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HERMITE-HADAMARD TYPE INEQUALITIES FOR GENERALIZED (s, m, φ)-PREINVEX GODUNOVA-LEVIN FUNCTIONS

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Abstract. In the present paper, a new class of generalized (s, m, φ)-preinvex Godunova-Levin function of the second kind is introduced. Moreover, some left inequalities of Gauss-Jacobi type quadrature formula are given and some Hermite-Hadamard type inequalities for generalized (s, m, φ)-preinvex Godunova-Levin functions of the second kind via classical integrals are established. At the end, some applications to special means are given.

1. Introduction and preliminaries

The following notation is used throughout this paper. We use I to denote an interval on the real line $\mathbb{R} = (-\infty, +\infty)$ and I° to denote the interior of I. For any subset $K \subseteq \mathbb{R}^n, K^\circ$ is used to denote the interior of K. $\mathbb{R}^n$ is used to denote a generic n-dimensional vector space. The nonnegative real numbers are denoted by $\mathbb{R}_\circ = [0, +\infty)$.

The following inequality, named Hermite-Hadamard inequality, is one of the most famous inequalities in the literature for convex functions.

Theorem 1.1. *Let $f : I \subseteq \mathbb{R} \longrightarrow \mathbb{R}$ be a convex function on an interval I of real numbers and $a, b \in I$ with $a < b$. Then the following inequality holds:*

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2}. \tag{1.1}$$

In recent years, various generalizations, extensions and variants of such inequalities have been obtained. For other recent results concerning Hermite-Hadamard type inequalities through various classes of convex functions see

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[10] and the references cited therein, also see [8] and the references cited therein.

DEFINITION 1.2 (see [4]). *A nonnegative function $f : I \subseteq \mathbb{R} \longrightarrow \mathbb{R}_\circ$ is said to be P-function or P-convex, if*

$$f(tx + (1-t)y) \leq f(x) + f(y), \quad \forall x, y \in I,\ t \in [0,1].$$

DEFINITION 1.3 (see [6]). *A function $f : I \subseteq \mathbb{R} \longrightarrow \mathbb{R}_\circ$ is said to be a Godunova-Levin function or $f \in Q(I)$, if f is nonnegative and for all $x, y \in I$, $t \in (0,1)$, we have that*

$$f(tx + (1-t)y) \leq \frac{f(x)}{t} + \frac{f(y)}{1-t}.$$

The class $Q(I)$ was firstly described in [7] by Godunova and Levin. Some further properties of it are given in [4, 11, 12]. Among others, it is noted that nonnegative monotone and nonnegative convex functions belong to this class of functions.

DEFINITION 1.4 (see [13]). *A function $f : I \subseteq \mathbb{R} \longrightarrow \mathbb{R}_\circ$ is said to be (s,m)-Godunova-Levin functions of the second kind or $f \in Q^2_{(s,m)}$, if $s \in [0,1]$, $m \in (0,1]$, we have*

$$f(tx + m(1-t)y) \leq \frac{f(x)}{t^s} + \frac{mf(y)}{(1-t)^s}, \quad \forall x, y \in I,\ t \in (0,1).$$

It is obvious that for $s = 0, m = 1, (s,m)$-Godunova-Levin functions of the second kind reduces to Definition 1.2 of P-functions. If $s = 1, m = 1$, it then reduces to Godunova-Levin functions. For $m = 1$, we have the definition of s-Godunova-Levin function of the second kind introduced and studied by Dragomir (see [2, 3]).

The Gauss-Jacobi type quadrature formula has the following form

$$\int_a^b (x-a)^p (b-x)^q f(x) dx = \sum_{k=0}^{+\infty} B_{m,k} f(\gamma_k) + R^\star_m |f|, \tag{1.2}$$

for certain $B_{m,k}$, γ_k and $R^\star_m|f|$ (see [15]).

Recently, Liu (see [9]) obtained several integral inequalities for the left hand side of (1.2) under the Definition 1.2 of P-function. Also in [14], Özdemir et al. established several integral inequalities concerning the left-hand side of (1.2) via some kinds of convexity.

Motivated by these results, the aim of this paper is to establish left type inequalities of (1.2) and right Hermite-Hadamard type inequalities as (1.1) for classical integrals using new identity given in Section 3. The paper is organized as follows. In Section 2, a new class of generalized (s, m, φ)-preinvex

Godunova-Levin functions of the second kind is introduced and some new integral inequalities for the left hand side of (1.2) involving generalized (s, m, φ)-preinvex Godunova-Levin functions of the second kind along with beta function are given. In Section 3, some Hermite-Hadamard type inequalities for generalized (s, m, φ)-preinvex Godunova-Levin functions of the second kind via classical integrals are given. In Section 4, some applications to special means are given.

2. New integral inequalities

Definition 2.1 (see [5]). *A set $K \subseteq \mathbb{R}^n$ is said to be m-invex with respect to the mapping $\eta : K \times K \times (0, 1] \longrightarrow \mathbb{R}^n$ for some fixed $m \in (0, 1]$, if $mx + t\eta(y, x, m) \in K$ holds for each $x, y \in K$ and any $t \in [0, 1]$.*

We next give new definition, to be referred as generalized (s, m, φ)-preinvex Godunova-Levin function of the second kind.

Definition 2.2. *Let $\varphi : I \longrightarrow \mathbb{R}$ be a continuous function. Let $K \subseteq \mathbb{R}^n$ be an open m-invex set with respect to $\eta : K \times K \times (0, 1] \longrightarrow \mathbb{R}^n$. For $f : K \longrightarrow \mathbb{R}$ and some fixed $s \in [0, 1]$, $m \in (0, 1]$, if*

$$f(m\varphi(y) + t\eta(\varphi(x), \varphi(y), m)) \le \frac{f(\varphi(x))}{t^s} + \frac{mf(\varphi(y))}{(1-t)^s}, \tag{2.1}$$

is valid for all $x, y \in K$, $t \in (0, 1)$, then we say that f is a generalized (s, m, φ)-preinvex Godunova-Levin function of the second kind with respect to η, or $f \in Q^{\star 2}_{(s,m,\varphi)}$.

Remark 2.3. In Definition 2.2, it is worthwhile to note that generalized (s, m, φ)-preinvex Godunova-Levin function of the second kind is an (s, m)-Godunova-Levin functions of the second kind on $K = I$ with respect to $\eta(\varphi(x), \varphi(y), m) = \varphi(x) - m\varphi(y)$, $\varphi(x) = x$, $\forall x \in K$.

In this section, in order to prove our main results regarding some new integral inequalities involving generalized (s, m, φ)-preinvex Godunova-Levin functions of the second kind along with beta function, we need the following lemma

Lemma 2.4. *Let $\varphi : I \longrightarrow \mathbb{R}$ be a continuous function. Assume that a function $f : K = [m\varphi(a), m\varphi(a) + \xi_1] \longrightarrow \mathbb{R}$ is continuous with respect to $\eta : K \times K \times (0, 1] \longrightarrow \mathbb{R}$. Then, for some fixed $m \in (0, 1]$ and any fixed $p, q > 0$,*

$$\int_{m\varphi(a)}^{m\varphi(a)+\xi_1} (x - m\varphi(a))^p (m\varphi(a) + \xi_1 - x)^q f(x)dx$$
$$= \xi_1^{p+q+1} \int_0^1 t^p (1-t)^q f(m\varphi(a) + t\xi_1)dt,$$

where

$$\xi_1 = \eta(\varphi(b), \varphi(a), m).$$

PROOF. It is easy to observe that

$$\begin{aligned}
&\int_{m\varphi(a)}^{m\varphi(a)+\xi_1} (x - m\varphi(a))^p (m\varphi(a) + \xi_1 - x)^q f(x) dx \\
&= \xi_1 \int_0^1 (m\varphi(a) + t\xi_1 - m\varphi(a))^p \\
&\quad \times (m\varphi(a) + \xi_1 - m\varphi(a) - t\xi_1)^q f(m\varphi(a) + t\xi_1) dt \\
&= \xi_1^{p+q+1} \int_0^1 t^p (1-t)^q f(m\varphi(a) + t\xi_1) dt.
\end{aligned}$$

□

The following definition will be used in the sequel.

DEFINITION 2.5. *The Euler Beta function is defined for $x, y > 0$ as*

$$\beta(x, y) = \int_0^1 t^{x-1}(1-t)^{y-1} dt = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}.$$

THEOREM 2.6. *Let $\varphi : I \longrightarrow \mathbb{R}$ be a continuous function. Assume that a function $f : K = [m\varphi(a), m\varphi(a) + \xi_1] \longrightarrow \mathbb{R}$ is continuous on the interval of real numbers K° with respect to $\eta : K \times K \times (0, 1] \longrightarrow \mathbb{R}$ and $\varphi(a), \varphi(b) \in K$, $a < b$ with $\xi_1 > 0$. Let $k > 1$. If $|f|^{\frac{k}{k-1}}$ is a generalized (s, m, φ)-preinvex Godunova-Levin function of the second kind on K for any fixed $m \in (0, 1]$ and $s \in [0, 1)$, then for some fixed $p, q > 0$,*

$$\int_{m\varphi(a)}^{m\varphi(a)+\xi_1} (x - m\varphi(a))^p (m\varphi(a) + \xi_1 - x)^q f(x) dx$$

$$\leq \frac{|\xi_1|^{p+q+1}}{(1-s)^{\frac{k-1}{k}}} \Big[\beta(kp+1, kq+1)\Big]^{\frac{1}{k}} \Big(m|f(\varphi(a))|^{\frac{k}{k-1}} + |f(\varphi(b))|^{\frac{k}{k-1}}\Big)^{\frac{k-1}{k}},$$

where

$$\xi_1 = \eta(\varphi(b), \varphi(a), m).$$

PROOF. Since $|f|^{\frac{k}{k-1}}$ is a generalized (s, m, φ)-preinvex Godunova-Levin function of the second kind on K, combining with Lemma 2.4, Definition 2.5 and Hölder inequality for all $t \in (0, 1)$ and for some fixed $m \in (0, 1]$ and

$s \in [0, 1)$, we get

$$\int_{m\varphi(a)}^{m\varphi(a)+\xi_1} (x - m\varphi(a))^p (m\varphi(a) + \xi_1 - x)^q f(x)dx$$

$$\leq |\xi_1|^{p+q+1} \left[\int_0^1 t^{kp}(1-t)^{kq}dt\right]^{\frac{1}{k}} \left[\int_0^1 \left|f(m\varphi(a) + t\xi_1)\right|^{\frac{k}{k-1}} dt\right]^{\frac{k-1}{k}}$$

$$\leq |\xi_1|^{p+q+1} \Big[\beta(kp+1, kq+1)\Big]^{\frac{1}{k}} \left[\int_0^1 \Big(\frac{m|f(\varphi(a))|^{\frac{k}{k-1}}}{(1-t)^s} + \frac{|f(\varphi(b))|^{\frac{k}{k-1}}}{t^s}\Big)dt\right]^{\frac{k-1}{k}}$$

$$= \frac{|\xi_1|^{p+q+1}}{(1-s)^{\frac{k-1}{k}}} \Big[\beta(kp+1, kq+1)\Big]^{\frac{1}{k}} \Big(m|f(\varphi(a))|^{\frac{k}{k-1}} + |f(\varphi(b))|^{\frac{k}{k-1}}\Big)^{\frac{k-1}{k}}.$$

The proof of Theorem 2.6 is completed. □

THEOREM 2.7. *Let $\varphi : I \longrightarrow \mathbb{R}$ be a continuous function. Assume that a function $f : K = [m\varphi(a), m\varphi(a) + \xi_1] \longrightarrow \mathbb{R}$ is continuous on the interval of real numbers K° with respect to $\eta : K \times K \times (0, 1] \longrightarrow \mathbb{R}$ and $\varphi(a), \varphi(b) \in K$, $a < b$ with $\xi_1 > 0$. Let $l \geq 1$. If $|f|^l$ is a generalized (s, m, φ)-preinvex Godunova-Levin function of the second kind on K for any fixed $m \in (0, 1]$ and $s \in [0, 1]$, then for some fixed $p, q > 0$,*

$$\int_{m\varphi(a)}^{m\varphi(a)+\xi_1} (x - m\varphi(a))^p (m\varphi(a) + \xi_1 - x)^q f(x)dx$$

$$\leq |\xi_1|^{p+q+1} \Big[\beta(p+1, q+1)\Big]^{\frac{l-1}{l}}$$

$$\times \Big[m|f(\varphi(a))|^l \beta\left(p+1, q-s+1\right) + |f(\varphi(b))|^l \beta\left(p-s+1, q+1\right)\Big]^{\frac{1}{l}},$$

where

$$\xi_1 = \eta(\varphi(b), \varphi(a), m).$$

PROOF. Since $|f|^l$ is a generalized (s, m, φ)-preinvex Godunova-Levin function of the second kind on K, combining with Lemma 2.4, Definition 2.5 and Hölder inequality for all $t \in (0, 1)$ and for some fixed $m \in (0, 1]$ and $s \in [0, 1]$, we get

$$\int_{m\varphi(a)}^{m\varphi(a)+\xi_1} (x - m\varphi(a))^p (m\varphi(a) + \xi_1 - x)^q f(x)dx$$

$$= \xi_1^{p+q+1} \int_0^1 \Big[t^p(1-t)^q\Big]^{\frac{l-1}{l}} \Big[t^p(1-t)^q\Big]^{\frac{1}{l}} f(m\varphi(a) + t\xi_1)dt$$

$$\leq |\xi_1|^{p+q+1} \left[\int_0^1 t^p(1-t)^q dt\right]^{\frac{l-1}{l}} \left[\int_0^1 t^p(1-t)^q \left|f(m\varphi(a) + t\xi_1)\right|^l dt\right]^{\frac{1}{l}}$$

$$\leq |\xi_1|^{p+q+1}\Big[\beta(p+1,q+1)\Big]^{\frac{l-1}{l}}\left[\int_0^1 t^p(1-t)^q\left(\frac{m|f(\varphi(a))|^l}{(1-t)^s}+\frac{|f(\varphi(b))|^l}{t^s}\right)dt\right]^{\frac{1}{l}}$$

$$= |\xi_1|^{p+q+1}\Big[\beta(p+1,q+1)\Big]^{\frac{l-1}{l}}$$

$$\times\Big[m|f(\varphi(a))|^l\beta(p+1,q-s+1)+|f(\varphi(b))|^l\beta(p-s+1,q+1)\Big]^{\frac{1}{l}}.$$

The proof of Theorem 2.7 is completed. □

3. Hermite-Hadamard type integral inequalities

In this section, in order to prove our main results regarding some Hermite-Hadamard type inequalities for generalized (s,m,φ)-preinvex Godunova-Levin function of the second kind via classical integrals, we need the following lemma:

Lemma 3.1. *Let $\varphi : I \longrightarrow \mathbb{R}$ be a continuous function. Let $K \subseteq \mathbb{R}$ be an open m-invex subset with respect to $\eta : K \times K \times (0,1] \longrightarrow \mathbb{R}$ for some fixed $m \in (0,1]$ and let $\varphi(a), \varphi(b) \in K$, $a < b$ with $\eta_1 > 0$. Assume that $f : K \longrightarrow \mathbb{R}$ is a differentiable function on K° and f' is integrable on $[m\varphi(b), m\varphi(b)+\eta_1]$. Then, we have*

$$\frac{f(m\varphi(b))+f(m\varphi(b)+\eta_1)}{2}-\frac{1}{\eta_1}\int_{m\varphi(b)}^{m\varphi(b)+\eta_1}f(x)dx \tag{3.1}$$

$$=\frac{\eta_1}{4}\left\{\int_0^1 tf'\left(m\varphi(b)+\frac{(t+1)\eta_1}{2}\right)dt-\int_0^1(1-t)f'\left(m\varphi(b)+\frac{t\eta_1}{2}\right)dt\right\},$$

where

$$\eta_1=\eta(\varphi(a),\varphi(b),m).$$

Proof. Denote

$$I=\frac{\eta_1}{4}(I_1-I_2),$$

where

$$I_1=\int_0^1 tf'\left(m\varphi(b)+\frac{(t+1)\eta_1}{2}\right)dt \text{ and } I_2=\int_0^1(1-t)f'\left(m\varphi(b)+\frac{t\eta_1}{2}\right)dt.$$

By integration by parts, we get

$$I_1=\frac{2tf\left(m\varphi(b)+\frac{(t+1)\eta_1}{2}\right)}{\eta_1}\Bigg|_0^1-\frac{2}{\eta_1}\int_0^1 f\left(m\varphi(b)+\frac{(t+1)\eta_1}{2}\right)dt$$

$$=\frac{2f(m\varphi(b)+\eta_1)}{\eta_1}-\frac{4}{\eta_1^2}\int_{m\varphi(b)+\frac{\eta_1}{2}}^{m\varphi(b)+\eta_1}f(x)dx.$$

Similarly

$$I_2 = -\frac{2f(m\varphi(b))}{\eta_1} + \frac{4}{\eta_1^2}\int_{m\varphi(b)}^{m\varphi(b)+\frac{\eta_1}{2}} f(x)dx.$$

Subtracting I_2 from I_1 and multiplying the result by the factor $\frac{\eta_1}{4}$ we get equality (3.1). ∎

Using Lemma 3.1, the following results can be obtained for the corresponding version for power of the absolute value of the first derivative.

THEOREM 3.2. *Let $\varphi : I \longrightarrow \mathbb{R}$ be a continuous function. Let $A \subseteq \mathbb{R}$ be an open m-invex subset with respect to $\eta : A \times A \times (0,1] \longrightarrow \mathbb{R}$ for some fixed $s \in [0,1)$, $m \in (0,1]$ and let $\varphi(a), \varphi(b) \in A$, $a < b$ with $\eta_1 > 0$. Assume that $f : A \longrightarrow \mathbb{R}$ is a differentiable function on A°. If $|f'|^q$ is a generalized (s,m,φ)-preinvex Godunova-Levin function of the second kind on $[m\varphi(b), m\varphi(b)+\eta_1]$, $q > 1$, $p^{-1}+q^{-1} = 1$, then we have*

$$\begin{aligned}(3.2)\qquad &\left|\frac{f(m\varphi(b)) + f(m\varphi(b)+\eta_1)}{2} - \frac{1}{\eta_1}\int_{m\varphi(b)}^{m\varphi(b)+\eta_1} f(x)dx\right| \\ &\leq \frac{1}{(p+1)^{1/p}}\left(\frac{2^s}{1-s}\right)^{\frac{1}{q}}\frac{|\eta_1|}{4} \\ &\times\left\{\left[(2^{1-s}-1)|f'(\varphi(a))|^q + m|f'(\varphi(b))|^q\right]^{\frac{1}{q}}\right. \\ &\left.+\left[|f'(\varphi(a))|^q + m(2^{1-s}-1)|f'(\varphi(b))|^q\right]^{\frac{1}{q}}\right\},\end{aligned}$$

where $\eta_1 = \eta(\varphi(a), \varphi(b), m)$.

PROOF. Suppose that $q > 1$. Using Lemma 3.1, the fact that $|f'|^q$ is a generalized (s,m,φ)-preinvex Godunova-Levin function of the second kind and Hölder inequality, we have

$$\left|\frac{f(m\varphi(b)) + f(m\varphi(b)+\eta_1)}{2} - \frac{1}{\eta_1}\int_{m\varphi(b)}^{m\varphi(b)+\eta_1} f(x)dx\right|$$

$$\le \frac{|\eta_1|}{4}\left\{ \int_0^1 t\left|f'\left(m\varphi(b)+\frac{(t+1)\eta_1}{2}\right)\right|dt + \int_0^1 (1-t)\left|f'\left(m\varphi(b)+\frac{t\eta_1}{2}\right)\right|dt\right\}$$

$$\le \frac{|\eta_1|}{4}\left\{ \left(\int_0^1 t^p dt\right)^{\frac{1}{p}} \left(\int_0^1 \left|f'\left(m\varphi(b)+\frac{(t+1)\eta_1}{2}\right)\right|^q dt\right)^{\frac{1}{q}} + \left(\int_0^1 (1-t)^p dt\right)^{\frac{1}{p}} \left(\int_0^1 \left|f'\left(m\varphi(b)+\frac{t\eta_1}{2}\right)\right|^q dt\right)^{\frac{1}{q}}\right\}$$

$$\le \frac{|\eta_1|}{4}\left\{ \frac{1}{(p+1)^{1/p}}\left[\int_0^1 \left(\frac{|f'(\varphi(a))|^q}{\left(\frac{t+1}{2}\right)^s} + \frac{m|f'(\varphi(b))|^q}{\left(1-\frac{t+1}{2}\right)^s}\right)dt\right]^{\frac{1}{q}} + \frac{1}{(p+1)^{1/p}}\left[\int_0^1 \left(\frac{|f'(\varphi(a))|^q}{\left(\frac{t}{2}\right)^s} + \frac{m|f'(\varphi(b))|^q}{\left(1-\frac{t}{2}\right)^s}\right)dt\right]^{\frac{1}{q}}\right\}$$

$$= \frac{1}{(p+1)^{1/p}}\left(\frac{2^s}{1-s}\right)^{\frac{1}{q}} \frac{|\eta_1|}{4} \times \left\{\left[(2^{1-s}-1)|f'(\varphi(a))|^q + m|f'(\varphi(b))|^q\right]^{\frac{1}{q}} + \left[|f'(\varphi(a))|^q + m(2^{1-s}-1)|f'(\varphi(b))|^q\right]^{\frac{1}{q}}\right\}.$$

The proof of Theorem 3.2 is completed. ∎

COROLLARY 3.3. *Under the conditions of Theorem 3.2, if* $m = 1$ *and* $\eta(\varphi(a), \varphi(b), 1) = \varphi(a) - \varphi(b)$, *we have*

$$\left|\frac{f(\varphi(a))+f(\varphi(b))}{2} - \frac{1}{\varphi(b)-\varphi(a)}\int_{\varphi(a)}^{\varphi(b)} f(x)dx\right| \le \frac{1}{(p+1)^{1/p}}\left(\frac{2^s}{1-s}\right)^{\frac{1}{q}}\left(\frac{\varphi(b)-\varphi(a)}{4}\right) \times \left\{\left[(2^{1-s}-1)|f'(\varphi(a))|^q + |f'(\varphi(b))|^q\right]^{\frac{1}{q}} + \left[|f'(\varphi(a))|^q + (2^{1-s}-1)|f'(\varphi(b))|^q\right]^{\frac{1}{q}}\right\}. \tag{3.3}$$

THEOREM 3.4. *Let* $\varphi : I \longrightarrow \mathbb{R}$ *be a continuous function. Let* $A \subseteq \mathbb{R}$ *be an open* m*-invex subset with respect to* $\eta : A \times A \times (0,1] \longrightarrow \mathbb{R}$ *for some fixed* $s \in [0,1)$, $m \in (0,1]$ *and let* $\varphi(a), \varphi(b) \in A$, $a < b$ *with* $\eta_1 > 0$. *Assume that* $f : A \longrightarrow \mathbb{R}$ *is a differentiable function on* A°. *If* $|f'|^q$ *is a generalized* (s,m,φ)*-preinvex Godunova-Levin function of the second kind on* $[m\varphi(b), m\varphi(b)+\eta_1]$, $q \geq 1$, *then we have*

$$\left|\frac{f(m\varphi(b))+f(m\varphi(b)+\eta_1)}{2}-\frac{1}{\eta_1}\int_{m\varphi(b)}^{m\varphi(b)+\eta_1} f(x)dx\right| \tag{3.4}$$
$$\leq \left(\frac{1}{2}\right)^{1-\frac{1}{q}}\left(\frac{2^s}{(s-1)(s-2)}\right)^{\frac{1}{q}}\frac{|\eta_1|}{4}$$
$$\times\Bigg\{\Big[(1-2^{1-s}s)|f'(\varphi(a))|^q+m|f'(\varphi(b))|^q\Big]^{\frac{1}{q}}$$
$$+\Big[|f'(\varphi(a))|^q+m(1-2^{1-s}s)|f'(\varphi(b))|^q\Big]^{\frac{1}{q}}\Bigg\},$$

where $\eta_1 = \eta(\varphi(a), \varphi(b), m)$.

PROOF. Using Lemma 3.1, the fact that $|f'|^q$ is a generalized (s,m,φ)-preinvex Godunova-Levin function of the second kind and the well-known power mean inequality, we have

$$\left|\frac{f(m\varphi(b))+f(m\varphi(b)+\eta_1)}{2}-\frac{1}{\eta_1}\int_{m\varphi(b)}^{m\varphi(b)+\eta_1} f(x)dx\right|$$

$$\leq \frac{|\eta_1|}{4}\Bigg\{\int_0^1 t\left|f'\left(m\varphi(b)+\frac{(t+1)\eta_1}{2}\right)\right|dt$$
$$+\int_0^1 (1-t)\left|f'\left(m\varphi(b)+\frac{t\eta_1}{2}\right)\right|dt\Bigg\}$$
$$\leq \frac{|\eta_1|}{4}\Bigg\{\left(\int_0^1 tdt\right)^{1-\frac{1}{q}}\left(\int_0^1 t\left|f'\left(m\varphi(b)+\frac{(t+1)\eta_1}{2}\right)\right|^q dt\right)^{\frac{1}{q}}$$
$$+\left(\int_0^1 (1-t)dt\right)^{1-\frac{1}{q}}\left(\int_0^1 (1-t)\left|f'\left(m\varphi(b)+\frac{t\eta_1}{2}\right)\right|^q dt\right)^{\frac{1}{q}}\Bigg\}$$
$$\leq \frac{|\eta_1|}{4}\Bigg\{\left(\frac{1}{2}\right)^{1-\frac{1}{q}}\left[\int_0^1 t\left(\frac{|f'(\varphi(a))|^q}{\left(\frac{t+1}{2}\right)^s}+\frac{m|f'(\varphi(b))|^q}{\left(1-\frac{t+1}{2}\right)^s}\right)dt\right]^{\frac{1}{q}}$$

$$+\left(\frac{1}{2}\right)^{1-\frac{1}{q}}\left[\int_0^1 (1-t)\left(\frac{|f'(\varphi(a))|^q}{\left(\frac{t}{2}\right)^s}+\frac{m|f'(\varphi(b))|^q}{\left(1-\frac{t}{2}\right)^s}\right)dt\right]^{\frac{1}{q}}\Bigg\}$$

$$=\left(\frac{1}{2}\right)^{1-\frac{1}{q}}\left(\frac{2^s}{(s-1)(s-2)}\right)^{\frac{1}{q}}\frac{|\eta_1|}{4}$$

$$\times\left\{\left[(1-2^{1-s}s)|f'(\varphi(a))|^q+m|f'(\varphi(b))|^q\right]^{\frac{1}{q}}\right.$$

$$\left.+\left[|f'(\varphi(a))|^q+m(1-2^{1-s}s)|f'(\varphi(b))|^q\right]^{\frac{1}{q}}\right\}.$$

The proof of Theorem 3.4 is completed. □

Corollary 3.5. *Under the conditions of Theorem 3.4, if $m = 1$ and $\eta(\varphi(a), \varphi(b), 1) = \varphi(a) - \varphi(b)$, we have*

$$\left|\frac{f(\varphi(a))+f(\varphi(b))}{2}-\frac{1}{\varphi(b)-\varphi(a)}\int_{\varphi(a)}^{\varphi(b)} f(x)dx\right| \tag{3.5}$$

$$\leq\left(\frac{1}{2}\right)^{1-\frac{1}{q}}\left(\frac{2^s}{(s-1)(s-2)}\right)^{\frac{1}{q}}\left(\frac{\varphi(b)-\varphi(a)}{4}\right)$$

$$\times\left\{\left[(1-2^{1-s}s)|f'(\varphi(a))|^q+|f'(\varphi(b))|^q\right]^{\frac{1}{q}}\right.$$

$$\left.+\left[|f'(\varphi(a))|^q+(1-2^{1-s}s)|f'(\varphi(b))|^q\right]^{\frac{1}{q}}\right\}.$$

4. Applications to special means

In the following we give certain generalizations of some notions for a positive valued function of a positive variable.

Definition 4.1 (see [1]). *A function $M : \mathbb{R}_+^2 \longrightarrow \mathbb{R}_+$ is called a Mean function if it has the following properties:*

1. *Homogeneity: $M(ax, ay) = aM(x, y)$, for all $a > 0$,*
2. *Symmetry: $M(x, y) = M(y, x)$,*
3. *Reflexivity: $M(x, x) = x$,*
4. *Monotonicity: If $x \leq x'$ and $y \leq y'$, then $M(x, y) \leq M(x', y')$,*
5. *Internality: $\min\{x, y\} \leq M(x, y) \leq \max\{x, y\}$.*

We consider some means for arbitrary positive real numbers α, β ($\alpha \neq \beta$).

1. The arithmetic mean:

$$A := A(\alpha, \beta) = \frac{\alpha+\beta}{2}.$$

2. The geometric mean:

$$G := G(\alpha, \beta) = \sqrt{\alpha\beta}.$$

3. The harmonic mean:

$$H := H(\alpha, \beta) = \frac{2}{\frac{1}{\alpha} + \frac{1}{\beta}}.$$

4. The power mean:

$$P_r := P_r(\alpha, \beta) = \left(\frac{\alpha^r + \beta^r}{2}\right)^{\frac{1}{r}}, \quad r \geq 1.$$

5. The identric mean:

$$I := I(\alpha, \beta) = \begin{cases} \frac{1}{e}\left(\frac{\beta^\beta}{\alpha^\alpha}\right), & \alpha \neq \beta; \\ \alpha, & \alpha = \beta. \end{cases}$$

6. The logarithmic mean:

$$L := L(\alpha, \beta) = \frac{\beta - \alpha}{\ln|\beta| - \ln|\alpha|}.$$

7. The generalized log-mean:

$$L_p := L_p(\alpha, \beta) = \left[\frac{\beta^{p+1} - \alpha^{p+1}}{(p+1)(\beta - \alpha)}\right]^{\frac{1}{p}}; \quad p \in \mathbb{R} \setminus \{-1, 0\}.$$

8. The weighted p-power mean:

$$M_p \begin{pmatrix} \alpha_1, & \alpha_2, & \cdots & ,\alpha_n \\ u_1, & u_2, & \cdots & ,u_n \end{pmatrix} = \left(\sum_{i=1}^{n} \alpha_i u_i^p\right)^{\frac{1}{p}}$$

where $0 \leq \alpha_i \leq 1$, $u_i > 0\,(i = 1, 2, \ldots, n)$ with $\sum_{i=1}^{n} \alpha_i = 1$.

It is well known that L_p is monotonic nondecreasing over $p \in \mathbb{R}$ with $L_{-1} := L$ and $L_0 := I$. In particular, we have the following inequality $H \leq G \leq L \leq I \leq A$. Now, let a and b be positive real numbers such that $a < b$. Consider the function $M := M(\varphi(a), \varphi(b)) : [\varphi(a), \varphi(a) + \eta(\varphi(b), \varphi(a))] \times [\varphi(a), \varphi(a) + \eta(\varphi(b), \varphi(a))] \longrightarrow \mathbb{R}_+$, which is one of the above mentioned means and $\varphi : I \longrightarrow \mathbb{R}$ be a continuous function, therefore one can obtain various inequalities using the results of Section 3 for these means as follows.

Replacing $\eta(\varphi(x), \varphi(y), m)$ with $\eta(\varphi(x), \varphi(y))$ and setting $\eta(\varphi(a), \varphi(b)) = M(\varphi(a), \varphi(b))$ for $m = 1$ in (3.2) and (3.4), one can obtain the following interesting inequalities involving means:

$$\left|\frac{f(\varphi(b))+f(\varphi(b)+M(\varphi(a),\varphi(b)))}{2}-\frac{1}{M(\varphi(a),\varphi(b))}\int_{\varphi(b)}^{\varphi(b)+M(\varphi(a),\varphi(b))}f(x)dx\right|$$

$$\tag{4.1} \begin{aligned}&\leq \frac{1}{(p+1)^{1/p}}\left(\frac{2^s}{1-s}\right)^{\frac{1}{q}}\frac{M(\varphi(a),\varphi(b))}{4}\\ &\times\left\{\left[(2^{1-s}-1)|f'(\varphi(a))|^q+|f'(\varphi(b))|^q\right]^{\frac{1}{q}}\right.\\ &\left.+\left[|f'(\varphi(a))|^q+(2^{1-s}-1)|f'(\varphi(b))|^q\right]^{\frac{1}{q}}\right\},\end{aligned}$$

$$\left|\frac{f(\varphi(b))+f(\varphi(b)+M(\varphi(a),\varphi(b)))}{2}-\frac{1}{M(\varphi(a),\varphi(b))}\int_{\varphi(b)}^{\varphi(b)+M(\varphi(a),\varphi(b))}f(x)dx\right|$$

$$\tag{4.2} \begin{aligned}&\leq \left(\frac{1}{2}\right)^{1-\frac{1}{q}}\left(\frac{2^s}{(s-1)(s-2)}\right)^{\frac{1}{q}}\frac{M(\varphi(a),\varphi(b))}{4}\\ &\times\left\{\left[(1-2^{1-s}s)|f'(\varphi(a))|^q+|f'(\varphi(b))|^q\right]^{\frac{1}{q}}\right.\\ &\left.+\left[|f'(\varphi(a))|^q+(1-2^{1-s}s)|f'(\varphi(b))|^q\right]^{\frac{1}{q}}\right\}.\end{aligned}$$

Letting $M(\varphi(a),\varphi(b)) = A, G, H, P_r, I, L, L_p, M_p$ in (4.1) and (4.2), we get the inequalities involving means for a particular choice of a differentiable generalized $(s,1,\varphi)$-preinvex Godunova-Levin functions f of the second kind. The details are left to the interested reader.

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Nejednakosti Hermite-Hadamardovog tipa za poopćene (s, m, φ)-preinveksne Godunova-Levinove funkcije

Artion Kashuri i Rozana Liko

Sažetak. U ovom članku uvodi se nova klasa poopćenih (s, m, φ)-preinveksnih Godunova-Levinovih funkcija druge vrste. Dane su neke lijeve nejednakosti za kvadraturne formule Gauss-Jacobijevog tipa te neke nejednakosti Hermite-Hadamardovog tipa za poopćene (s, m, φ)-preinveksne Godunova-Levinove funkcije druge vrste preko klasičnih integrala. Na kraju članka dane su neke primjene na specijalne sredine.

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