ON $p$–EXTENDED MATHIEU SERIES

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Abstract. Motivated by several generalizations of the well–known Mathieu series, the main object of this paper is to introduce new extension of generalized Mathieu series and to derive various integral representations of such series. Finally, master bounding inequality is established using the newly derived integral expression.

1. Introduction and Motivation

The series of the form
\[ S(r) = \sum_{n \geq 1} \frac{2n}{(n^2 + r^2)^2}, \quad r > 0, \]
is known in literature as Mathieu series. Émile Leonard Mathieu was the first who investigated such series in 1890 in his book [15]. A remarkable useful integral representation for $S(r)$ is given by Emersleben [7] in the following elegant form
\[ S(r) = \frac{1}{r} \int_{0}^{\infty} \frac{x \sin(rx)}{e^x - 1} \, dx, \]
which can also be written in terms of the Riemann Zeta function $\zeta(s) = \sum_{n \geq 1} n^{-s}, s > 1$ as [4, p. 863, Eq. (2.3)] (with replacing $n$ by $n + 1$)
\[ S(r) = 2 \sum_{n \geq 0} (-1)^n (n + 1) \frac{\zeta(2n + 3)}{2n}, \quad |r| < 1. \]

The so–called generalized Mathieu series with a fractional power reads [2, p. 2, Eq. (1.6)] (also see [16, p. 181])
\[ S_{\mu}(r) = \sum_{n \geq 1} \frac{2n}{(n^2 + r^2)^{\mu + 1}}, \quad r > 0, \mu > 0; \]

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such series has been widely considered in mathematical literature (see e.g. papers by Diananda [5], Cerone and Lenard [2] and Pogány et al. [20]). Cerone and Lenard also gave a series representation of $S_\mu(r)$ in terms of the Riemann Zeta function [2, p. 3, Eq. (2.1)]

$$S_\mu(r) = 2 \sum_{n \geq 0} (-1)^n \binom{\mu + n}{n} \zeta(2\mu + 2n + 1) r^{2n}, \quad |r| < 1;$$  

in [2] was not mentioned the convergence region $|r| < 1$. To show (1.3) it is enough to expand the summands in (1.2) into a binomial series for $r \in (-1, 1)$ (compare [20, p. 72, Proposition 1]). Cerone and Lenard derived also the next integral expression [2, p. 3, Theorem 2.1] (also consult [16, p. 181, Eq. (1.3)])

$$S_\mu(r) = \frac{\sqrt{\pi}}{(2r)^{\mu-rac{1}{2}} \Gamma(\mu + 1)} \int_0^\infty \frac{x^{\mu + \frac{1}{2}} e^{-p x} e^{-x} J_{\mu - \frac{1}{2}}(rx)}{e^x - 1} \, dx, \quad \mu > 0.$$

Motivated by the previous extension and by huge spectrum of other generalizations of the Mathieu series, the main aim of this paper is to study certain another types of, in new fashion generalized, Mathieu series.

Having in mind (1.3) let the $p$–extended Mathieu series be defined as

$$S_{\mu,p}(r) = 2 \sum_{n \geq 0} r^{2n} (-1)^n \binom{\mu + n}{n} \zeta_p(2\mu + 2n + 1),$$

where $p \geq 0$; $\mu > 0$, $|r| < 1$ and $\zeta_p$ stands for the $p$–extended Riemann Zeta function [3]

$$\zeta_p(\alpha) = \frac{1}{\Gamma(\alpha)} \int_0^\infty \frac{x^{\alpha - 1} e^{-\frac{x}{e^x - 1}}}{x} \, dx$$

defined for $\Re(p) > 0$ or $p = 0$ and $\Re(\alpha) > 0$, which reduces to Riemann Zeta function when $p = 0$. It is also important to quote that (1.5) reduces to (1.3) when $p = 0$, while taking $\mu = 1$ we yield (1.1).

2. INTEGRAL FORMS, INTEGRAL TRANSFORMS AND SERIES REPRESENTATIONS OF $S_{\mu,p}(r)$

In this section we derive an integral expression for the $p$–extended Mathieu series $S_{\mu,p}(r)$. Then its various Mellin and Laplace transforms are exposed.

**Theorem 2.1.** For all $\Re(p) > 0$ or $p = 0$, $\mu > 0$ and $r > 0$ the following integral representation for the extended generalized Mathieu series $S_{\mu,p}(r)$ holds true:

$$S_{\mu,p}(r) = \frac{\sqrt{\pi}}{(2r)^{\mu-rac{1}{2}} \Gamma(\mu + 1)} \int_0^\infty \frac{x^{\mu + \frac{1}{2}} e^{-\frac{x}{e^x - 1}} J_{\mu - \frac{1}{2}}(rx)}{e^x - 1} \, dx.$$
Proof. Using the series representation of $J_{\nu}$ [24, p. 40]

$$J_{\nu}(x) = \sum_{n \geq 0} \frac{(-1)^n}{n! \Gamma(n + \nu + 1)} \left( \frac{x}{2} \right)^{2n+\nu},$$

valid for all $\nu, x \in \mathbb{C}$ we can simplify an integral given in (2.1) as:

$$I = \int_{0}^{\infty} x^{\mu+\frac{1}{2}} e^{-\frac{x}{2}} J_{\mu-\frac{1}{2}}(rx) \, dx = \sum_{n \geq 0} \frac{(-1)^n (\frac{x}{2})^{2n+\frac{1}{2}}}{n! \Gamma(\mu + n + \frac{1}{2})} \int_{0}^{\infty} x^{2\mu+2n} e^{-\frac{x}{2}} \, dx,$$

which leads to the desired result.

Remark 2.2. The integral expression (2.1) one reduces to (1.4) when $p = 0$.

In what follows we derive Mellin and Laplace transforms of the newly constructed series $S_{\mu,p}(r)$.

The Mellin and Laplace transforms (respectively) of some suitably integrable function $f$ with index $s$ are defined by

$$\mathcal{M}_{s}\{f(x)\}(s) = \int_{0}^{\infty} x^{s-1} f(x) \, dx, \quad \mathcal{L}_{s}\{f(x)\}(s) = \int_{0}^{\infty} e^{-sx} f(x) \, dx,$$

provided that the corresponding integrals exist.

Theorem 2.3. The Mellin transform of the extended generalized Mathieu series $S_{\mu,p}(r)$ read as follows:

$$\mathcal{M}_{s}\{S_{\mu,p}(r)\}(s) = 2s \Gamma^2(s) \sum_{n \geq 0} (-1)^n \binom{\mu+n}{n} \left( \frac{2\mu+2n+s}{2\mu+2n} \right) \zeta(2\mu+2n+s+1),$$

in the range $|r| < 1$. Moreover, for $\mu > 0$; $0 < \Re(s) < \mu + 1$ and $\Re(p) > 0$,

$$\mathcal{M}_{s}\{S_{\mu,p}(r)\}(s) = B\left( \frac{s}{2}, \mu + 1 - \frac{s}{2} \right) \zeta(2\mu - s + 1),$$

where $B(x, y), \min\{\Re(x), \Re(y)\} > 0$ stands for the Euler Beta function.
Proof. Using the definition of the Mellin transform, we find from (1.5)

\[
\mathcal{M}_p\{S_{\mu,p}(r)\}(s) = \int_0^\infty p^{s-1}S_{\mu,p}(r)\,dp
\]

\[
= 2 \sum_{n \geq 0} r^{2n}(-1)^n \binom{\mu+n}{n} \int_0^\infty p^{s-1}\zeta_p(2\mu+2n+1)\,dp
\]

\[
= 2\Gamma(s)\Gamma(s+1) \sum_{n \geq 0} r^{2n}(-1)^n \binom{\mu+n}{n} \left(\frac{2\mu+2n+s}{2\mu+2n}\right)\zeta(2\mu+2n+s+1),
\]

where in the last equality we used the formula [3, p. 1244, Eq. (3.6)]

\[
\int_0^\infty p^{s-1}\zeta_p(\alpha)\,dp = \frac{\Gamma(s)\Gamma(\alpha+s)}{\Gamma(\alpha)}\zeta(\alpha+s), \quad \Re(\alpha) > 0, \ \Re(s) > 0.
\]

Next, with the help of the Weber–Sonine integral [24, p. 391, Eq. 13.24(1)]

\[
\int_0^\infty x^{\mu-\nu-1}J_\nu(x)\,dx = \frac{\Gamma\left(\frac{\nu}{2}\right)}{2^{\nu-\mu+1}\Gamma\left(1+\nu-\frac{\mu}{2}\right)}, \quad 0 < \Re(\mu) < \Re(\nu) + \frac{1}{2},
\]

the integral representation (2.1) derived in Theorem 2.1 and the definition of extended Riemman Zeta \(\zeta_p\), we find that

\[
\mathcal{M}_r\{S_{\mu,p}(r)\}(s) = \int_0^\infty r^{s-1}S_{\mu,p}(r)\,dr
\]

\[
= \sqrt{2\pi} 2^{-\mu} \Gamma(\mu+1) \int_0^\infty \frac{x^{\mu+\frac{1}{2}}e^{-\frac{x}{2}}}{e^{x} - 1} \left(\int_0^\infty r^{s-\mu-\frac{1}{2}}J_{\mu-\frac{1}{2}}(rx)\,dr\right)\,dx
\]

\[
= \frac{\sqrt{\pi} \Gamma\left(\frac{\mu}{2}\right)}{2^{2\mu-s}\Gamma(\mu+1)\Gamma\left(\mu+\frac{1-s}{2}\right)} \int_0^\infty \frac{x^{\mu-s+\frac{1}{2}}e^{-\frac{x}{2}}}{e^{x} - 1} \,dx
\]

\[
= \frac{\sqrt{\pi} \Gamma\left(\frac{\mu}{2}\right)\Gamma(2\mu-s+1)\zeta_p(2\mu-s+1)}{2^{2\mu-s}\Gamma(\mu+1)\Gamma\left(\mu+\frac{1-s}{2}\right)},
\]

which gives (2.2) with the help of the duplication formula for the Gamma function and the relation between Beta and Gamma functions. \(\Box\)

**Theorem 2.4.** For the \(p\)-extended Mathieu series \(S_{\mu,p}(r)\) we have the Laplace transform formula

(2.3) \(\mathcal{L}_r\{S_{\mu,p}(r)\}(x) = \frac{2}{x\Gamma(2\mu+1)} \int_0^\infty \frac{t^{2\mu}e^{-\frac{x}{2}t}}{e^{t} - 1} \,F_1\left[ 1, \frac{1}{2} + \frac{1}{2}, \frac{1}{2} \right] + \frac{\theta^2}{x^2} \,dt\)

provided that the each member of (2.3) exists.
**Proof.** Using the Laplace transform formula [8, p. 49, Eq. 7.7.3(16)]

\[
\mathcal{L}_x\{x^{\lambda-1}J_\nu(\rho x)\}(s) = \frac{\rho^\nu \Gamma(\nu + \lambda)}{2^{\nu+\lambda}\Gamma(\nu + 1)} \binom{\frac{1}{2}(\nu + \lambda), \frac{1}{2}(\nu + \lambda + 1)}{\nu + 1} - \frac{\rho^2}{s^2},
\]

valid for all \(|\Re(s)| > |\Im(\rho)|, \Re(\nu + \lambda) > 0\) and the integral representation (2.1), we get

\[
\mathcal{L}_r\{S_{\mu,p}(r)\}(x) = \int_0^\infty e^{-rx} S_{\mu,p}(r) \, dr
\]

\[
= \frac{\sqrt{\pi}}{2^{2-\mu}} \frac{2^{-\mu}}{\Gamma(\mu + 1)} \int_0^\infty \frac{t^\mu}{e^t - 1} \left( \int_0^\infty e^{-rt} r^{\frac{1}{2}-\mu} J_{\mu-\frac{1}{2}}(rt) \, dr \right) \, dt
\]

\[
= \frac{\sqrt{\pi}}{2^{1-2\mu}} \frac{2^{1-2\mu}}{x^{\mu+1} \Gamma(\mu + 1)} \int_0^\infty e^{\frac{t^2}{x^2}} \left( \int_0^\infty e^{-rt} r^{\frac{1}{2}} J_{\mu-\frac{1}{2}}(rt) \, dr \right) \, dt,
\]

which becomes (2.3) with the help of duplication formula. \(\Box\)

**Remark 2.5.** It is interesting to note when \(p = 0\), (2.2) and (2.3) reduce to known results in [6].

### 3. Master bounding inequality upon \(S_{\mu,p}(r)\)

A set of bounding inequalities exist for the generalized Mathieu series \(S_\mu(r)\); as their main source we can list the articles [20], [21], [23]. To give upper bounds for \(S_{\mu,p}(r)\) via \(S_\mu(r)\), since the oscillatory behavior of \(J_\mu\) in the integrand of the integral representation (1.4), we are forced to consider the modulus of the input series.

Observing the \(p\)-kernel \(e^{-\frac{p}{x}}, x > 0\) introduced by Chaudhary et al. [3], instead of the obvious bound \(e^{-\frac{p}{x}} \leq 1\), for non-negative parameter \(p\) we infer the more precise estimate

\[
e^{-\frac{p}{x}} \leq \mathcal{C}_p(x) = \begin{cases} 
\frac{2x}{pe^2} & x \in \left(0, \frac{p}{2}\right) \\
\frac{4x}{pe^2} & x \in \left[\frac{p}{2}, \frac{p}{4}(1 + e^2)\right) \\
1 & x \geq \frac{p}{4}(1 + e^2)
\end{cases}, \quad x > 0.
\]

Indeed, being \(I(p/2, e^{-2})\) the inflection point in which the kernel is changing behavior from convex into concave in growing \(x\), the secant line joining the origin and \(I\) is above the kernel’s arc, while the tangent line in \(I\) bounds the kernel from above in the middle interval. The structure of \(\mathcal{C}_p(x)\) *mutatis*
mutandis splits the integration domain in (2.1) getting

$$|S_{\mu,p}(r)| \leq \frac{\sqrt{\pi}}{(2r)^{\mu-\frac{1}{2}}\Gamma(\mu+1)} \int_0^\infty \frac{x^{\mu+\frac{1}{2}} e_{\mu}(x) |J_{\mu-\frac{1}{2}}(rx)|}{e^x - 1} dx$$

$$= 2\sqrt{\pi} (2r)^{\frac{1}{2}-\mu} \int_0^\infty \frac{x^{\mu+\frac{1}{2}} |J_{\mu-\frac{1}{2}}(rx)|}{e^x - 1} dx$$

$$+ 4\sqrt{\pi} (2r)^{\frac{1}{2}-\mu} \int_\frac{1}{2}^\infty \frac{x^{\mu+\frac{1}{2}} |J_{\mu-\frac{1}{2}}(rx)|}{e^x - 1} \left( x - \frac{p}{4} \right) dx$$

(3.1)

$$+ \frac{\sqrt{\pi}}{(2r)^{\mu-\frac{1}{2}}\Gamma(\mu+1)} \int_\frac{1}{2}^\infty \frac{x^{\mu+\frac{1}{2}} |J_{\mu-\frac{1}{2}}(rx)|}{e^x - 1} dx.$$
absolute constant $C = C(\mu)$ and the power $q$ are changing with the different kind bounds pointing out that the application of estimates mentioned, and by the sake of simplicity not used in evaluating $S_{\mu,p}(r)$, we plane at another address.

At this point we establish the master inequality by virtue of the newly established integral expression (2.1) covering all above listed cases of Bessel function bounding inequalities.

**Theorem 3.1.** For all $p \geq 0$, $\mu > 0$; $q > -\frac{1}{2}$ and for all $r > 0$ there holds

$$|S_{\mu,p}(r)| \leq \frac{C \sqrt{\pi} p^{\mu+q+\frac{1}{2}}}{e^{2} 2^{2\mu+q} r^{\mu-\frac{1}{2}} \Gamma(\mu + 1)} \left\{ \frac{1}{2(\mu + q) + 5} \left( \frac{4}{2(\mu + q) + 3} + \frac{p}{\alpha^2 - 1} \right) 
\right. 
\left. + \frac{2p}{2(\mu + q) + 5} \left( \left( \frac{1 + e^2}{2} \right)^{\mu + q + \frac{1}{2}} - 1 \right) 
\right. 
\left. + \frac{4}{2(\mu + q) + 3} \left( \left( \frac{1 + e^2}{2} \right)^{\mu + q + \frac{1}{2}} - 1 \right) 
\right. 
\left. - \frac{(1 + e^2)p}{2[2(\mu + q) + 1] \left( e^{\frac{\ell(1+\ell^2)}{2}} - 1 \right)} \left( \left( \frac{1 + e^2}{2} \right)^{\mu + q + \frac{1}{2}} - 1 \right) 
\right. 
\left. - \frac{e^2p(1 + e^2)^{\mu + q + \frac{1}{2}}}{2^{\mu+q+\frac{1}{2}}[2(\mu + q) + 1] \left( e^{\frac{\ell(1+\ell^2)}{2}} - 1 \right)} \right\} 
\right. 
\left. + \frac{C \sqrt{\pi}}{(2r)^{\mu+q} \Gamma(\mu + 1)} \Gamma(\mu + q + \frac{3}{2}) \zeta(\mu + q + \frac{3}{2}). \right.$$  

(3.5)

**Proof.** Consider the auxiliary integral

$$\mathcal{K}(\alpha, a, b) = \int_{a}^{b} \frac{x^{\alpha-1}}{e^{x} - 1} \, dx; \quad \alpha > 1; 0 \leq a < b < \infty.$$

Being the function $x \mapsto x(e^x - 1)^{-1}$ monotone decreasing and convex for $x > 0$ we estimate this function’s arc from above with secant line crossing $A(a, a(e^a - 1)^{-1})$ and $B(b, b(e^b - 1)^{-1})$.

Next, taking the lower bound $x(e^x - 1)^{-1} \geq b(e^b - 1)^{-1}$ on the whole $(a, b)$ for $\alpha > 1$ we achieve

$$\frac{b(b^{\alpha-1} - a^{\alpha-1})}{(\alpha - 1)(e^{b} - 1)} \leq \mathcal{K}(\alpha, a, b) \leq \frac{K_1}{\alpha} (b^{\alpha} - a^{\alpha}) + \frac{K_2}{\alpha} \left( b^{\alpha-1} - a^{\alpha-1} \right),$$

(3.6)

where

$$K_1 = \left( \frac{b}{e^{b} - 1} - \frac{a}{e^{a} - 1} \right) \frac{1}{b - a}; \quad K_2 = \left( \frac{1}{e^{b} - 1} - \frac{1}{e^{a} - 1} \right) \frac{ab}{b - a}.$$

Letting here $a \to 0+$, (3.6) one reduces to

$$\frac{b^{\alpha}}{(\alpha - 1)(e^{b} - 1)} \leq \mathcal{K}(\alpha, 0, b) \leq \frac{b^{\alpha-1}}{\alpha} \left( \frac{1}{\alpha - 1} + \frac{b}{e^{b} - 1} \right), \quad \alpha > 1.$$  

(3.7)
On the other hand, we get
\[
\mathcal{K}(\alpha, b) = \int_b^\infty \frac{x^{\alpha-1}}{e^x - 1} \, dx = \int_0^\infty \frac{x^{\alpha-1}}{e^{b^\alpha x} - 1} \, dx = \mathcal{K}(\alpha, 0, b)
\]
where for all \(b > 1\) the right-hand-side estimate is not redundant, namely in this \(b\)-domain the upper bound is strict positive.

In the introductory part of this section we list diverse bounding inequalities for the Bessel function of the first kind of positive argument. Bearing in mind (3.1) we conclude
\[
|S_{\mu,p}(r)| \leq \frac{2C \sqrt{\pi}}{pe^2 (2r)^{\mu-\frac{1}{2}}} \Gamma(\mu + 1) \{ K_1 \left( \mu + q + \frac{5}{2}, 0, \frac{p}{2} \right) + 2 \mathcal{K} \left( \mu + q + \frac{5}{2}, \frac{p}{2}, \frac{p}{4} (1 + e^2) \right) \}
\]
where \(K_1^1, K_2^2\) are the restricted values of \(K_1, K_2\) for specific \(\alpha, a, b\) used in (3.9). Now routine steps lead to the assertion.

The specific estimates upon \(J_{\mu-\frac{1}{2}}(x)\) in (3.5) form a set of respective particular bounds:
A. Taking \( C = 2^{-\frac{1}{2}}, q = 0 \), we infer a Lommel–type bound from (3.5) if \( \mu > \frac{1}{2} \).

B. When \( q = \mu - \frac{1}{2} \geq 0 \) and respectively
\[
C(r) = \frac{r^{\mu - \frac{1}{2}}}{2^{\mu - \frac{1}{2}} \Gamma (\mu + \frac{1}{2})},
\]
we arrive at the Minakshisundaram–Szász–type bound, which surprisingly becomes \( r \)-independent.

C. Making use of Landau’s first estimate with \( q = 0 \) and
\[
C(r) = \frac{b_L}{\sqrt{\mu - \frac{1}{2}}}, \quad \mu > \frac{1}{2},
\]
where \( b_L \) was defined in (3.2), we get a bound of the same magnitude (in \( r \)) then von Lommel’s one which is now equal to \( O(r^{-\mu + \frac{1}{2}}) \).

D. Next, using Landau’s second estimate (3.3) with \( q = -\frac{1}{3} \) and
\[
C(r) = \frac{c_L}{\sqrt{r}}, \quad \mu > \frac{1}{3}
\]
increases the magnitude of (3.5) into \( O(r^{-\mu + \frac{1}{2}}) \), \( c_L \) being described around (3.3).

E. Finally, putting \( q = -\frac{1}{2} \) and according to (3.4)
\[
C(r) = \frac{d_O}{\sqrt{r}}, \quad \mu > \frac{1}{2},
\]
implies the Olenko bound which magnitude reads \( O(r^{-\mu}) \).

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O $p$–poopćenom Mathieuovom redu

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