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ON p–EXTENDED MATHIEU SERIES

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Abstract. Motivated by several generalizations of the well–known Mathieu series, the main object of this paper is to introduce new extension of generalized Mathieu series and to derive various integral representations of such series. Finally, master bounding inequality is established using the newly derived integral expression.

1. Introduction and motivation

The series of the form

$$S(r) = \sum_{n\geq 1} \frac{2n}{(n^2+r^2)^2}, \qquad r > 0,$$

is known in literature as Mathieu series. Émile Leonard Mathieu was the first who investigated such series in 1890 in his book [15]. A remarkable useful integral representation for $S(r)$ is given by Emersleben [7] in the following elegant form

$$S(r) = \frac{1}{r} \int_0^\infty \frac{x \sin(rx)}{\mathrm{e}^x - 1}\, \mathrm{d}x\,,$$

which can also be written in terms of the Riemann Zeta function $\zeta(s) = \sum_{n\geq 1} n^{-s}, s > 1$ as [4, p. 863, Eq. (2.3)](with replacing n by $n+1$)

$$S(r) = 2 \sum_{n\geq 0} (-1)^n \, (n+1)\, \zeta(2n+3)\, r^{2n}, \qquad |r| < 1. \tag{1.1}$$

The so–called generalized Mathieu series with a fractional power reads [2, p. 2, Eq. (1.6)] (also see [16, p. 181])

$$S_\mu(r) = \sum_{n\geq 1} \frac{2n}{(n^2+r^2)^{\mu+1}}, \qquad r > 0,\ \mu > 0; \tag{1.2}$$

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such series has been widely considered in mathematical literature (see e.g. papers by Diananda [5], Cerone and Lenard [2] and Pogány *et al.* [20]). Cerone and Lenard also gave a series representation of $S_\mu(r)$ in terms of the Riemann Zeta function [2, p. 3, Eq. (2.1)]

$$S_\mu(r) = 2\sum_{n\geq 0}(-1)^n \binom{\mu+n}{n}\zeta(2\mu+2n+1)\,r^{2n}, \qquad |r|<1\,; \tag{1.3}$$

in [2] was not mentioned the convergence region $|r|<1$. To show (1.3) it is enough to expand the summands in (1.2) into a binomial series for $r \in (-1,1)$ (compare [20, p. 72, Proposition 1]). Cerone and Lenard derived also the next integral expression [2, p. 3, Theorem 2.1] (also consult [16, p. 181, Eq. (1.3)])

$$S_\mu(r) = \frac{\sqrt{\pi}}{(2r)^{\mu-\frac12}\Gamma(\mu+1)}\int_0^\infty \frac{x^{\mu+\frac12}}{\mathrm{e}^x-1}J_{\mu-\frac12}(rx)\,\mathrm{d}x, \qquad \mu>0. \tag{1.4}$$

Motivated by the previous extension and by huge spectrum of other generalizations of the Mathieu series, the main aim of this paper is to study certain another types of, in new fashion generalized, Mathieu series.

Having in mind (1.3) let the *p–extended Mathieu series* be defined as

$$S_{\mu,p}(r) = 2\sum_{n\geq 0} r^{2n}(-1)^n\binom{\mu+n}{n}\zeta_p(2\mu+2n+1), \tag{1.5}$$

where $p\geq 0$; $\mu>0$, $|r|<1$ and ζ_p stands for the p–extended Riemann Zeta function [3]

$$\zeta_p(\alpha) = \frac{1}{\Gamma(\alpha)}\int_0^\infty \frac{x^{\alpha-1}\,\mathrm{e}^{-\frac{p}{x}}}{\mathrm{e}^x-1}\,\mathrm{d}x \tag{1.6}$$

defined for $\Re(p)>0$ or $p=0$ and $\Re(\alpha)>0$, which reduces to Riemann Zeta function when $p=0$. It is also important to quote that (1.5) reduces to (1.3) when $p=0$, while taking $\mu=1$ we yield (1.1).

2. Integral forms, integral transforms and series representations of $S_{\mu,p}(r)$

In this section we derive an integral expression for the p–extended Mathieu series $S_{\mu,p}(r)$. Then its various Mellin and Laplace transforms are exposed.

Theorem 2.1. *For all $\Re(p)>0$ or $p=0$, $\mu>0$ and $r>0$ the following integral representation for the extended generalized Matheiu series $S_{\mu,p}(r)$ holds true:*

$$S_{\mu,p}(r) = \frac{\sqrt{\pi}}{(2r)^{\mu-\frac12}\Gamma(\mu+1)}\int_0^\infty \frac{x^{\mu+\frac12}\mathrm{e}^{-\frac{p}{x}}}{\mathrm{e}^x-1}J_{\mu-\frac12}(rx)\,\mathrm{d}x\,. \tag{2.1}$$

Proof. Using the series representation of J_ν [24, p. 40]

$$J_\nu(x) = \sum_{n\geq 0} \frac{(-1)^n}{n!\,\Gamma(n+\nu+1)} \left(\frac{x}{2}\right)^{2n+\nu},$$

valid for all $\nu, x \in \mathbb{C}$ we can simplify an integral given in (2.1) as:

$$\mathcal{I} = \int_0^\infty \frac{x^{\mu+\frac{1}{2}}\mathrm{e}^{-\frac{p}{x}}}{e^t - 1} J_{\mu-\frac{1}{2}}(rx)\,\mathrm{d}x = \sum_{n\geq 0} \frac{(-1)^n(\frac{r}{2})^{\mu+2n-\frac{1}{2}}}{n!\Gamma(\mu+n+\frac{1}{2})} \int_0^\infty \frac{x^{2\mu+2n}\mathrm{e}^{-\frac{p}{x}}}{\mathrm{e}^x - 1}\,\mathrm{d}x$$

$$= \sum_{n\geq 0} \frac{(-1)^n(\frac{r}{2})^{\mu+2n-\frac{1}{2}}}{n!\Gamma(\mu+n+\frac{1}{2})}\,\Gamma(2\mu+2n+1)\,\zeta_p(2\mu+2n+1),$$

where in the last equality the definition of the p-extended Riemann Zeta function (1.6) was used.

By the Legendre duplication formula $\sqrt{\pi}\,\Gamma(2z) = 2^{2z-1}\Gamma(z)\Gamma(z+\frac{1}{2})$ we get

$$\mathcal{I} = \frac{2\,(2r)^{\mu-\frac{1}{2}}\Gamma(\mu+1)}{\sqrt{\pi}} \sum_{n\geq 0} r^{2n}(-1)^n \binom{\mu+n}{n} \zeta_p(2\mu+2n+1)$$

which leads to the desired result. □

Remark 2.2. The integral expression (2.1) one reduces to (1.4) when $p = 0$.

In what follows we derive Mellin and Laplace transforms of the newly constructed series $S_{\mu,p}(r)$.

The Mellin and Laplace transforms (respectively) of some suitably integrable function f with index s are defined by

$$\mathcal{M}_x\{f(x)\}(s) = \int_0^\infty x^{s-1}\, f(x)\,\mathrm{d}x, \qquad \mathcal{L}_x\{f(x)\}(s) = \int_0^\infty \mathrm{e}^{-sx}\, f(x)\,\mathrm{d}x.$$

provided that the corresponding integrals exist.

Theorem 2.3. *The Mellin transform of the extended generalized Matheiu series $S_{\mu,p}(r)$ read as follows:*

$$\mathcal{M}_p\{S_{\mu,p}(r)\}(s) = 2s\Gamma^2(s)\sum_{n\geq 0}(-r^2)^n\binom{\mu+n}{n}\binom{2\mu+2n+s}{2\mu+2n}\zeta(2\mu+2n+s+1)$$

in the range $|r| < 1$. Moreover, for $\mu > 0$; $0 < \Re(s) < \mu+1$ and $\Re(p) > 0$,

$$\mathcal{M}_r\{S_{\mu,p}(r)\}(s) = \mathrm{B}\left(\frac{s}{2}, \mu+1-\frac{s}{2}\right)\,\zeta_p(2\mu-s+1), \tag{2.2}$$

where $\mathrm{B}(x,y), \min\{\Re(x),\Re(y)\} > 0$ stands for the Euler Beta function.

Proof. Using the definition of the Mellin transform, we find from (1.5)

$$\begin{aligned}\mathcal{M}_p\left\{S_{\mu,p}(r)\right\}(s) &= \int_0^\infty p^{s-1} S_{\mu,p}(r)\,\mathrm{d}p\\ &= 2\sum_{n\geq 0} r^{2n}(-1)^n\binom{\mu+n}{n}\int_0^\infty p^{s-1}\zeta_p(2\mu+2n+1)\,\mathrm{d}p\\ &= 2\Gamma(s)\Gamma(s+1)\sum_{n\geq 0} r^{2n}(-1)^n\binom{\mu+n}{n}\\ &\qquad\times\binom{2\mu+2n+s}{2\mu+2n}\zeta(2\mu+2n+s+1),\end{aligned}$$

where in the last equality we used the formula [3, p. 1244, Eq. (3.6)]

$$\int_0^\infty p^{s-1}\zeta_p(\alpha)\,\mathrm{d}p = \frac{\Gamma(s)\Gamma(\alpha+s)}{\Gamma(\alpha)}\zeta(\alpha+s), \qquad \Re(\alpha)>0,\ \Re(s)>0.$$

Next, with the help of the Weber–Sonine integral [24, p. 391, Eq. 13.24(1)]

$$\int_0^\infty x^{\mu-\nu-1}J_\nu(x)\,\mathrm{d}x = \frac{\Gamma\left(\frac{\mu}{2}\right)}{2^{\nu-\mu+1}\Gamma\left(1+\nu-\frac{\mu}{2}\right)}, \qquad 0<\Re(\mu)<\Re(\nu)+\frac{1}{2},$$

the integral representation (2.1) derived in Theorem 2.1 and the definition of extended Riemman Zeta ζ_p, we find that

$$\begin{aligned}\mathcal{M}_r\left\{S_{\mu,p}(r)\right\}(s) &= \int_0^\infty r^{s-1}S_{\mu,p}(r)\,\mathrm{d}r\\ &= \frac{\sqrt{2\pi}\;2^{-\mu}}{\Gamma(\mu+1)}\int_0^\infty\frac{x^{\mu+\frac{1}{2}}\mathrm{e}^{-\frac{p}{x}}}{\mathrm{e}^x-1}\left(\int_0^\infty r^{s-\mu-\frac{1}{2}}J_{\mu-\frac{1}{2}}(rx)\,\mathrm{d}r\right)\mathrm{d}x\\ &= \frac{\sqrt{\pi}\,\Gamma\left(\frac{s}{2}\right)}{2^{2\mu-s}\Gamma(\mu+1)\,\Gamma\left(\mu+\frac{1-s}{2}\right)}\int_0^\infty\frac{x^{2\mu-s}\mathrm{e}^{-\frac{p}{x}}}{\mathrm{e}^x-1}\,\mathrm{d}x\\ &= \frac{\sqrt{\pi}\,\Gamma\left(\frac{s}{2}\right)\,\Gamma(2\mu-s+1)\,\zeta_p(2\mu-s+1)}{2^{2\mu-s}\Gamma(\mu+1)\,\Gamma\left(\mu+\frac{1-s}{2}\right)},\end{aligned}$$

which gives (2.2) with the help of the duplication formula for the Gamma function and the relation between Beta and Gamma functions. □

Theorem 2.4. *For the p–extended Mathieu series $S_{\mu,p}(r)$ we have the Laplace transform formula*

$$\mathcal{L}_r\{S_{\mu,p}(r)\}(x) = \frac{2}{x\Gamma(2\mu+1)}\int_0^\infty\frac{t^{2\mu}\mathrm{e}^{-\frac{p}{t}}}{\mathrm{e}^t-1}\,{}_2F_1\left[\begin{matrix}1,\frac{1}{2}\\ \mu+\frac{1}{2}\end{matrix}\,\middle|\,-\frac{t^2}{x^2}\right]\mathrm{d}t. \tag{2.3}$$

provided that the each member of (2.3) *exists.*

Proof. Using the Laplace transform formula [8, p. 49, Eq. 7.7.3(16)]

$$\mathcal{L}_x\{x^{\lambda-1}J_\nu(\rho x)\}(s) = \frac{\rho^\nu\Gamma(\nu+\lambda)}{2s^{\nu+\lambda}\Gamma(\nu+1)}\, {}_2F_1\left[\begin{array}{c} \frac12(\nu+\lambda), \frac12(\nu+\lambda+1) \\ \nu+1 \end{array} \middle| -\frac{\rho^2}{s^2}\right],$$

valid for all $|\Re(s)| > |\Im(\rho)|$, $\Re(\nu+\lambda) > 0$ and the integral representation (2.1), we get

$$\begin{aligned} \mathcal{L}_r\left\{S_{\mu,p}(r)\right\}(x) &= \int_0^\infty \mathrm{e}^{-xr}S_{\mu,p}(r)\,\mathrm{d}r \\ &= \frac{\sqrt{2\pi}\;2^{-\mu}}{\Gamma(\mu+1)}\int_0^\infty \frac{t^{\mu+\frac12}\mathrm{e}^{-\frac{p}{t}}}{\mathrm{e}^t-1}\left(\int_0^\infty \mathrm{e}^{-xr}r^{\frac12-\mu}J_{\mu-\frac12}(rt)\,\mathrm{d}r\right)\mathrm{d}t \\ &= \frac{\sqrt{\pi}\;2^{1-2\mu}}{x\Gamma(\mu+1)\Gamma(\mu+\frac12)}\int_0^\infty \frac{t^{2\mu}\mathrm{e}^{-\frac{p}{t}}}{\mathrm{e}^t-1}\, {}_2F_1\left[\begin{array}{c} 1, \frac12 \\ \mu+\frac12 \end{array}\middle| -\frac{t^2}{x^2}\right]\mathrm{d}t, \end{aligned}$$

which becomes (2.3) with the help of duplication formula. ∎

Remark 2.5. It is interesting to note when $p=0$, (2.2) and (2.3) reduce to known results in [6].

3. Master bounding inequality upon $S_{\mu,p}(r)$

A set of bounding inequalities exist for the generalized Mathieu series $S_\mu(r)$; as their main source we can list the articles [20], [21], [23]. To give upper bounds for $S_{\mu,p}(r)$ *via* $S_\mu(r)$, since the oscillatory behavior of J_μ in the integrand of the integral representation (1.4), we are forced to consider the modulus of the input series.

Observing the p–kernel $\mathrm{e}^{-\frac{p}{x}}, x>0$ introduced by Chaudhary *et al.* [3], instead of the obvious bound $\mathrm{e}^{-\frac{p}{x}} \le 1$, for non–negative parameter p we infer the more precise estimate

$$\mathrm{e}^{-\frac{p}{x}} \le \mathcal{C}_p(x) = \begin{cases} \dfrac{2x}{p\,\mathrm{e}^2} & x \in \left(0, \dfrac{p}{2}\right) \\ \dfrac{4x}{p\,\mathrm{e}^2} - \dfrac{1}{\mathrm{e}^2} & x \in \left[\dfrac{p}{2}, \dfrac{p}{4}(1+\mathrm{e}^2)\right) \\ 1 & x \ge \dfrac{p}{4}(1+\mathrm{e}^2) \end{cases}, \qquad x>0\,.$$

Indeed, being $I(p/2, \mathrm{e}^{-2})$ the inflection point in which the kernel is changing behavior from convex into concave in growing x, the secant line joining the origin and I is above the kernel's arc, while the tangent line in I bounds the kernel from above in the middle interval. The structure of $\mathcal{C}_p(x)$ *mutatis* *mutandis* splits the integration domain in (2.1) getting

$$\begin{aligned}
|S_{\mu,p}(r)| &\le \frac{\sqrt{\pi}}{(2r)^{\mu-\frac12}\Gamma(\mu+1)}\int_0^\infty \frac{x^{\mu+\frac12}\,\mathcal{C}_p(x)}{\mathrm{e}^x-1}|J_{\mu-\frac12}(rx)|\,\mathrm{d}x \\
&= \frac{2\sqrt{\pi}\,(2r)^{\frac12-\mu}}{p\,\mathrm{e}^2\,\Gamma(\mu+1)}\int_0^{\frac{p}{2}} \frac{x^{\mu+\frac32}\,|J_{\mu-\frac12}(rx)|}{\mathrm{e}^x-1}\,\mathrm{d}x \\
&\quad+ \frac{4\sqrt{\pi}\,(2r)^{\frac12-\mu}}{p\,\mathrm{e}^2\,\Gamma(\mu+1)}\int_{\frac{p}{2}}^{\frac{p}{4}(1+\mathrm{e}^2)} \frac{x^{\mu+\frac12}\,|J_{\mu-\frac12}(rx)|}{\mathrm{e}^x-1}\Big(x-\frac{p}{4}\Big)\,\mathrm{d}x \\
&\quad+ \frac{\sqrt{\pi}}{(2r)^{\mu-\frac12}\Gamma(\mu+1)}\int_{\frac{p}{4}(1+\mathrm{e}^2)}^\infty \frac{x^{\mu+\frac12}}{\mathrm{e}^x-1}|J_{\mu-\frac12}(rx)|\,\mathrm{d}x\,. \qquad (3.1)
\end{aligned}$$

All three integrals in (3.1) contain the same fashion integrands, but for $p>0$ we couldn't express these integrals in terms *via* a finite linear combination of elementary and/or higher transcendental, special functions. That is the reason why we list here various uniform and functional upper bounds upon $|J_\nu(x)|$, preferably the ones of polynomial decay compare for instance [1] and [19, Subsection 3.2]. For the sake of simplicity we recall here only few of them. Firstly, we mention von Lommel's uniform bounds [13], [14, pp. 548–549], [24, p. 406]:

$$|J_\nu(x)| \le 1, \qquad |J_{\nu+1}(x)| \le \frac{1}{\sqrt{2}}, \qquad \nu>0,\ x\in\mathbb{R},$$

and the bound by Minakshisundaram and Szász [17, p. 37, Corollary]

$$|J_\nu(x)| \le \frac{|x|^\nu}{2^\nu\,\Gamma(\nu+1)}, \qquad \nu\ge 0,\ x\in\mathbb{R}.$$

Further estimates were given by Landau [12] with respect to ν and x which are in a sense best possible (outside of Bessel function's transition region)

$$|J_\nu(x)| \le b_L\,\nu^{-1/3}, \qquad b_L = \sqrt[3]{2}\sup_{x\ge 0}\mathrm{Ai}(t), \qquad (3.2)$$

$$|J_\nu(x)| \le c_L\,|x|^{-1/3}, \qquad c_L = \sup_{x\ge 0} x^{1/3}J_0(x)\,; \qquad (3.3)$$

here $\mathrm{Ai}(\cdot)$ denotes Airy function. In turn, Olenko answered to this challenge by [18, Theorem 1]

$$\sup_{x\ge 0}\sqrt{x}\,|J_\nu(x)| \le b_L\sqrt{\nu^{1/3}+\frac{\alpha_1}{\nu^{1/3}}+\frac{3\alpha_1^2}{10\nu}} = d_O, \qquad \nu>0\,. \qquad (3.4)$$

Here α_1 is the smallest positive zero of Ai, being b_L the first Landau's constant. Further considerable upper bounds are listed e.g. in the works by Baricz *et al.* [1] and by Srivastava and Pogány [22]. Also one draws the reader's attention to the sophisticated functional bounds by Krasikov [10], [11]. We cover all these cases with a generic bound $|J_{\mu-\frac12}(x)| \le C\cdot x^q, x>0$, where both, the absolute constant $C = C(\mu)$ and the power q are changing with the different kind bounds pointing out that the application of estimates mentioned, and by the sake of simplicity not used in evaluating $S_{\mu,p}(r)$, we plane at another address.

At this point we establish the master inequality by virtue of the newly established integral expression (2.1) covering all above listed cases of Bessel function bounding inequalities.

Theorem 3.1. *For all $p \geq 0, \mu > 0; q > -\mu - \frac{1}{2}$ and for all $r > 0$ there holds*

$$\begin{aligned}
|S_{\mu,p}(r)| \leq \frac{C\sqrt{\pi}\, p^{\mu+q+\frac{1}{2}}}{\mathrm{e}^2\, 2^{2\mu+q} r^{\mu-\frac{1}{2}}\, \Gamma(\mu+1)} \bigg\{ & \frac{1}{2(\mu+q)+5} \Big(\frac{4}{2(\mu+q)+3} + \frac{p}{\mathrm{e}^{\frac{p}{2}}-1} \Big) \\
& + \frac{2p}{2(\mu+q)+5} \Big(\Big(\frac{1+\mathrm{e}^2}{2} \Big)^{\mu+q+\frac{5}{2}} - 1 \Big) \\
& + \frac{4}{2(\mu+q)+3} \Big(\Big(\frac{1+\mathrm{e}^2}{2} \Big)^{\mu+q+\frac{3}{2}} - 1 \Big) \\
& - \frac{(1+\mathrm{e}^2)\, p}{2\, [2(\mu+q)+1] (\mathrm{e}^{\frac{p}{4}(1+\mathrm{e}^2)} - 1)} \Big(\Big(\frac{1+\mathrm{e}^2}{2} \Big)^{\mu+q+\frac{1}{2}} - 1 \Big) \\
& - \frac{\mathrm{e}^2 p (1+\mathrm{e}^2)^{\mu+q+\frac{3}{2}}}{2^{\mu+q+\frac{3}{2}} [2(\mu+q)+1]\, (\mathrm{e}^{\frac{p}{4}(1+\mathrm{e}^2)} - 1)} \bigg\} \\
& + \frac{C\sqrt{\pi}}{(2r)^{\mu-\frac{1}{2}}\, \Gamma(\mu+1)}\, \Gamma\Big(\mu+q+\frac{3}{2}\Big)\, \zeta\Big(\mu+q+\frac{3}{2}\Big)\,.
\end{aligned} \tag{3.5}$$

Proof. Consider the auxiliary integral

$$\mathcal{K}(\alpha, a, b) = \int_a^b \frac{x^{\alpha-1}}{\mathrm{e}^x - 1}\, \mathrm{d}x; \qquad \alpha > 1; 0 \leq a < b < \infty\,.$$

Being the function $x \mapsto x(\mathrm{e}^x - 1)^{-1}$ monotone decreasing and convex for $x > 0$ we estimate this function's arc from above with secant line crossing $A(a, a(\mathrm{e}^a - 1)^{-1})$ and $B(b, b(\mathrm{e}^b - 1)^{-1})$.

Next, taking the lower bound $x(\mathrm{e}^x - 1)^{-1} \geq b(\mathrm{e}^b - 1)^{-1}$ on the whole (a, b) for $\alpha > 1$ we achieve

$$\frac{b(b^{\alpha-1} - a^{\alpha-1})}{(\alpha-1)\,(\mathrm{e}^b - 1)} \leq \mathcal{K}(\alpha, a, b) \leq \frac{K_1}{\alpha} \big(b^\alpha - a^\alpha \big) + \frac{K_2}{\alpha - 1} \big(b^{\alpha-1} - a^{\alpha-1} \big), \tag{3.6}$$

where

$$K_1 = \Big(\frac{b}{\mathrm{e}^b - 1} - \frac{a}{\mathrm{e}^a - 1} \Big) \frac{1}{b-a}; \qquad K_2 = \Big(\frac{1}{\mathrm{e}^b - 1} - \frac{1}{\mathrm{e}^a - 1} \Big) \frac{ab}{b-a}\,.$$

Letting here $a \to 0+$, (3.6) one reduces to

$$\frac{b^\alpha}{(\alpha-1)\,(\mathrm{e}^b - 1)} \leq \mathcal{K}(\alpha, 0, b) \leq \frac{b^{\alpha-1}}{\alpha} \Big(\frac{1}{\alpha-1} + \frac{b}{\mathrm{e}^b - 1} \Big), \qquad \alpha > 1\,. \tag{3.7}$$

On the other hand, we get

$$\begin{aligned}\overline{\mathcal{K}}(\alpha, b) &= \int_b^\infty \frac{x^{\alpha-1}}{\mathrm{e}^x - 1}\,\mathrm{d}x = \int_0^\infty \frac{x^{\alpha-1}}{\mathrm{e}^x - 1}\,\mathrm{d}x - \mathcal{K}(\alpha, 0, b) \\ &\le \Gamma(\alpha)\,\zeta(\alpha) - \frac{b^\alpha}{(\alpha-1)\,(\mathrm{e}^b - 1)}\,, \end{aligned} \tag{3.8}$$

where for all $b > 1$ the right–hand–side estimate is not redundant, namely in this b–domain the upper bound is strict positive.

In the introductory part of this section we list diverse bounding inequalities for the Bessel function of the first kind of positive argument. Bearing in mind (3.1) we conclude

$$\begin{aligned}|S_{\mu,p}(r)| \le \frac{2C\sqrt{\pi}}{p\,\mathrm{e}^2\,(2r)^{\mu-\frac12}\,\Gamma(\mu+1)} &\Big\{\mathcal{K}\Big(\mu+q+\frac52, 0, \frac{p}{2}\Big) \\ &+ 2\,\mathcal{K}\Big(\mu+q+\frac52, \frac{p}{2}, \frac{p}{4}(1+\mathrm{e}^2)\Big) \\ &- \frac{p}{2}\,\mathcal{K}\Big(\mu+q+\frac32, \frac{p}{2}, \frac{p}{4}(1+\mathrm{e}^2)\Big) \\ &+ \frac{p\,\mathrm{e}^2}{2}\,\overline{\mathcal{K}}\Big(\mu+q+\frac32, \frac{p}{4}(1+\mathrm{e}^2)\Big)\Big\}\,. \end{aligned} \tag{3.9}$$

Applying the estimates (3.6), (3.7) upon $\mathcal{K}$ and (3.8) for $\overline{\mathcal{K}}$, it follows

$$\begin{aligned}|S_{\mu,p}(r)| \le \frac{2C\sqrt{\pi}}{\mathrm{e}^2(2r)^{\mu-\frac12}\,\Gamma(\mu+1)} &\left\{\frac{\left(\frac{p}{2}\right)^{\mu+q+\frac12}}{2[2(\mu+q)+5]}\,\Big(\frac{4}{2(\mu+q)+3} + \frac{p}{\mathrm{e}^{\frac{p}{2}}-1}\Big)\right. \\ &+ \frac{2\,K_1^*\,\left(\frac{p}{2}\right)^{\mu+q+\frac32}}{2(\mu+q)+5}\Big(\Big(\frac{1+\mathrm{e}^2}{2}\Big)^{\mu+q+\frac52} - 1\Big) \\ &+ \frac{2\,K_2^*\,\left(\frac{p}{2}\right)^{\mu+q+\frac12}}{2(\mu+q)+3}\Big(\Big(\frac{1+\mathrm{e}^2}{2}\Big)^{\mu+q+\frac32} - 1\Big) \\ &- \frac{(1+\mathrm{e}^2)\,\left(\frac{p}{2}\right)^{\mu+q+\frac52}}{p[2(\mu+q)+1](\mathrm{e}^{\frac{p}{4}(1+\mathrm{e}^2)}-1)}\Big(\Big(\frac{1+\mathrm{e}^2}{2}\Big)^{\mu+q+\frac12} - 1\Big) \\ &+ \frac{\mathrm{e}^2}{2}\,\Gamma\Big(\mu+q+\frac32\Big)\,\zeta\Big(\mu+q+\frac32\Big) \\ &\left.- \frac{\mathrm{e}^2 p^{\mu+q+\frac32}(1+\mathrm{e}^2)^{\mu+q+\frac32}}{2^{2(\mu+q)+3}[2(\mu+q)+1]\,(\mathrm{e}^{\frac{p}{4}(1+\mathrm{e}^2)}-1)}\right\}, \end{aligned}$$

where K_1^*, K_2^* are the restricted values of K_1, K_2 for specific α, a, b used in (3.9). Now routine steps lead to the assertion. $\square$

The specific estimates upon $J_{\mu-\frac12}(x)$ in (3.5) form a set of respective particular bounds:

A. Taking $C = 2^{-\frac{1}{2}}, q = 0$, we infer a Lommel–type bound from (3.5) if $\mu > \frac{1}{2}$.

B. When $q = \mu - \frac{1}{2} \geq 0$ and respectively

$$C(r) = \frac{r^{\mu-\frac{1}{2}}}{2^{\mu-\frac{1}{2}}\,\Gamma\left(\mu+\frac{1}{2}\right)},$$

we arrive at the Minakshisundaram–Szász–type bound, which surprisingly becomes r–independent.

C. Making use of Landau's first estimate with $q = 0$ and

$$C(r) = \frac{b_L}{\sqrt[3]{\mu - \frac{1}{2}}}, \qquad \mu > \tfrac{1}{2},$$

where b_L was defined in (3.2), we get a bound of the same magnitude (in r) then von Lommel's one which is now equal to $\mathcal{O}(r^{-\mu+\frac{1}{2}})$.

D. Next, using Landau's second estimate (3.3) with $q = -\frac{1}{3}$ and

$$C(r) = \frac{c_L}{\sqrt[3]{r}}, \qquad \mu > \frac{1}{3}$$

increases the magnitude of (3.5) into $\mathcal{O}(r^{-\mu+\frac{1}{6}})$, c_L being described around (3.3).

E. Finally, putting $q = -\frac{1}{2}$ and according to (3.4)

$$C(r) = \frac{d_O}{\sqrt{r}}, \qquad \mu > \frac{1}{2},$$

implies the Olenko bound which magnitude reads $\mathcal{O}(r^{-\mu})$.

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O p–poopćenom Mathieuovom redu

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Sažetak. Pobuđeni mnogobrojnim poopćenjima Mathieuovog reda uveli smo jedno novo proširenje pojma generaliziranog Mathieuvog reda. Za novodefinirani red u članku su izvedene raznovrsne integralne reprezentacije. Napokon općenita funkcionalna gornja granica je dokazana pomoću prethodno uspostavljenih integralnih reprezentacija.

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