abcs^2$.
This case leads to $t^2-2abcs^2 = 1$, which contradicts the minimality of the fundamental solution (U_0, V_0).

$3^+)$ $U_0 + 1 = 2abs^2$, $U_0 - 1 = 2^2ct^2$.
In this case we get $abs^2 - 2ct^2 = 1$, which implies $7s^2 - 2t^2 \equiv 1 \pmod 8$, and this is not satisfied for any $s, t \in \mathbb{Z}$.

$3^-)$ $U_0 + 1 = 2^2ct^2$, $U_0 - 1 = 2abs^2$.
This factorization gives $2ct^2 - abs^2 = 1$, which implies $(2ct)^2 = 2abcs^2 + 2c$. If we set

$$\left(\frac{2c}{a}\right) = \left(\frac{2}{a}\right)\left(\frac{c}{a}\right) = -\left(\frac{c}{a}\right) = -1 \;\Rightarrow\; \left(\frac{c}{a}\right) = 1$$

or

$$\left(\frac{2c}{b}\right) = \left(\frac{2}{b}\right)\left(\frac{c}{b}\right) = -\left(\frac{c}{b}\right) = -1 \;\Rightarrow\; \left(\frac{c}{b}\right) = 1,$$

then this factorization is impossible.

$4^+)$ $U_0 + 1 = 2acs^2$, $U_0 - 1 = 2^2bt^2$.
We obtain $acs^2 - 2bt^2 = 1$, which implies $5s^2 - 6t^2 \equiv 1 \pmod 8$, and this is not satisfied for any $s, t \in \mathbb{Z}$.

$4^-)$ $U_0 + 1 = 2^2bt^2$, $U_0 - 1 = 2acs^2$.
This factorization gives $(2bt)^2 = 2abcs^2 + 2b$. If we set

$$\left(\frac{2b}{a}\right) = \left(\frac{2}{a}\right)\left(\frac{b}{a}\right) = -\left(\frac{b}{a}\right) = -1 \;\Rightarrow\; \left(\frac{b}{a}\right) = 1,$$

or

$$\left(\frac{2b}{c}\right) = \left(\frac{2}{c}\right)\left(\frac{b}{c}\right) = \left(\frac{b}{c}\right) = -1 \;\Rightarrow\; \left(\frac{b}{c}\right) = -1,$$

then this factorization is impossible.

$5^+)$ $U_0 + 1 = 2bcs^2$, $U_0 - 1 = 2^2at^2$.
This factorization gives $2bcs^2 - 2^2at^2 = 2$, which implies $(2at)^2 = 2abcs^2 - 2a$. For

$$\left(\frac{-2a}{b}\right) = \left(\frac{-1}{b}\right)\left(\frac{2}{b}\right)\left(\frac{a}{b}\right) = \left(\frac{a}{b}\right) \;\Rightarrow\; \left(\frac{a}{b}\right) = -1$$

or

$$\left(\frac{-2a}{c}\right) = \left(\frac{-1}{c}\right)\left(\frac{2}{c}\right)\left(\frac{a}{c}\right) = \left(\frac{a}{c}\right) \;\Rightarrow\; \left(\frac{a}{c}\right) = -1,$$

this factorization is impossible.

$5^-)$ $U_0 + 1 = 2^2at^2$, $U_0 - 1 = 2bcs^2$.
In this case we get $2at^2 - bcs^2 = 1$, which implies $2t^2 - 3s^2 \equiv 1 \pmod 8$, and this is not satisfied for any $s, t \in \mathbb{Z}$, so this factorization is not possible.

$6^+)$ $U_0 + 1 = 2as^2$, $U_0 - 1 = 2^2bct^2$.
We get $as^2 - 2bct^2 = 1$, which leads to $2t^2 - 3s^2 \equiv 1 \pmod 8$, and this is not satisfied for any $s, t \in \mathbb{Z}$.

$6^-)$ $U_0+1=2^2bct^2,\ U_0-1=2as^2$.
This factorization gives $2bct^2-as^2=1$, which implies $(as)^2=2abct^2-a$. If we set

$$\left(\frac{-a}{b}\right)=-1 \quad\Rightarrow\quad \left(\frac{a}{b}\right)=1$$

or

$$\left(\frac{-a}{c}\right)=-1 \quad\Rightarrow\quad \left(\frac{a}{c}\right)=-1,$$

then this factorization is not possible.

$7^+)$ $U_0+1=2bs^2,\ U_0-1=2^2act^2$.
This case gives $2bs^2-2act^2=2$, which implies $(bs)^2=2abct^2+b$. For

$$\left(\frac{b}{a}\right)=-1 \text{ or } \left(\frac{b}{c}\right)=-1$$

the equation $(bs)^2=2abct^2+b$ is not possible.

$7^-)$ $U_0+1=2^2act^2,\ U_0-1=2bs^2$.
The equation $2act^2-bs^2=1$ implies $2t^2-3s^2\equiv 1 \pmod 8$, which is not satisfied for any $s,t\in\mathbb{Z}$.

$8^+)$ $U_0+1=2cs^2,\ U_0-1=2^2abt^2$.
This factorization gives $2=2cs^2-2^2abt^2$, which implies $(cs)^2=2abct^2+c$. For

$$\left(\frac{c}{a}\right)=-1 \text{ or } \left(\frac{c}{b}\right)=-1,$$

the equation $(cs)^2=2abct^2+c$ is not possible.

$8^-)$ $U_0+1=2^2abt^2,\ U_0-1=2cs^2$.
We get $2abt^2-cs^2=1$, which implies $6t^2-s^2\equiv 1 \pmod 8$, and this is not satisfied for any $s,t\in\mathbb{Z}$.

From the above observations we notice that factorizations $3^-)$, $4^-)$, $5^+)$, $6^-)$, $7^+)$, $8^+)$ are possible. If we set

$$\left(\frac{a}{c}\right)=\left(\frac{c}{a}\right)=-1 \quad\text{and}\quad \left(\frac{c}{b}\right)=\left(\frac{b}{c}\right)=1,$$

the only possible factorizations are $4^-)$ and $7^+)$. For $\left(\frac{b}{a}\right)=\left(\frac{a}{b}\right)=-1$ the only possible case is $4^-)$ and for $\left(\frac{b}{a}\right)=\left(\frac{a}{b}\right)=1$ the only possible case is $7^+)$.

We conditionally prove that the above conditions can be fulfilled if a famous conjecture holds.

Let k be an integer that satisfies the following conditions:

(i) $k \equiv 3 \pmod 8$,

(ii) $\left(\frac{\delta k-\delta+1}{A}\right) = -1$ for $A = \frac{\delta^2}{2} - \delta + 1$,

(iii) $\left(\frac{\delta k-\delta+1}{B}\right) = 1$ for $B = \frac{\delta}{2} - 1$,

(iv) $a = 2k^2 - 2k + 1$ is prime,

(v) $b = \delta k - 1$ is prime,

(vi) $c = \delta k - \delta + 1$ is prime.

The condition (i) implies that a, b, c defined by (iv), (v), (vi) satisfy

$$a \equiv 5 \pmod 8, \quad b \equiv 3 \pmod 8, \quad c \equiv 1 \pmod 8.$$

We show that the condition (ii) is equivalent to $\left(\frac{a}{c}\right) = -1$ and the condition (iii) is equivalent to $\left(\frac{b}{c}\right) = 1$. More precisely, we have

$$\begin{aligned}\left(\frac{a}{c}\right) &= \left(\frac{2k^2-2k+1}{\delta k-\delta+1}\right) = \left(\frac{2\delta^2k^2-2\delta^2k+\delta^2}{\delta k-\delta+1}\right)\\ &= \left(\frac{2\delta k(\delta k-\delta+1)-2\delta k+\delta^2}{\delta k-\delta+1}\right) = \left(\frac{-2\delta k+\delta^2}{\delta k-\delta+1}\right)\\ &= \left(\frac{-2(\delta k-\delta+1)-2\delta+2+\delta^2}{\delta k-\delta+1}\right) = \left(\frac{\delta^2/2-\delta+1}{\delta k-\delta+1}\right)\\ &= \left(\frac{\delta k-\delta+1}{\delta^2/2-\delta+1}\right) = \left(\frac{c}{A}\right),\end{aligned}$$

where $A = \frac{\delta^2}{2} - \delta + 1$.

Furthermore, we have

$$\begin{aligned}\left(\frac{c}{b}\right) &= \left(\frac{\delta k-\delta+1}{\delta k-1}\right) = \left(\frac{\delta k-1}{\delta k-\delta+1}\right) = \left(\frac{\delta k-\delta+1+\delta-2}{\delta k-\delta+1}\right)\\ &= \left(\frac{\delta-2}{\delta k-\delta+1}\right) = \left(\frac{\delta/2-1}{\delta k-\delta+1}\right) = \left(\frac{\delta k-\delta+1}{\delta/2-1}\right) = \left(\frac{c}{B}\right),\end{aligned}$$

where $B = \delta/2 - 1$.

First, we check whether the conditions (i), (ii) and (iii) can all be fulfilled simultaneously. It can be easily shown that $\gcd(AB, \delta) = 1$. Indeed, since $A = B\delta+1$, we have $\gcd(A, B) = \gcd(A, \delta) = 1$. Furthermore, since $2B = \delta+2$ and B is odd, we obtain $\gcd(B, \delta) = 1$. Consequently, we get $\gcd(AB, \delta) = 1$.

Let

$$A = p_1^{a_1} p_2^{a_2} \cdot \dots \cdot p_r^{a_r}$$

be the canonical prime factorization of A. We have $A \equiv 5 \pmod 8$, so A is not a perfect square. Furthermore,

$$A \not\equiv 0 \pmod 3. \tag{3.10}$$

Since A is not a perfect square, some of the exponents a_i in its canonical prime factorization are odd. Without a loss of generality, we can assume that

a_1 is odd. Let x_1 be some quadratic nonresidue modulo p_1. Since $p_1 \geq 5$, there are $(p_1-1)/2 \geq 2$ quadratic nonresidues modulo p_1, so we can choose x_1 such that

$$x_1 \not\equiv 2-\delta \pmod{p_1}.$$

Since $\gcd(A,B)=1$, according to Chinese remainder theorem, we conclude that there exist infinitely many integers x that satisfy the congruences

$$x \equiv x_1 \pmod{p_1},\ \ x\equiv 1 \pmod{p_i},\ \ i=2,\dots r,\quad x\equiv 1 \pmod{B}.$$

We define k as

$$k=\frac{x+\delta-1}{\delta},$$

where x is some solution of the above system of congruences. Since $k \equiv 3 \pmod 8$, we have $x=\delta k-\delta+1 \equiv 2\delta+1 \pmod{8\delta}$.

We find $\gcd(AB, 8\delta)=1$, which implies that the system of congruences

$$x \equiv x_1 \pmod{p_1},\ \ x\equiv 1 \pmod{p_i},\ \ i=2,\dots,r,$$

$$x\equiv 1 \pmod{B},\ \ x\equiv 2\delta+1 \pmod{8\delta}$$

is solvable. If x_0 is one solution of the above system, then all solutions x are given by

$$x \equiv x_0 \pmod{8p_1\dots p_r B\delta}.$$

Obviously, all the solutions x of the above system satisfy the conditions

$$\left(\frac{x}{A}\right)=\left(\frac{x_1}{p_1}\right)=-1 \quad\text{and}\quad \left(\frac{x}{B}\right)=\left(\frac{1}{B}\right)=1,$$

especially those of the form $x=\delta k-\delta+1$, where $k\equiv 3 \pmod 8$. This shows us that the conditions (i), (ii) and (iii) can be simultaneously fulfilled for infinitely many such positive integers k.

It remains to answer whether conditions (iv), (v) and (vi) can be simultaneously satisfied while conditions (i), (ii) and (iii) are fulfilled, too. In order to answer that question, we use Schinzel's hypothesis H [5].

Conjecture 3.1 (Schinzel's Hypothesis H). *Let $f_1(x),\dots,f_m(x)$ be polynomials with integer coefficients and positive leading coefficients. If the following conditions hold*

i) $f_i(x)$ is irreducible for all $i=1,2,\dots,m$,

ii) for each prime p there exists a positive integer n such that

$$f_1(n)f_2(n)\dots f_m(n)\not\equiv 0 \pmod{p},$$

then there exist infinitely many positive integers t such that

$$f_1(t), f_2(t),\dots,f_m(t)$$

are simultaneously prime numbers.

Proposition 3.2. *If Schinzel's Hypothesis H holds, then for all positive integers* $\delta \equiv 4 \pmod 8$ *there exist infinitely many odd positive integers* n *for which there are two divisors* d_1, d_2 *of* $\frac{n^2+1}{2}$ *such that*

$$d_1 + d_2 = \delta n + \delta - 2.$$

Proof. We have already shown that positive integers k defined before simultaneously satisfy conditions (i), (ii) and (iii). In what follows, we show if Schinzel's Hypothesis H holds, then there exist infinitely many such positive integers k for which

$$a = 2k^2 - 2k + 1, \ \ b = \delta k - 1 \ \text{ and } \ c = \delta k - \delta + 1 \tag{3.11}$$

are simultaneously prime. We assume

$$k \equiv y_0 \pmod{(8p_1p_2\dots p_rB)}, \ \text{ i.e. } \ k = se + y_0, \ \ e \in \mathbb{N}, \tag{3.12}$$

where $s = 8p_1p_2\dots p_rB$. We deal with polynomials of the form

$$g_1(k) = 2k^2 - 2k + 1, \ \ g_2(k) = \delta k - 1, \ \ g_3(k) = \delta k - \delta + 1. \tag{3.13}$$

Since $k = se + y_0$, then polynomials $g_1(k), g_2(k), g_3(k)$ are polynomials in the variable e of the form

$$f_1(e) = 2s^2e^2 + 2s(2y_0 - 1)e + 2y_0^2 - 2y_0 + 1,$$

$$f_2(e) = \delta se + \delta y_0 - 1,$$

$$f_3(e) = \delta se + \delta y_0 - \delta + 1,$$

respectively.

We next prove $f_1(e), f_2(e), f_3(e)$ satisfy conditions of Schinzel's Hypothesis H. Polynomials f_1, f_2, f_3 are irreducible with positive leading coefficients, so they satisfy the first condition of Schinzel's hypothesis H.

Now we prove that for every prime number p there exists a positive integer n for which

$$f_1(n)f_2(n)f_3(n) \not\equiv 0 \pmod p.$$

We distinguish three cases: $p = 2$, $p = 3$ and $p \geq 5$, p prime.

Since δ is even, for $p = 2$ we have $f_1(e) \equiv f_2(e) \equiv f_3(e) \equiv 1 \pmod 2$, so we conclude that for every positive integer e we have

$$f_1(e)f_2(e)f_3(e) \not\equiv 0 \pmod 2.$$

Thus, the second condition of Schinzel's Hypothesis H is satisfied for $p = 2$.

Let $p = 3$. We show that $f_1(e) \not\equiv 0 \pmod 3$ for every positive integer e. Indeed, if the congruence $f_1(e) \equiv 0 \pmod 3$ is satisfied, then

$$2f_1(e) \equiv (2se + (2y_0 - 1))^2 + 1 \equiv 0 \pmod 3$$

which implies $\left(\frac{-1}{3}\right) = 1$, a contradiction.

We distinguish two cases: $3 | s$ and $3 \nmid s$. For $3 | s$ the congruence (3.10) implies that $3 \nmid A$ which implies $3|B$, so $\delta \equiv 2 \pmod 3$. From $x \equiv 1 \pmod B$ we have $x \equiv 1 \pmod 3$. On the other side, since

$$x = \delta k - \delta + 1, \tag{3.14}$$

we have

$$x \equiv \delta y_0 - \delta + 1 \equiv 2y_0 - 1 \equiv 1 \pmod 3.$$

Consequently, $3 \nmid (\delta y_0 - 1)$ and $3 \nmid (\delta y_0 - \delta + 1)$, so congruences $f_2(e) \equiv 0 \pmod 3$ and $f_3(e) \equiv 0 \pmod 3$ are unsolvable.

If $3 \nmid s$, then each of the congruences $f_2(e) \equiv 0 \pmod 3$ and $f_3(e) \equiv 0 \pmod 3$ has at most one solution modulo 3. But this means that there exists at least one residue class modulo 3 such that each element of that class does not satisfy any of these two congruences. So, there are infinitely many positive integers e that satisfy

$$f_1(e)f_2(e)f_3(e) \not\equiv 0 \pmod 3.$$

Hence, the second condition of Schinzel's Hypothesis H is satisfied for $p = 3$.

Now, let $p \geq 5$ be a prime. Again, we distinguish two cases: $p|s$ and $p \nmid s$. If $p|s$, then $p|A$ or $p|B$. We have

$$f_1(e) \equiv 2y_0^2 - 2y_0 + 1 \pmod p,$$
$$f_2(e) \equiv \delta y_0 - 1 \pmod p,$$
$$f_3(e) \equiv \delta y_0 - \delta + 1 \pmod p.$$

If $p|B$, then $\delta \equiv 2 \pmod p$, so from $x \equiv 1 \pmod p$ and (3.14) we get

$$x \equiv \delta y_0 - \delta + 1 \equiv 2y_0 - 1 \equiv 1 \pmod p.$$

We conclude $y_0 \equiv 1 \pmod p$. So, we have

$$2y_0^2 - 2y_0 + 1 \equiv 1 \pmod p, \quad \delta y_0 - 1 \equiv 1 \pmod p, \quad \delta y_0 - \delta + 1 \equiv 1 \pmod p$$

which implies that congruences $f_1(e) \equiv 0 \pmod p$, $f_2(e) \equiv 0 \pmod p$, $f_3(e) \equiv 0 \pmod p$ do not have solutions.

If $p|A$, we distinguish two cases: $p = p_1$ and $p = p_i$ for $i \in \{2, \ldots, r\}$. Let $p = p_i$ for $i \in \{2, \ldots, r\}$. From (3.14) we get

$$x \equiv \delta y_0 - \delta + 1 \equiv 1 \pmod{p_i},$$

so we have $\delta(y_0 - 1) \equiv 0 \pmod{p_i}$. From $2A = \delta(\delta - 2) + 2 = (\delta - 1)^2 + 1 \equiv 0 \pmod{p_i}$, we get $p_i \nmid \delta$ and $p_i \nmid (\delta - 1)$, so we have $y_0 \equiv 1 \pmod{p_i}$. Since

$$2y_0^2 - 2y_0 + 1 \equiv 1 \not\equiv 0 \pmod{p_i}, \quad \delta y_0 - \delta + 1 \equiv 1 \not\equiv 0 \pmod{p_i},$$
$$\delta y_0 - 1 \equiv \delta - 1 \not\equiv 0 \pmod{p_i},$$

congruences $f_1(e) \equiv 0 \pmod{p_i}$, $f_2(e) \equiv 0 \pmod{p_i}$, $f_3(e) \equiv 0 \pmod{p_i}$ do not have solutions.

Finally, let $p = p_1$. From (3.14) we have

$$x \equiv \delta y_0 - \delta + 1 \equiv x_1 \pmod{p_1}, \tag{3.15}$$

where x_1 is a quadratic nonresidue modulo p_1 and $x_1 \not\equiv 2-\delta \pmod{p_1}$. Since $f_2(e) \equiv x_1+\delta-2 \pmod{p_1}$ and $f_2(e) \equiv x_1 \pmod{p_1}$, the congruences $f_2(e) \equiv 0 \pmod{p_1}$ and $f_3(e) \equiv 0 \pmod{p_1}$ do not have solutions. It remains to deal with the congruence $f_1(e) \equiv 0 \pmod{p_1}$, or more precisely with

$$2y_0^2-2y_0+1 \equiv 0 \pmod{p_1}.$$

From (3.14) and (3.15) we have

$$\delta^2(2y_0^2-2y_0+1) \equiv 2x_1^2+2x_1\delta-4x_1+\delta^2-2\delta+2 \equiv 2x_1(x_1+\delta-2) \not\equiv 0 \pmod{p_1}$$

so the congruence $f_1(e) \equiv 0 \pmod{p_1}$ does not have solutions.

If $p \nmid s$, the congruence $f_1(e) \equiv 0 \pmod{p}$ has at most two solutions modulo p, while each of congruences $f_2(e) \equiv 0 \pmod{p}$ and $f_3(e) \equiv 0 \pmod{p}$ has at most one solution modulo p. Hence, there exists at least one residue class modulo p such that each element of that class does not satisfy any of these three congruences. Therefore, for each prime number $p \geq 5$ there are infinitely many positive integers e that satisfy $f_1(e)f_2(e)f_3(e) \not\equiv 0 \pmod{p}$. Consequently, if Schinzel's Hypothesis H holds, then there exist infinitely many positive integers k satisfying conditions (i)-(vi). This implies that there are infinitely many solutions (X,Y) of the equation (3.4) that satisfy the condition (3.6), which again implies that there exist infinitely many odd positive integers n with given property. □

EXAMPLE 3.3. For $\delta = 12$ we get $A = 61$, $B = 5$ and $x_1 \not\equiv 51 \pmod{61}$. For $x_1 = 24$ the corresponding system of congruences is

$$x \equiv 24 \pmod{61}, \quad x \equiv 1 \pmod{5}, \quad x \equiv 25 \pmod{96}.$$

Solutions of the above system of congruences are given by

$$x \equiv 16921 \pmod{29280}.$$

Let $x = 29280e+16921$, $e \in \mathbb{Z}$. From (3.14) we get $k \equiv 1411 \pmod{2440}$ i.e. $k = 2440e+1411$, $e \in \mathbb{Z}$. By inserting k into (iv), (v) and (vi), we obtain three polynomials

$$\begin{aligned} a &= f_1(e) = 11907200e^2+13766480e+3979021, \\ b &= f_2(e) = 29280e+16931, \\ c &= f_3(e) = 29280e+16921. \end{aligned}$$

The first condition of the Schinzel's Hypothesis is satisfied. We next explicitly check the second condition of Schinzel's Hypothesis H.
For $n = 1$ we get

$$f_1(1)\cdot f_2(1)\cdot f_3(1) = (13\cdot 2280977)\cdot(11\cdot 4201)\cdot(47\cdot 983).$$

For $n = 2$ we obtain

$$f_1(2)\cdot f_2(2)\cdot f_3(2) = 79140781\cdot(13\cdot 5807)\cdot(7\cdot 41\cdot 263),$$

while for $n=3$ we get

$$f_1(3)\cdot f_2(3)\cdot f_3(3)=(641\cdot 237821)\cdot(17\cdot 6163)\cdot 104761.$$

We notice that

$$\gcd(f_1(1)\cdot f_2(1)\cdot f_3(1),\ \ f_1(2)\cdot f_2(2)\cdot f_3(2), f_1(3)\cdot f_2(3)\cdot f_3(3))=1,$$

so we have shown that prime p that divides each of the three products $f_1(n)f_2(n)f_3(n)$, $n=1,2,3$, does not exist. Therefore, if Schinzel's Hypothesis H holds, then there are infinitely many positive integers $k=2440e+1411, e\in\mathbb{Z}$ such that conditions (i)-(vi) are simultaneously satisfied.

For $k\le 10^9$ there are 153 positive integers k that satisfy given conditions. The first few of them are

$$1411, 16051, 240531, 360091, 425971, 626051, 1314131, 1975371, 2241331, \ldots$$

For $k=1411$, the corresponding Pell equation is

$$U^2-2279895083614942V^2=1.$$

The fundamental solution (U_0,V_0) of the above equation satisfies

$$U_0\approx 2.58023\cdot 10^{1502988},\ \ V_0\approx 1.54982\cdot 10^{1502980}.$$

Since $X_0=2g(2k-1)U_0$, where $g=\frac{\delta^2-2\delta+2}{2}$, from

$$n=\frac{X_0-d\delta(\delta-2)}{d\delta^2-2g},$$

we get

$$n\approx 1.54982\cdot 10^{1502985},$$

while divisors of $(n^2+1)/2$ are

$$d_1\approx 9.89977\cdot 10^{1502978},\ \ d_2\approx 1.85979\cdot 10^{1502986}.$$

3.2. *Case* $\delta\equiv 6 \pmod 8$. Let $\delta\equiv 6 \pmod 8$ and $k\equiv 2 \pmod 8$. For integers a,b,c we get

$$\begin{aligned} a&=2k^2-2k+1\equiv 5 \pmod 8,\\ b&=\delta k-1\equiv 3 \pmod 8,\\ c&=\delta k-\delta+1\equiv 7 \pmod 8.\end{aligned}$$

Like in the previous subsection, we want to check which of the following factorizations are possible:

Cases $1^{\pm}$) lead to $(2t)^2=2abcs^2\mp 2$. Since $\left(\frac{\mp 2}{a}\right)=-1$, these factorizations are impossible.

In case 2^{+}) we obtain $t^2=2abcs^2-1$. Since, $\left(\frac{-1}{b}\right)=-1$, this factorization is impossible, too.

In case 2^{-}) we get $t^2-2abcs^2=1$, which is in contradiction with minimality of (U_0,V_0).

Case 3^+) leads to $(2ct)^2 = 2abcs^2 - 2c$. If we set

$$\left(\frac{-2c}{a}\right) = -1 \quad \Rightarrow \quad \left(\frac{c}{a}\right) = 1,$$

or

$$\left(\frac{-2c}{b}\right) = -1 \quad \Rightarrow \quad \left(\frac{c}{b}\right) = -1,$$

then this factorization is impossible.

Case 3^-) leads to $(2ct)^2 = 2abcs^2 + 2c$. If we set

$$\left(\frac{2c}{a}\right) = -1 \;\Rightarrow\; \left(\frac{c}{a}\right) = 1$$

or

$$\left(\frac{2c}{b}\right) = -1 \;\Rightarrow\; \left(\frac{c}{b}\right) = 1,$$

then this factorization is impossible.

We find that cases $4^+), 4^-), 5^+), 5^-), 6^+), 6^-), 7^+), 7^-)$ lead to the equations that are impossible modulo 8.

In case 8^+) we obtain $(cs)^2 = 2abct^2 + c$. If we set

$$\left(\frac{c}{a}\right) = -1 \text{ or } \left(\frac{c}{b}\right) = 1,$$

then this case is impossible.

In case 8^-) we get $(cs)^2 = 2abct^2 - c$. If we set

$$\left(\frac{-c}{a}\right) = -1 \;\Rightarrow\; \left(\frac{c}{a}\right) = -1$$

or

$$\left(\frac{-c}{b}\right) = -1 \;\Rightarrow\; \left(\frac{c}{b}\right) = 1,$$

then this factorization is impossible.

Thus, the only possible factorizations are $3^+), 3^-), 8^+), 8^-)$. If we set

$$\left(\frac{c}{a}\right) = \left(\frac{a}{c}\right) = 1, \; \left(\frac{c}{b}\right) = \left(\frac{b}{c}\right) = -1,$$

then only possible factorization is 8^-) and (3.8) holds.

For $\delta \equiv 6 \pmod 8$ and $k \equiv 2 \pmod 8$ we have

$$\left(\frac{c}{a}\right) = \left(\frac{-c}{A}\right),$$

where $A = \delta^2/2 - \delta + 1$. Note that $A \equiv 1 \pmod 4$ and A can be composite number. We also obtain

$$\left(\frac{c}{b}\right) = \begin{cases} \left(\frac{\delta k - \delta + 1}{B}\right) = \left(\frac{c}{B}\right) = -\left(\frac{-c}{B}\right), & \text{for } \delta \equiv 14 \pmod{16},\ B \equiv 3 \pmod 4 \\ -\left(\frac{\delta k - \delta + 1}{B}\right) = -\left(\frac{c}{B}\right) = -\left(\frac{-c}{B}\right), & \text{for } \delta \equiv 6 \pmod{16},\ B \equiv 1 \pmod 4 \end{cases}$$

where $B = (\delta - 2)/4$ and B can be composite, too.

Let k be positive integer with the following properties:

(i) $k \equiv 2 \pmod 8$,

(ii) $\left(\frac{-c}{A}\right) = 1$ for $A = \frac{\delta^2}{2} - \delta + 1$,

(iii) $\left(\frac{-c}{B}\right) = 1$ for $B = (\delta-2)/4$,

(iv) $a = 2k^2 - 2k + 1$ is prime,

(v) $b = \delta k - 1$ is prime,

(vi) $c = \delta k - \delta + 1$ is prime.

We have already shown that the condition (ii) is equivalent with the condition $(\frac{c}{a}) = 1$ and the condition (iii) is equivalent with $(\frac{c}{b}) = -1$. Now, we check whether the conditions (i), (ii) and (iii) can be simultaneously satisfied.

Since $A = 4B\delta + 1$, we have $\gcd(A, B) = \gcd(A, \delta) = 1$. Furthermore, since $4B = \delta - 2$ and B is odd, we have $\gcd(B, \delta) = 1$ which implies $\gcd(AB, \delta) = 1$.

By Chinese remainder theorem we conclude that there exist infinitely many integers x that satisfy the following system of congruences

$$x \equiv x_i \pmod{p_i},\ \ x \equiv 1 \pmod B,\ \ i = 1, 2, \dots, r,$$

where $p_i,\ i = 1, 2, \dots, r$ are all different prime factors of A and x_i is a quadratic residue modulo p_i. We get

$$A = \frac{\delta^2}{2} - \delta + 1 \not\equiv 0 \pmod 3. \tag{3.16}$$

Like in the previous section, for every prime factor p_i of A we have $p_i \geq 5$, so there are $(p_i - 1)/2 \geq 2$ quadratic residues modulo p_i and we choose x_i such that

$$x_i \not\equiv \delta - 2 \pmod{p_i}.$$

We define k by $x = -(\delta k - \delta + 1) = \delta - \delta k - 1$, i.e.

$$k = \frac{\delta - x - 1}{\delta},$$

where x is a solution of the above system of the congruences. Let $k \equiv 2 \pmod 8$. In this case we have

$$x \equiv -\delta - 1 \pmod{8\delta}.$$

Since $\gcd(AB, 8\delta) = 1$, the system of the congruences

$$x \equiv x_i \pmod{p_i},\ \ x \equiv 1 \pmod B,\ \ x \equiv -\delta - 1 \pmod{8\delta},\ \ i = 1, 2, \dots, r,$$

has solutions. If x_0 is one solution, then all solutions x are given by

$$x \equiv x_0 \pmod{8AB\delta}.$$

Obviously, all solutions x satisfy

$$\left(\frac{x}{A}\right) = \left(\frac{x_i}{A}\right) = 1 \quad \text{and} \quad \left(\frac{x}{B}\right) = \left(\frac{1}{B}\right) = 1,\ \ i = 1, 2, \dots, r,$$

especially those of the form $x = \delta k - \delta - 1$, where $k \equiv 2 \pmod 8$. Hence, conditions (i), (ii) and (iii) can be simultaneously satisfied.

PROPOSITION 3.4. *If Schinzel's Hypothesis H is true, then for all positive integers $\delta \equiv 6 \pmod 8$ there are infinitely many odd positive integers n such that there exist divisors d_1, d_2 of $\frac{n^2+1}{2}$ such that $d_1 + d_2 = \delta n + \delta - 2$.*

PROOF. We follow the proof of Proposition 3.2. Let y_0, e and s be defined by (3.12). We apply Schinzel's Hypothesis H to the same polynomials $f_1(e), f_2(e), f_3(e)$. Since those polynomials are irreducible with positive leading coefficients, it remains to show that the second condition of Schinzel's Hypothesis H is satisfied.

We consider three cases: $p = 2$, $p = 3$ and $p \geq 5$, p prime. Cases for $p = 2, 3$ can be proven completely analogously as in the proof of Proposition 3.2 and for $p \geq 5$, p prime, again we distinguish two cases: $p|s$ and $p \nmid s$.

Let $p|s$. In this case we have $p|A$ or $p|B$ and

$$\begin{aligned} f_1(e) &\equiv 2y_0^2 - 2y_0 + 1 \pmod p, \\ f_2(e) &\equiv \delta y_0 - 1 \pmod p, \\ f_3(e) &\equiv \delta y_0 - \delta + 1 \pmod p. \end{aligned}$$

If $p|B$, then we get

$$x \equiv \delta - \delta y_0 - 1 \equiv 1 - \delta y_0 \equiv 1 - 2y_0 \equiv 1 \pmod p,$$

which implies $y_0 \equiv 0 \pmod p$ and congruences $f_1(e) \equiv 0 \pmod p$, $f_2(e) \equiv 0 \pmod p$ and $f_3(e) \equiv 0 \pmod p$ do not have solutions.

If $p|A$, then we have $p = p_i$ for some $i = 1, \dots, r$, which implies

$$x \equiv \delta - \delta y_0 - 1 \equiv x_i \pmod p. \tag{3.17}$$

Since $x_i \not\equiv \delta - 2 \pmod p$, we have $1 - \delta y_0 \equiv x_i - \delta + 2 \not\equiv 0 \pmod p$ so congruences $f_2(e) \equiv 0 \pmod p$ and $f_3(e) \equiv 0 \pmod p$ do not have solutions. Finally, we deal with the congruence $f_1(e) \equiv 0 \pmod p$, i.e. with

$$2y_0^2 - 2y_0 + 1 \equiv 0 \pmod p.$$

Analogously as in Proposition 3.2 we get

$$\delta^2(2y_0^2 - 2y_0 + 1) \equiv 2x_i^2 - 2x_i\delta + 4x_i + \delta^2 - 2\delta + 2 \equiv 2x_i(x_i - \delta + 2) \pmod p$$

so $f_1(e) \equiv 0 \pmod p$ does not have any solutions.

If $p \nmid s$, similarly as in the proof of Proposition 3.2, we conclude that there exists at least one residue class modulo p such that each element of that class does not satisfy any of above three congruences.

So, polynomials f_1, f_2, f_3 satisfy the second condition of Schinzel's Hypothesis H.

As in the proof of Proposition 3.2, again we conclude that if Schinzel's Hypothesis H holds, then there exist infinitely many positive integers k satisfying conditions (i)-(vi) which implies that there exist infinitely many odd positive integers n with given property. □

EXAMPLE 3.5. For $\delta = 14$ we get $A = 85$, $B = 3$. We exclude $x_1 \equiv 2 \pmod 5$ and $x_2 \equiv 12 \pmod{17}$. So, let $x_1 = x_2 = 1$. The system of the congruences we deal with is

$$x \equiv 1 \pmod 5,\quad x \equiv 1 \pmod{17},\quad x \equiv 1 \pmod 3,\quad x \equiv -15 \pmod{112}.$$

Solutions of the above system are given by

$$x \equiv 8161 \pmod{28560}.$$

Since $k = 2040e - 582$, $e \in \mathbb{Z}$, the polynomials f_1, f_2, f_3 are of the form:

$$\begin{aligned} a &= f_1(e) = 8323200e^2 - 4753200e + 678613,\\ b &= f_2(e) = 28560e - 8149,\\ c &= f_3(e) = 28560e - 8161. \end{aligned}$$

The first condition of Schinzel's Hypothesis H is satisfied. We next explicitly show that the second condition of the hypothesis is satisfied, too.

For $n = 1$ we get

$$f_1(1)\cdot f_2(1)\cdot f_3(1) = (181\cdot 23473)\cdot(20411)\cdot(20399).$$

For $n = 2$ we get

$$f_1(2)\cdot f_2(2)\cdot f_3(2) = 24465013\cdot(13\cdot 3767)\cdot(173\cdot 283).$$

Obviously, we have

$$\gcd(f_1(1)\cdot f_2(1)\cdot f_3(1), f_1(2)\cdot f_2(2)\cdot f_3(2)) = 1,$$

so the second condition of Schinzel's hypothesis H is satisfied. Hence, there exist infinitely many positive integers e such that $f_1(e)$, $f_2(e)$, $f_3(e)$ are simultaneously prime.

We obtain

$$k = 119778, 519618, 1101018, 1200978, 1313178, 1531458, \ldots.$$

As we can see, these are relatively large values of k, so calculating solutions of the corresponding Pell equation (3.7) would take a large amount of CPU time. Thus, we want to find some smaller values for k. For that purpose, we choose other, more convenient quadratic residues x_1 and x_2.

If we set $x_1 = x_2 = 16$, we obtain the system

$$x \equiv 16 \pmod{85},\quad x \equiv 1 \pmod 3,\quad x \equiv -15 \pmod{112}.$$

We get

$$x \equiv 28321 \pmod{28560} \quad\text{and}\quad k \equiv -2022 \equiv 18 \pmod{2040}.$$

For $k = 18$ we have

$$a = 613, \quad b = 251, \quad c = 239.$$

The corresponding Pell equation is

$$U^2 - 73546514V^2 = 1,$$

while

$$U_0 \approx 2.91573 \cdot 10^{691}, \; V_0 \approx 3.39990 \cdot 10^{687}.$$

Finally, we get

$$n \approx 1.44598 \cdot 10^{690},$$

and divisors d_1, d_2 of $(n^2+1)/2$ are

$$d_1 \approx 7.16336 \cdot 10^{689}, \; d_2 \approx 2.02366 \cdot 10^{692}.$$

4. Open problems

Remark 4.1. Let $\delta \equiv 0 \pmod 8$. For $k \equiv 3 \pmod 8$ we get

$$a \equiv 5 \pmod 8, \; b \equiv 7 \pmod 8, \; c \equiv 1 \pmod 8.$$

In this case we are not able to eliminate the factorization 5^-). More precisely, in that case we have

$$U_0 + 1 = 2^2 at^2, \; U_0 - 1 = 2bcs^2$$

which implies

$$U_0 \equiv 1 \pmod{(\delta k - 1)} \quad \text{and} \quad U_0 \equiv 1 \pmod{(\delta k - \delta + 1)},$$

contradicting the assumption (3.8). For example, let $\delta = 8$. Case 5^-) leads to $(2at)^2 = 2abcs^2 + 2a$. Since

$$\begin{aligned}\left(\frac{2a}{b}\right) &= \left(\frac{a}{b}\right) = \left(\frac{2k^2-2k+1}{8k-1}\right) = \left(\frac{8k^2-8k+4}{8k-1}\right) = \left(\frac{-7k+4}{8k-1}\right) \\ &= \left(\frac{-112k+64}{8k-1}\right) = \left(\frac{50}{8k-1}\right) = \left(\frac{2}{8k-1}\right) = 1,\end{aligned}$$

$$\begin{aligned}\left(\frac{2a}{c}\right) &= \left(\frac{a}{c}\right) = \left(\frac{2k^2-2k+1}{8k-7}\right) = \left(\frac{8k^2-8k+4}{8k-7}\right) = \left(\frac{-k+4}{8k-7}\right) \\ &= \left(\frac{-16k+64}{8k-7}\right) = \left(\frac{50}{8k-7}\right) = \left(\frac{2}{8k-7}\right) = 1,\end{aligned}$$

we are not able to eliminate the factorization 5^-) using the Legendre symbols. Furthermore, even though we have found sporadic solutions for relatively small δ's, we have not found any solutions for $\delta = 40$. So, we are not sure whether there exist infinitely many odd positive integers n for which there exist divisors d_1, d_2 of $(n^2+1)/2$ such that

$$d_1 + d_2 = \delta n + \delta - 2, \; \delta \equiv 0 \pmod 8.$$

Remark 4.2. Let $\delta \equiv 2 \pmod 8$. If we apply the same method as in the cases $\delta \equiv 4, 6 \pmod 8$, we get more complicated conditions on Legendre symbols. For example, we get

$$\left(\frac{a}{c}\right) = -1, \ \left(\frac{a}{b}\right) = 1, \ \left(\frac{c}{b}\right) = 1,$$

so we cannot use Chinese remainder theorem and Schinzel's Hypothesis H in order to get similar conclusions.

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Dva djelitelja od $(n^2+1)/2$ čiji je zbroj $\delta n + \delta \pm 2$ za δ paran

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Sažetak. U članku dokazujemo da postoji beskonačno mnogo neparnih prirodnih brojeva n za koje postoji par djelitelja d_1, d_2 od $(n^2+1)/2$ takvih da vrijedi

$$d_1 + d_2 = \delta n + \varepsilon, \quad \varepsilon = \delta + 2,$$

gdje je δ paran prirodan broj. Nadalje, analiziramo isti problem u slučaju kad je $\varepsilon = \delta - 2$ i $\delta \equiv 4, 6 \pmod 8$ te koristeći različite pristupe i metode uvjetno dokazujemo slične rezultate oslanjajući se na valjanost Schinzelove hipoteze H.

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