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WEIGHTED HARDY-TYPE INEQUALITIES INVOLVING FRACTIONAL CALCULUS OPERATORS

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Abstract. The aim of this paper is to give a new class of general weighted Hardy-type inequalities involving an arbitrary convex function with some applications of generalized fractional calculus convolutive operators which contain Gauss-hypergeometric function, generalized Mittag-Leffler function and Hilfer fractional derivative operator, in the kernel.

1. Introduction

In recent years many mathematicians gave generalizations and improvements of Hardy-type inequalities. They discover important and useful Hardy-type integral inequalities for convex functions which has great importance in mathematical analysis. The Hardy inequality has fundamental importance in the mathematical analysis and lot of rich literature and informations concerning Hardy-type inequalities have been published (see for example [1–3, 5, 11–13]).

Let $(\Omega_1, \Sigma_1, \mu_1)$ and $(\Omega_2, \Sigma_2, \mu_2)$ be measure spaces with σ-finite measures and A_k be an integral operator defined by

$$A_k f(x) := \frac{1}{K(x)} \int_{\Omega_2} k(x,y) f(y) d\mu_2(y), \tag{1.1}$$

where $k : \Omega_1 \times \Omega_2 \to \mathbb{R}$ is measurable and non-negative kernel, f is measurable function on Ω_2, and

$$0 < K(x) := \int_{\Omega_2} k(x,y) d\mu_2(y), \qquad x \in \Omega_1. \tag{1.2}$$

The following results of this section are given in [3].

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THEOREM 1.1. *Let $0 < p \leq q < \infty$. Let $(\Omega_1, \Sigma_1, \mu_1)$ and $(\Omega_2, \Sigma_2, \mu_2)$ be measure spaces with σ-finite measures, u be a weight function on Ω_1, w be a μ_2- a.e. positive function on Ω_2, k be a non-negative measurable function on $\Omega_1 \times \Omega_2$, and K be defined on Ω_1 by (1.2). Suppose that $K(x) > 0$ for all $x \in \Omega_1$ and that the function $x \mapsto u(x)\left(\frac{k(x,y)}{K(x)}\right)^{\frac{q}{p}}$ is integrable on Ω_1 for each fixed $y \in \Omega_2$. Let Φ be a non-negative convex function on an interval $I \subseteq \mathbb{R}$. If*

$$A = \sup_{y\in\Omega_2} w^{\frac{-1}{p}}(y) \left(\int\limits_{\Omega_1} u(x) \left(\frac{k(x,y)}{K(x)}\right)^{\frac{q}{p}} d\mu_1(x) \right)^{\frac{1}{q}} < \infty,$$

then there exists a positive constant C, such that the inequality

$$\left(\int\limits_{\Omega_1} u(x)[\Phi\left(A_k f(x)\right)]^{\frac{q}{p}} d\mu_1(x) \right)^{\frac{1}{q}} \leq C \left(\int\limits_{\Omega_2} w(y)\Phi\left(f(y)\right) d\mu_2(y) \right)^{\frac{1}{p}} \tag{1.3}$$

holds for all measurable functions $f : \Omega_2 \to \mathbb{R}$ with values in I and $A_k f$ be defined by (1.1). *Moreover, if C is smallest constant for* (1.3) *to hold, then $C \leq A$.*

COROLLARY 1.2. *Let $-\infty < q \leq p < 0$, and let the assumption of Theorem* 1.1 *be satisfied with a positive convex function Φ. If*

$$B = \inf_{y\in\Omega_2} w^{\frac{-1}{p}}(y) \left(\int\limits_{\Omega_1} u(x) \left(\frac{k(x,y)}{K(x)}\right)^{\frac{q}{p}} d\mu_1(x) \right)^{\frac{1}{q}} < \infty,$$

then there exists a positive real constant C, such that the inequality

$$\left(\int\limits_{\Omega_1} u(x)[\Phi\left(A_k f(x)\right)]^{\frac{q}{p}} d\mu_1(x) \right)^{\frac{1}{q}} \geq C \left(\int\limits_{\Omega_2} w(y)\Phi\left(f(y)\right) d\mu_2(y) \right)^{\frac{1}{p}} \tag{1.4}$$

holds for all measurable functions $f : \Omega_2 \to \mathbb{R}$ with values in Ω_2. Moreover, if C is smallest constant for (1.4) *to hold, then $C \geq B$.*

THEOREM 1.3. *Let $1 < p \leq q < \infty$. Let $(\Omega_1, \Sigma_1, \mu_1)$ and $(\Omega_2, \Sigma_2, \mu_2)$ be measure spaces with σ-finite measures, u be a weight function on Ω_1, v be a measurable μ_2-a.e. positive function on Ω_2, k be a non-negative measurable function on $\Omega_1 \times \Omega_2$, and K be defined on Ω_1 by* (1.2). *Let $K(x) > 0$ for all $x \in \Omega_1$ and let the function $x \mapsto u(x)\left(\frac{k(x,y)}{K(x)}\right)^{q}$ be integrable on Ω_1 for each* *fixed* $y \in \Omega_2$. *Suppose that* $\Phi : I \to [0,\infty)$ *is a bijective convex function on an interval* $I \subseteq \mathbb{R}$. *If there exist a real parameter* $s \in (1,p)$ *and a positive measurable function* $V : \Omega_2 \to \mathbb{R}$ *such that*

$$A(s,V) \; = \; F(V,v) \sup_{y\in\Omega_2} V^{\frac{s-1}{p}}(y) \left(\int_{\Omega_1} u(x) \left(\frac{k(x,y)}{K(x)}\right)^q d\mu_1(x) \right)^{\frac{1}{q}} \; < \; \infty,$$

where

$$F(V,v) = \left(\int_{\Omega_2} V^{\frac{-p'(s-1)}{p}}(y) v^{1-p'}(y) d\mu_2(y) \right)^{\frac{1}{p'}},$$

then there is a positive real constant C, *such that the inequality*

$$\left(\int_{\Omega_1} u(x)\Phi^q \left(A_k f(x)\right) d\mu_1(x) \right)^{\frac{1}{q}} \le C \left(\int_{\Omega_2} v(y)\Phi^p \left(f(y)\right) d\mu_2(y) \right)^{\frac{1}{p}} \tag{1.5}$$

holds for all measurable functions $f : \Omega_2 \to \mathbb{R}$ *with values in* I *where* $A_k f$ *is defined on* Ω_1 *by* (1.1). *Moreover, if* C *is smallest constant for* (1.5) *to hold, then*

$$C \le \inf_{\substack{1<s<p \\ V>0}} A(s,V).$$

2. Hardy-type inequalities for fractional integral operator which contains Gauss-hypergeometric function

First we give the definition of fractional hypergeometric operator (see [4]).

Definition 2.1. *Let* $\alpha > 0, \mu > -1, \beta, \eta \in \mathbb{R}$. *Then the generalized fractional integral* $I_{a,t}^{\alpha,\beta,\eta,\mu}$ *of order* α, *for a real-valued continuous function f is defined by:*

$$I_{0,x}^{\alpha,\beta,\eta,\mu} f(x) = \frac{x^{-\alpha-\beta-2\mu}}{\Gamma(\alpha)} \int_0^x t^{\mu}(x-t)^{\alpha-1} {}_2F_1\left(\alpha+\beta+\mu, -\eta; \alpha; 1-\frac{t}{x}\right) f(t)dt, \;\; x \in [0,b], \tag{2.1}$$

where, the function ${}_2F_1(.\,,\,.\,,\,;\,.)$ *appearing as kernel for operator (2.1) is the Gaussian hypergeometric function defined by*

$${}_2F_1(a,b;c;t) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n n!} t^n,$$

and $(a)_n$ *is the Pochhammer symbol:* $(a)_n = a(a+1)...(a+n-1), (a)_0 = 1$.

The operator (2.1) includes Saigo, Riemann-Liouville and Erdélyi-Kober fractional integral operators i.e.

$$\begin{aligned} I_{0,x}^{\alpha,\beta,\eta} f(x) &= I_{0,x}^{\alpha,\beta,\eta,0} f(x) \\ &= \frac{x^{-\alpha-\beta}}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1} {}_2F_1\left(\alpha+\beta,-\eta;\alpha;1-\frac{t}{x}\right) f(t)dt,\ x\in[0,b]. \end{aligned}$$

$$R^\alpha f(x) = I_{0,x}^{\alpha,-\alpha,\eta} f(x) = \frac{1}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1} f(t)dt,\ x\in[0,b].$$

and

$$I^{\alpha,\eta} f(x) = I_{0,x}^{\alpha,0,\eta} f(x) = \frac{x^{-\alpha-\eta}}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1} t^\eta f(t)dt,\ x\in[0,b].$$

Our first result for generalized fractional integral operator is given in the upcoming theorem.

THEOREM 2.2. *Let* $0<p\le q<\infty$, $s>1$, $\alpha>0$, $\mu>-1$, $\beta,\eta\in\mathbb{R}$. *Let* u *be a weight function on* $(0,b)$, w *be a.e. positive function on* $(0,b)$. *If*

$$A = \sup_{t\in(0,b)} w^{\frac{-1}{p}}(t) \left(\frac{\Gamma(1-\beta)\Gamma(\alpha+\mu+1+\eta)}{\Gamma(\mu+1)\Gamma(1-\beta+\eta)}\right)^{\frac{1}{p}} \tag{2.2}$$

$$\times\left(\int_t^b u(x)\left(x^{-\alpha-\mu} {}_2F_1\left(\alpha+\beta+\mu,-\eta,\alpha;1-\frac{t}{x}\right) t^\mu (x-t)^{\alpha-1}\right)^{\frac{q}{p}} dx\right)^{\frac{1}{q}} < \infty.$$

Then there exists a positive constant C, *such that the inequality*

$$\left(\int_0^b u(x)\left(\frac{I_{0,x}^{\alpha,\beta,\eta,\mu} f(x)\Gamma(1-\beta)\Gamma(\alpha+\mu+1+\eta)}{x^{-\mu-\beta}\Gamma(\mu+1)\Gamma(1-\beta+\eta)}\right)^{\frac{sq}{p}} dx\right)^{\frac{1}{q}} \tag{2.3}$$

$$\le C\left(\int_0^b w(t) f^s(t)dt\right)^{\frac{1}{p}}$$

holds for all measurable functions $f: I\to\mathbb{R}$ *with values in* I. *Moreover, if* C *is the smallest constant for* (2.3) *to hold, then* $C\le A$.

PROOF. Applying Theorem 1.1 with $\Omega_1=\Omega_2=(0,b)$, $d\mu_1(x)=dx$, $d\mu_2(t)=dt$,

$$\hat{k}(x,t) = \begin{cases} \frac{x^{-\alpha-\beta-2\mu}}{\Gamma(\alpha)} {}_2F_1(\alpha+\beta+\mu,-\eta,\alpha;1-\frac{t}{x}) t^\mu (x-t)^{\alpha-1}, & 0\le t\le x; \\ 0, & x<t\le b, \end{cases}$$

and we calculate $\hat{K}(x)$ as:

$$\hat{K}(x) = \frac{1}{\Gamma(\alpha)} \int_0^x x^{-\alpha-\beta-2\mu} {}_2F_1\left(\alpha+\beta+\mu, -\eta, \alpha; 1-\frac{t}{x}\right) t^{\mu}(x-t)^{\alpha-1} dt \tag{2.4}$$

Substituting $1 - \frac{t}{x} = y$ and using the formula given in [6, page 813], i.e.

$$\int_0^1 x^{\gamma-1}(1-x)^{\rho-1} F(\alpha, \beta; \gamma; x) dx = \frac{\Gamma(\gamma)\Gamma(\rho)\Gamma(\gamma+\rho-\alpha-\beta)}{\Gamma(\gamma+\rho-\alpha)\Gamma(\gamma+\rho-\beta)},$$

we get

$$\hat{K}(x) = \frac{x^{-\mu-\beta}\Gamma(\mu+1)\Gamma(1-\beta+\eta)}{\Gamma(1-\beta)\Gamma(\alpha+\mu+1+\eta)}. \tag{2.5}$$

So that $A_k f(x)$ becomes

$$A_k f(x) = \frac{I_{0,x}^{\alpha,\beta,\eta,\mu} f(x)\Gamma(1-\beta)\Gamma(\alpha+\mu+1+\eta)}{x^{-\mu-\beta}\Gamma(\mu+1)\Gamma(1-\beta+\eta)}.$$

Then the inequality (1.3) takes the form

$$\left(\int_0^b u(x)\left[\Phi\left(\frac{I_{0,x}^{\alpha,\beta,\eta,\mu} f(x)\Gamma(1-\beta)\Gamma(\alpha+\mu+1+\eta)}{x^{-\mu-\beta}\Gamma(\mu+1)\Gamma(1-\beta+\eta)}\right)\right]^{\frac{q}{p}} dx\right)^{\frac{1}{q}} \tag{2.6}$$
$$\leq C\left(\int_0^b w(t)\Phi(f(t))dt\right)^{\frac{1}{p}}.$$

If we choose the function $\Phi : \mathbb{R}^+ \to \mathbb{R}$ defined by $\Phi(x) = x^s$, $s > 1$, then (2.6) reduces to (2.3). ∎

Corollary 2.3. *Let* $-\infty < q \leq p < 0$, *and let the assumptions of the Theorem 2.2 be satisfied. If*

$$B = \left(\frac{\Gamma(1-\beta)\Gamma(\alpha+\mu+1+\eta)}{\Gamma(\mu+1)\Gamma(1-\beta+\eta)}\right)^{\frac{1}{p}} \inf_{t\in(0,b)} w^{\frac{-1}{p}}(t)$$
$$\times\left(\int_t^b u(x)\left(x^{-\alpha-\mu} {}_2F_1\left(\alpha+\beta+\mu, -\eta, \alpha; 1-\frac{t}{x}\right) t^{\mu}(x-t)^{\alpha-1}\right)^{\frac{q}{p}} dx\right)^{\frac{1}{q}} < \infty,$$

then there exists a positive constant C, *such that the inequality*

$$\left(\int_0^b u(x) \left(\frac{I_{0,x}^{\alpha,\beta,\eta,\mu} f(x)\Gamma(1-\beta)\Gamma(\alpha+\mu+1+\eta)}{x^{-\mu-\beta}\Gamma(\mu+1)\Gamma(1-\beta+\eta)} \right)^{\frac{sq}{p}} dx \right)^{\frac{1}{q}} \geq C \left(\int_0^b w(t) f^s(t) dt \right)^{\frac{1}{p}} \tag{2.7}$$

holds for all measurable functions $f : I \to \mathbb{R}$ *with values in* I. *Moreover, if* C *is the smallest constant for* (2.7) *to hold, then* $C \geq B$.

THEOREM 2.4. *Let* $1 < p \leq q < \infty$, *and* $\alpha > 0$, $\mu > -1$, $\beta, \eta \in \mathbb{R}$. *Let* u *be a weight function on* $(0, b)$, w *be a.e. positive function on* $(0, b)$. *Suppose that* $\Phi : I \to [0, \infty)$ *is a bijective convex function on an interval* $I \subseteq \mathbb{R}$. *If there exist a real parameter* $s \in (1, p)$ *and* $V : (0, b) \to \mathbb{R}$ *is a positive measurable function such that*

$$A(s, V) = \left(\frac{\Gamma(1-\beta)\Gamma(\alpha+\mu+1+\eta)}{\Gamma(\mu+1)\Gamma(1-\beta+\eta)} \right) \left(\int_0^b V^{\frac{-p'(s-1)}{p}}(t) v^{1-p'}(t) dt \right)^{\frac{1}{p'}} \sup_{t \in (0,b)} V^{\frac{s-1}{p}}(t)$$

$$\times \left(\int_t^b u(x) \left(x^{-\alpha-\mu} {}_2F_1 \left(\alpha+\beta+\mu, -\eta, \alpha; 1 - \frac{t}{x} \right) t^{\mu} (x-t)^{\alpha-1} \right)^q dx \right)^{\frac{1}{q}} < \infty,$$

then there exists a positive constant C, *such that the inequality*

$$\left(\int_0^b u(x) \left[\Phi \left(\frac{I_{0,x}^{\alpha,\beta,\eta,\mu} f(x)\Gamma(1-\beta)\Gamma(\alpha+\mu+1+\eta)}{x^{-\mu-\beta}\Gamma(\mu+1)\Gamma(1-\beta+\eta)} \right) \right]^q dx \right)^{\frac{1}{q}} \leq C \left(\int_0^b v(t) \Phi^p(f(t)) dt \right)^{\frac{1}{p}}. \tag{2.8}$$

holds. Moreover, if C *is the smallest constant for* (2.8) *to hold, then*

$$C \leq \inf_{\substack{1<s<p \\ V>0}} A(s, V).$$

PROOF. Applying Theorem 1.3 with the same techniques as used in Theorem 2.2. ∎

COROLLARY 2.5. *If we take* $\mu = 0$ *in Theorem 2.2, Corollary 2.3, Theorem 2.4, we get the results for Saigo fractional derivative.*

Corollary 2.6. *If along $\mu = 0$ we take $\beta = -\alpha$ in Theorem 2.2, Corollary 2.3, Theorem 2.4, we get the inequalities for Riemann-Liouville's fractional integral which are given in [8] and [7].*

Corollary 2.7. *If we take $\beta = 0$ and $\mu = 0$ in Theorem 2.2, Corollary 2.3, Theorem 2.4, we get the inequalities for Erdélyi-Kober fractional integral operator.*

3. Hardy-type inequalities for fractional integral operator which contains generalized Mittag-Leffler functions

Now we give the definition of Mittag-Leffler function [14] and fractional integral operator involving generalized Mittag-Leffler function appearing in the kernel [17].

Definition 3.1. *Let $\alpha, \beta, \gamma, \delta \in \mathbb{C}$, $min\{\Re(\alpha), \Re(\beta), \Re(\gamma), \Re(\delta)\} > 0$, $m, k > 0$ and $k < \Re\alpha + m$. Then the generalized Mittag-Leffler function defined in [17] is given by*

$$E^{\gamma,\delta,k}_{\alpha,\beta,m}(z) = \sum_{n=0}^{\infty} \frac{(\gamma)_{kn}}{\Gamma(\alpha n + \beta)} \frac{z^n}{(\delta)_{mn}}, \tag{3.1}$$

where $(\gamma)_n$ represents the Pochhammer symbol, defined by $(\gamma)_n = \gamma(\gamma-1)(\gamma-2)\ldots(\gamma-n+1)$. The function (3.1) represents all the previous generalizations of Mittag-Leffler function by setting

- $m = k = 1$, *it reduces to* $E^{\gamma,\delta}_{\alpha,\beta}(z) = \sum\limits_{n=0}^{\infty} \frac{(\gamma)_n}{\Gamma(\alpha n+\beta)} \frac{z^n}{(\delta)_n}$ *defined by Salim [16].*
- $\delta = m = 1$, *it represents* $E^{\gamma,k}_{\alpha,\beta}(z) = \sum\limits_{n=0}^{\infty} \frac{(\gamma)_{kn}}{\Gamma(\alpha n+\beta)} \frac{z^n}{n!}$ *which was introduced by A. K. Shukla and J. C. Prajapati in [18]. In [19] H. M. Srivastava and Z. Tomovski investigated the properties of this function and its existence for a wider set of parameters.*
- $\delta = m = k = 1$, *the operator (3.1) is defined by Prabhakar in [15] and is denoted as:* $E^{\gamma}_{\alpha,\beta}(z) = \sum\limits_{n=0}^{\infty} \frac{(\gamma)_n}{\Gamma(\alpha n+\beta)} \frac{z^n}{n!}$.
- $\gamma = \delta = m = k = 1$, *it reduces to Wiman's function presented in [21], moreover if $\beta = 1$, Mittag-Leffler function $E_\alpha(z)$ will be the result.*

Definition 3.2. *Let $\alpha, \beta, \gamma, \delta \in \mathbb{C}$, $min\{\Re(\alpha), \Re(\beta), \Re(\gamma), \Re(\delta)\} > 0$, $m, k > 0$ and $k < \Re\alpha + m$. For all $f \in L(a, b)$ we introduce an integral operator, which contain a Mittag-Leffler function (3.1) in the kernel*

$$\left(\varepsilon^{\gamma,\delta,k}_{\alpha,\beta,m,\omega;a^+} f\right)(x) = \int_a^x (x-t)^{\beta-1} E^{\gamma,\delta,k}_{\alpha,\beta,m}(\omega(x-t)^{\alpha}) f(t)dt, \tag{3.2}$$

which contains the generalized Mittag-Leffler function (3.1) in its kernel is investigated and its boundedness is proved under certain conditions.

THEOREM 3.3. *Let $0 < p \leq q < \infty$, and $\alpha, \beta, \gamma, \delta, m, k$ be as in Definition 3.2. Let u be a weight function on (a,b) and w be a.e. positive function on (a,b). Suppose that the function $x \mapsto u(x)\left(\frac{(x-t)^{\beta-1}E^{\gamma,\delta,k}_{\alpha,\beta,m}(\omega(x-t)^{\alpha})}{(x-a)^{\beta}E^{\gamma,\delta,k}_{\alpha,\beta+1,m}(\omega(x-a)^{\alpha})}\right)^{\frac{q}{p}}$ is integrable on (a,b) for each fixed $t \in (a,b)$. Let Φ be a non-negative convex function on an interval $I \subseteq \mathbb{R}$. If*

$$A = \sup_{t\in(a,b)} w^{\frac{-1}{p}}(t)\left(\int_t^b u(x)\left(\frac{(x-t)^{\beta-1}E^{\gamma,\delta,k}_{\alpha,\beta,m}(\omega(x-t)^{\alpha})}{(x-a)^{\beta}E^{\gamma,\delta,k}_{\alpha,\beta+1,m}(\omega(x-a)^{\alpha})}\right)^{\frac{q}{p}} dx\right)^{\frac{1}{q}} < \infty,$$

then there exists a positive constant C, such that the inequality

$$\left(\int_{\Omega_1} u(x)\left[\Phi\left(\frac{\left(\varepsilon^{\gamma,\delta,k}_{\alpha,\beta,m,\omega,a^+}f\right)(x)}{(x-a)^{\beta}E^{\gamma,\delta,k}_{\alpha,\beta+1,m}(\omega(x-a)^{\alpha})}\right)\right]^{\frac{q}{p}} dx\right)^{\frac{1}{q}} \leq C\left(\int_{\Omega_2} w(t)\Phi\left(f(t)\right)dt\right)^{\frac{1}{p}} \tag{3.3}$$

holds for all measurable functions $f : (a,b) \to \mathbb{R}$ with values in I. Moreover, if C is smallest constant for (3.3) *to hold, then $C \leq A$.*

PROOF. Applying Theorem 1.1 with $\Omega_1 = \Omega_2 = (a,b)$, $d\mu_1(x) = dx$, $d\mu_2(t) = dt$, where

$$\tilde{k}(x,t) = \begin{cases} (x-t)^{\beta-1}E^{\gamma,\delta,k}_{\alpha,\beta,m}(\omega(x-t)^{\alpha}), & a \leq t \leq x\,; \\ 0, & x < t \leq b\,, \end{cases}$$

(see Lemma 3.2 in [10]), and

$$\tilde{K}(x) = \int_a^x (x-t)^{\beta-1}E^{\gamma,\delta,k}_{\alpha,\beta,m}\left(\omega\left(x-t\right)^{\alpha}\right)dt = (x-a)^{\beta}\,E^{\gamma,\delta,k}_{\alpha,\beta+1,p}\left(\omega\left(x-a\right)^{\alpha}\right),$$

we get inequality (3.3). ∎

COROLLARY 3.4. *Let $-\infty < q \leq p < 0$, and let the assumption of Theorem* 3.3 *be satisfied with a positive convex function Φ. If*

$$B = \inf_{y\in(a,b)} w^{\frac{-1}{p}}(y)\left(\int_t^b u(x)\left(\frac{(x-t)^{\beta-1}E^{\gamma,\delta,k}_{\alpha,\beta,m}(\omega(x-t)^{\alpha})}{(x-a)^{\beta}E^{\gamma,\delta,k}_{\alpha,\beta+1,p}(\omega(x-a)^{\alpha})}\right)^{\frac{q}{p}} dx\right)^{\frac{1}{q}} < \infty,$$

then there exists a positive real constant C, such that the inequality

$$\left(\int_a^b u(x)\left[\Phi\left(\frac{\left(\varepsilon^{\gamma,\delta,k}_{\alpha,\beta,m,\omega,a^+}f\right)(x)}{(x-a)^{\beta}E^{\gamma,\delta,k}_{\alpha,\beta+1,p}(\omega(x-a)^{\alpha})}\right)\right]^{\frac{q}{p}}dx\right)^{\frac{1}{q}} \geq C\left(\int_a^b w(y)\Phi\left(f(y)\right)dy\right)^{\frac{1}{p}} \tag{3.4}$$

holds for all measurable functions $f:(a,b)\to\mathbb{R}$ with values in (a,b). Moreover, if C is smallest constant for (3.4) to hold, then $C\geq B$.

Theorem 3.5. *Let $1<p\leq q<\infty$, and $\alpha,\beta,\gamma,\delta,m,k$ be as in Definition 3.2. Let u be a weight function on (a,b), and v be a measurable positive function on (a,b). Suppose that the function $x\mapsto u(x)\left(\frac{\left(\varepsilon^{\gamma,\delta,k}_{\alpha,\beta,m,\omega,a^+}f\right)(x)}{(x-a)^{\beta}E^{\gamma,\delta,k}_{\alpha,\beta+1,m}(\omega(x-a)^{\alpha})}\right)^{q}$ be integrable on (a,b) for each fixed $t\in(a,b)$. Suppose that $\Phi:I\to[0,\infty)$ is a bijective convex function on an interval $I\subseteq\mathbb{R}$. If there exist a real parameter $s\in(1,p)$ and a positive measurable function $V:(a,b)\to\mathbb{R}$ such that*

$$A(s,V)=\left(\int_a^b V^{\frac{-p'(s-1)}{p}}(t)v^{1-p'}(t)dt\right)^{\frac{1}{p'}}$$

$$\times\sup_{t\in(a,b)}V^{\frac{s-1}{p}}(t)\left(\int_t^b u(x)\left(\frac{\left(\varepsilon^{\gamma,\delta,k}_{\alpha,\beta,m,\omega,a^+}f\right)(x)}{(x-a)^{\beta}E^{\gamma,\delta,k}_{\alpha,\beta+1,m}(\omega(x-a)^{\alpha})}\right)^{q}dx\right)^{\frac{1}{q}}<\infty,$$

then there is a positive real constant C, such that the inequality

$$\left(\int_a^b u(x)\Phi^q\left(\frac{\left(\varepsilon^{\gamma,\delta,k}_{\alpha,\beta,m,\omega,a^+}f\right)(x)}{(x-a)^{\beta}E^{\gamma,\delta,k}_{\alpha,\beta+1,m}(\omega(x-a)^{\alpha})}\right)dx\right)^{\frac{1}{q}} \leq C\left(\int_a^b v(t)\Phi^p\left(f(t)\right)dt\right)^{\frac{1}{p}} \tag{3.5}$$

holds for all measurable functions $f:(a,b)\to\mathbb{R}$ with values in I. Moreover, if C is smallest constant for (3.5) to hold, then

$$C\leq\inf_{\substack{1<s<p\\ V>0}}A(s,V).$$

Proof. Applying Theorem 1.3 with the same technique as used in Theorem 3.3. ∎

Remark 3.6. Here some special cases of above discussed results are given.

- If we take $m = k = 1$, in Theorem 3.3, Corollary 3.4 and in Theorem 3.5, the inequality reduces for $E^{\gamma,\delta}_{\alpha,\beta}(z) = \sum\limits_{n=0}^{\infty} \frac{(\gamma)_n}{\Gamma(\alpha n+\beta)} \frac{z^n}{(\delta)_n}$.
- If we take $\delta = m = 1$, in Theorem 3.3, Corollary 3.4 and in Theorem 3.5 the inequality reduces for $E^{\gamma,k}_{\alpha,\beta}(z) = \sum\limits_{n=0}^{\infty} \frac{(\gamma)_{kn}}{\Gamma(\alpha n+\beta)} \frac{z^n}{n!}$.
- If we take $\delta = m = k = 1$, in Theorem 3.3, Corollary 3.4 and in Theorem 3.5 the inequality reduces for $E^{\gamma}_{\alpha,\beta}(z) = \sum\limits_{n=0}^{\infty} \frac{(\gamma)_n}{\Gamma(\alpha n+\beta)} \frac{z^n}{n!}$.
- If we take $\gamma = \delta = m = k = 1$, in Theorem 3.3, Corollary 3.4 and in Theorem 3.5 the inequality reduces to Wiman's function, moreover if $\beta = 1$, Mittag-Leffler function $E_\alpha(z)$ will be the result.

4. Hardy-type inequalities for Hilfer fractional derivative operator

Let $x > 0$. By $L^1(0,x)$ we denote the space of all Lebesgue integrable functions on the interval $(0,x)$. For any $f \in L^1(0,x)$ the Riemann-Liouville fractional integral of f of order ν is defined by

$$(4.1)\quad (I^{\nu}_{a_+} f)(s) = \frac{1}{\Gamma(\alpha)} \int_a^s (x-t)^{\nu-1} f(t)dt = (f * K_\nu)(s), \quad s \in [0,x] \quad (\nu > 0),$$

where $K_\nu(s) = \frac{s^{\nu-1}}{\Gamma(\nu)}$. The integral on the right side of (4.1) exists for almost $s \in [0,x]$ and $I^{\nu}_{a_+} f \in L^1(0,x)$. The Riemann-Liouville fractional derivative of $f \in L^1(0,x)$ of order ν is defined by

$$(D^{\nu}_{a_+} f)(s) = \left(\frac{d}{dx}\right)^n (I^{n-\nu}_{a_+} f)(x), (\nu > 0, n = [\nu + 1])$$

By $C^m[0,x]$ we denote the space of all functions on $[0,x]$ which have continuous derivatives up to order m, and $AC[0,x]$ is the space of all absolutely continuous functions on $[0,x]$. By $AC^m[0,x]$ we denote the space of all functions $f \in C^m[0,x]$ with $f^{(m-1)} \in AC[0,x]$. By $L_\infty(0,x)$ we denote the space of all measurable functions essentially bounded on $[0,x]$. Let $\mu > 0, m = [\mu]+1$ and $f \in AC^m[a,b]$. The Caputo derivative of order $\mu > 0$ is defined as

$$({}^C D^{\mu}_{a_+} f)(x) = \left(I^{m-\mu}_{a_+} \frac{d^m}{dx^m} f\right)(x) = \frac{1}{\Gamma(m-\mu)} \int_a^x (x-s)^{m-\mu-1} \frac{d^m}{dx^m} f(s)ds.$$

Let us recall the definition of Hilfer fractional derivative presented in [20].

Definition 4.1. *Let $f \in L^1[a,b], f * K_{(1-\nu)(1-\mu)} \in AC^1[a,b]$. The fractional derivative operator $D^{\mu,\nu}_{a+}$ of order $0 < \mu < 1$ and type $0 < \nu \leq 1$ with* *respect to $x \in [a,b]$ is defined by*

$$\left(D_{a+}^{\mu,\nu} f\right)(x) := I_{a+}^{\nu(1-\mu)} \frac{d}{dx}\left(I_{a+}^{(1-\nu)(1-\mu)} f(x)\right), \tag{4.2}$$

whenever the right hand side exists. The derivative (4.2) is usually called Hilfer fractional derivative.

The more general integral representation of equation (4.2) given in [7] define as: Let $f \in L^1[a,b], f * K_{(1-\nu)(n-\mu)} \in AC^n[a,b], n-1 < \mu < n, 0 < \nu \leq 1, n \in \mathbb{N}$ then the following equation holds true:

$$\left(D_{a+}^{\mu,\nu} f\right)(x) = \left(I_{a+}^{\nu(n-\mu)} \frac{d^n}{dx^n}\left(I_{a+}^{(1-\nu)(n-\mu)} f(x)\right)\right). \tag{4.3}$$

Specially for $\nu = 0$, $D_{a+}^{\mu,0} f = D_{a+}^{\mu} f$ is a Riemann- Liouvile fractional derivative of order μ, and for $\nu = 1$ it is a Caputo fractional derivative $D_{a+}^{\mu,1} f = {}^{C}D_{a+}^{\mu} f$ of order μ. Applying the properties of Riemann-Liouvile integral the relation (4.3) can be rewritten in the form:

$$\begin{aligned}\left(D_{a+}^{\mu,\nu} f\right)(x) &= \left(I_{a+}^{\nu(n-\mu)}\left(\left(D_{a+}^{n-(1-\nu)(n-\mu)} f\right)(x)\right)\right)\\ &= \frac{1}{\Gamma(\nu(n-\mu))}\int_a^x (x-t)^{\nu(n-\mu)-1}\left(\left(D_{a+}^{\mu+\nu(n-\mu)} f\right)(t)\right) dt. \end{aligned}\tag{4.4}$$

Theorem 4.2. *Let $0 < p \leq q < \infty$, $s > 1$. Let $f \in L^1[a,b]$ and the fractional derivative operator $D_{a+}^{\mu,\nu}$ of order $n-1 < \mu < n$ and type $0 < \nu \leq 1$. Let u be a weight function defined on (a,b), w be a.e. positive function on (a,b). If*

$$A = (\nu(n-\mu))^{\frac{1}{p}} \sup_{t\in(a,b)} w^{\frac{-1}{p}}(t)\left(\int_t^b u(x)\left(\frac{(x-t)^{\nu(n-\mu)-1}}{(x-a)^{\nu(n-\mu)}}\right)^{\frac{q}{p}} dx\right)^{\frac{1}{q}} < \infty,$$

then there exists a positive constant C, such that the inequality

$$\left(\int_a^b u(x)\left(\frac{\Gamma(\nu(n-\mu)+1)}{(x-a)^{\nu(n-\mu)}}\left(D_{a+}^{\mu,\nu} f\right)(x)\right)^{\frac{sq}{p}} dx\right)^{\frac{1}{q}} \leq C\left(\int_a^b w(t)\left(\left(D_{a+}^{\mu+\nu(n-\mu)} f\right)(t)\right)^{s} dt\right)^{\frac{1}{p}} \tag{4.5}$$

holds. Moreover, if C is the smallest constant for (4.5) to hold, then $C \leq A$.

PROOF. Applying Theorem 1.1 with $\Omega_1 = \Omega_2 = (a,b)$, $d\mu_1(x) = dx$, $d\mu_2(t) = dt$ with

$$\bar{k}(x,t) = \begin{cases} \frac{(x-t)^{\nu(n-\mu)-1}}{\Gamma(\nu(n-\mu))}, & a \le t \le x; \\ 0, & x < t \le b, \end{cases}$$

we get

$$\bar{K}(x) = \frac{(x-a)^{\nu(n-\mu)}}{\Gamma(\nu(n-\mu)+1)}. \tag{4.6}$$

The integral operator $A_k f(x)$ takes the form

$$A_k f(x) = \frac{\Gamma(\nu(n-\mu)+1)}{(x-a)^{\nu(n-\mu)}} \left(D_{a+}^{\mu,\nu} f\right)(x). \tag{4.7}$$

By using (4.7) in inequality (1.3) we have

$$\left(\int_a^b u(x) \left[\Phi\left(\frac{\Gamma(\nu(n-\mu)+1)}{(x-a)^{\nu(n-\mu)}} \left(D_{a+}^{\mu,\nu} f\right)(x) \right) \right]^{\frac{q}{p}} dx \right)^{\frac{1}{q}} \le C \left(\int_a^b w(y) \Phi\left(\left(D_{a+}^{\mu+\nu(n-\mu)} f \right)(t) \right) dy \right)^{\frac{1}{p}}. \tag{4.8}$$

For $s > 1$, the function $\Phi : \mathbb{R}^+ \to \mathbb{R}$ defined by $\Phi(x) = x^s$ reduced (4.8) in (4.5). ∎

COROLLARY 4.3. *Let $-\infty < q \le p < 0$, and let the assumption of Theorem 4.2 be satisfied with a positive convex function Φ. If*

$$B = \inf_{t \in (a,b)} w^{\frac{-1}{p}}(t) \left(\int_t^b u(x) \left(\frac{\nu(n-\mu)(x-t)^{\nu(n-\mu)-1}}{(x-a)^{\nu(n-\mu)}} \right)^{\frac{q}{p}} dx \right)^{\frac{1}{q}} < \infty,$$

then there exists a positive constant C, such that the inequality

$$\left(\int_a^b u(x) \left(\frac{\Gamma(\nu(n-\mu)+1)}{(x-a)^{\nu(n-\mu)}} \left(D_{a+}^{\mu,\nu} f\right)(x) \right)^{\frac{sq}{p}} dx \right)^{\frac{1}{q}} \ge C \left(\int_a^b w(t) \left(\left(D_{a+}^{\mu+\nu(n-\mu)} f \right)(t) \right)^s dt \right)^{\frac{1}{p}} \tag{4.9}$$

holds. Moreover, if C is the smallest constant for (4.9) to hold, then $C \ge B$.

THEOREM 4.4. *Let $1 < p \leq q < \infty$. Let $f \in L^1[a,b]$ and the fractional derivative operator $D_{a+}^{\mu,\nu}$ of order $n-1 < \mu < n$ and type $0 < \nu \leq 1$, u be a weight function defined on (a,b), w be a.e. positive function on (a,b). If there exist a real parameter $s \in (1,p)$ and a positive measurable function $V : (a,b) \to \mathbb{R}$ such that*

$$A(s,V) = \nu(n-\mu)\left(\int_a^b V^{\frac{-p'(s-1)}{p}}(t) v^{1-p'}(y)dt\right)^{\frac{1}{p'}}$$

$$\times \sup_{t\in(a,b)} V^{\frac{s-1}{p}}(y)\left(\int_t^b u(x)\left(\frac{(x-t)^{\nu(n-\mu)-1}}{(x-a)^{\nu(n-\mu)}}\right)^q dx\right)^{\frac{1}{q}} < \infty,$$

then there exists a positive constant C such that the inequality

$$\left(\int_a^b u(x)\left(\frac{\Gamma(\nu(n-\mu)+1)}{(x-a)^{\nu(n-\mu)}}\left(D_{a+}^{\mu,\nu}f\right)(x)\right)^{tq}dx\right)^{\frac{1}{q}} \tag{4.10}$$

$$\leq C\left(\int_a^b v(t)\left(\left(D_{a+}^{\mu+\nu(n-\mu)}f\right)(t)\right)^{tp}dt\right)^{\frac{1}{p}}$$

holds. Moreover, if C is the smallest constant for (4.10) *to hold, then*

$$C \leq \inf_{\substack{1<s<p\\ V>0}} A(s,V).$$

PROOF. Applying Theorem 1.3 with the same technique used in Theorem 4.2. ∎

REMARK 4.5. In particular if $\nu = 0$ in Theorem 3.3, Corollary 3.4 and in Theorem 3.5 the inequalities becomes for Riemann-Liouville fractional derivative of order μ and for $\nu = 1$ then results becomes for Caputo fractional derivative of order μ.

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Težinske nejednakosti Hardijevog tipe pomoću frakcijskog računa operatora

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Sažetak. Cilj ovog rada je izložiti novu generaliziranu klasu težinskih nejednakosti Hardijevog tipa, za klasu konveksnih funkcija, s primjenom generaliziranog frakcijskog računa konvolutivnih integralnih operatora, koje u svojoj jezgri sadrže Gaussovu hipergeometrijsku funkciju, generaliziranu funkciju Mittag-Lefflera, te Hilferov frakcijski diferencijalni operator.

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