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# KIEPERT HYPERBOLA IN AN ISOTROPIC PLANE

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Abstract. The concept of the Kiepert hyperbola of an allowable triangle in an isotropic plane is introduced in this paper. Important properties of the Kiepert hyperbola will be investigated in the case of the standard triangle. The relationships between the introduced concepts and some well known elements of a triangle will also be studied.

## 1. Introduction

Kiepert triangles of an allowable triangle $ABC$ in the isotropic plane $I_2$ are considered in [21]. Properties of the Kiepert hyperbola of a given triangle in the plane $I_2$ will be investigated in this paper on the basis of characteristics of these triangles.

It should be noted that in Euclidean geometry Kiepert triangles are defined in the following way. Let $ABC$ be a given triangle and $\varphi$ a given angle. If $BCA'$, $CAB'$, $ABC'$ are mutually similar isosceles triangles constructed on the bases $BC$, $CA$, $AB$ with the base angle $\varphi$, then $A'B'C'$ is a Kiepert triangle of the given triangle $ABC$. The Kiepert hyperbola of the triangle $ABC$ is one circumscribed rectangular hyperbola of that triangle which is the set of centers of homologies of this triangle with its Kiepert triangles (see Figure 1).

It is the isogonal image of the line $OK$ with respect to the triangle $ABC$ and the line $GK$ touches this hyperbola at the point $G$, where $G$, $O$, $K$ are the centroid, circumcenter and symmedian center of the triangle $ABC$,

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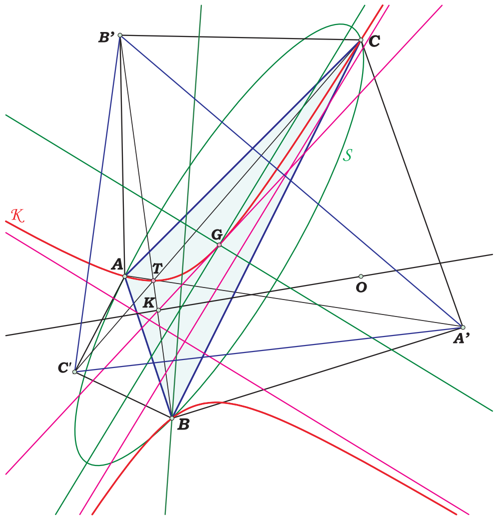


FIGURE 1. Kiepert hyperbola of the triangle $ABC$ in Euclidean case

respectively. Asymptotes of this hyperbola are parallel to the Steiner axes of this triangle (see Figure 1) (see e.g. [4]).

The isotropic (or Galilean) plane is a projective–metric plane, where the absolute consists of one line, the absolute line $\omega$, and one point on that line, the absolute point $\Omega$. The lines through the point $\Omega$ are isotropic lines, and the points on the line $\omega$ are isotropic points (the points at infinity). Two lines through the same isotropic point are parallel, and two points on the same isotropic line are parallel points. Therefore, an isotropic plane is in fact an affine plane with the pointed direction of isotropic lines where the duality principle is valid.

In an isotropic plane (see e.g. [17] and [18]) two points $P_i = (x_i, y_i)$ $(i = 1, 2)$ have the distance $P_1P_2 = x_2 - x_1$ and two lines with the equations $y = k_i x + l_i$ $(i = 1, 2)$ form the angle $k_2 - k_1$. For two parallel points $P_1$, $P_2$ their span is defined by $s(P_1, P_2) = y_2 - y_1$. Each isotropic line is perpendicular to each nonisotropic line. Two lines with $k_1 = k_2$ are parallel.

A triangle is said to be *allowable* if none of its sides is isotropic. Each allowable triangle $ABC$ can be set by a suitable choice of the coordinate system in the standard position, in which its circumscribed circle $\mathcal{K}_c$ has the equation $y = x^2$, its vertices are the points

$$A = (a, a^2), \quad B = (b, b^2), \quad C = (c, c^2), \tag{1.1}$$

and its sides $BC$, $CA$, $AB$ have the equations

$$y = -ax - bc, \quad y = -bx - ca, \quad y = -cx - ab, \tag{1.2}$$

where

$$a+b+c=0. \tag{1.3}$$

We shall say then that $ABC$ is a *standard triangle*. To prove the geometric facts for each allowable triangle it is sufficient to give a proof for the standard triangle (see [9]).

With the labels

$$p=abc, \quad q=bc+ca+ab, \tag{1.4}$$

a number of useful equalities are proved in [9], as e.g. $a^2+b^2+c^2=-2q$, $a^2=bc-q$, $b^2+bc+c^2=-q$, $2q-3ab=(b-c)(c-a)$, $q+3bc=-(b-c)^2$, $(b-c)^2(c-a)^2(a-b)^2=-(27p^2+4q^3)$. Therefore, $27p^2+4q^3\neq 0$.

## 2. Kiepert hyperbola in an isotropic plane

If the points $A_m$, $B_m$, $C_m$ are the midpoints of the sides of an allowable triangle $ABC$ and if $A'$, $B'$, $C'$ are the points on perpendicular bisectors of these sides such that the spans $\mathrm{s}(A_m,A')$, $\mathrm{s}(B_m,B')$, $\mathrm{s}(C_m,C')$ are proportional to the lengths of the sides $BC$, $CA$, $AB$, then $A'B'C'$ is the so-called *Kiepert triangle* of the triangle $ABC$. If we take the coefficient of proportionality such that

$$\frac{s(A_m,A')}{BC}=\frac{s(B_m,B')}{CA}=\frac{s(C_m,C')}{AB}=-\frac{1}{2}\,t,$$

then the real number $t$ is called *parameter* of the Kiepert triangle $A'B'C'$. In [21], it is shown that the triangles $ABC$ and $A'B'C'$ are homological, i.e. the lines $AA'$, $BB'$, $CC'$ pass through one point $T$ (see Figure 2). In the case of the standard triangle $ABC$ and the Kiepert triangle $A'B'C'$ with the parameter $t$ the point $T$ is given by the formula

$$T=\left(\frac{3pt}{q(2t+3\omega)},-\frac{3\omega t^2+2qt+6q\omega}{3(2t+3\omega)}\right), \tag{2.1}$$

where $\omega=-\frac{1}{3q}(b-c)(c-a)(a-b)$ is the Brocard angle of the triangle $ABC$. The point

$$T'=\left(\frac{3p}{2q},\frac{1}{2}\omega t-\frac{1}{3}q\right),$$

which lies on the Brocard diameter of this triangle with the equation $x=\frac{3p}{2q}$, is isogonal to the point $T$ with respect to the triangle $ABC$ (see [21]).

Theorem 2.1. *The point $T$ given by equality (2.1) determines one special hyperbola $\mathcal{K}$ (see Figure 2) with the equation*

$$3px^2+2qxy+2q^2x-3py-2pq=0. \tag{2.2}$$

Proof. The point $T$ from (2.1) has the coordinates

$$x = \frac{3pt}{q(2t+3\omega)}, \quad y = -\frac{3\omega t^2 + 2qt + 6q\omega}{3(2t+3\omega)}. \tag{2.3}$$

From the first equality in (2.3) there follows

$$t = \frac{3q\omega x}{3p - 2qx}.$$

We substitute $t$ in the second equality of (2.3) written in the form $3y(2t + 3\omega) + 3\omega t^2 + 2qt + 6q\omega = 0$. Then after multiplication by $(3p - 2qx)^2$ and division by $3\omega$, we obtain

$$y[6qx + 3(3p-2qx)](3p-2qx) + (3q\omega x)^2 + 2q^2x(3p-2qx) + 2q(3p-2qx)^2 = 0,$$

i.e. the equation

$$(9py - 2q^2x + 6pq)(3p - 2qx) + (b-c)^2(c-a)^2(a-b)^2x^2 = 0$$

or

$$4q^3x^2 - 18pqxy - 18pq^2x + 27p^2y + 18p^2q - (27p^2 + 4q^3)x^2 = 0,$$

which after dividing by $-9p$ yields the form (2.2). □

If equation (2.2) is written in the form

$$y = -\frac{3px^2 + 2q^2x - 2pq}{2qx - 3p}, \tag{2.4}$$

then it follows that the curve $\mathcal{K}$ has an isotropic asymptote given by the equation $x = \frac{3p}{2q}$, which is the equation of the Brocard diameter of the triangle $ABC$ and the curve $\mathcal{K}$ is one special hyperbola. By analogy with the Euclidean case we shall call it the *Kiepert hyperbola* of the triangle $ABC$. Equation (2.4) can also be written in the form

$$y = -\frac{3p}{2q}x - q - \frac{9p^2}{4q^2} - \frac{p}{4q^2}\,\frac{27p^2 + 4q^3}{2qx - 3p},$$

wherefrom it follows that the hyperbola $\mathcal{K}$ has a nonisotropic asymptote with the equation

$$y = -\frac{3p}{2q}x - q - \frac{9p^2}{4q^2}. \tag{2.5}$$

It is parallel to the Steiner axis of the triangle $ABC$ which, according to [20], has the equation

$$y = -\frac{3p}{2q}x - \frac{2}{3}q. \tag{2.6}$$

With $x = \frac{3p}{2q}$ from (2.5) we obtain

$$y = -q - \frac{9p^2}{2q^2},$$

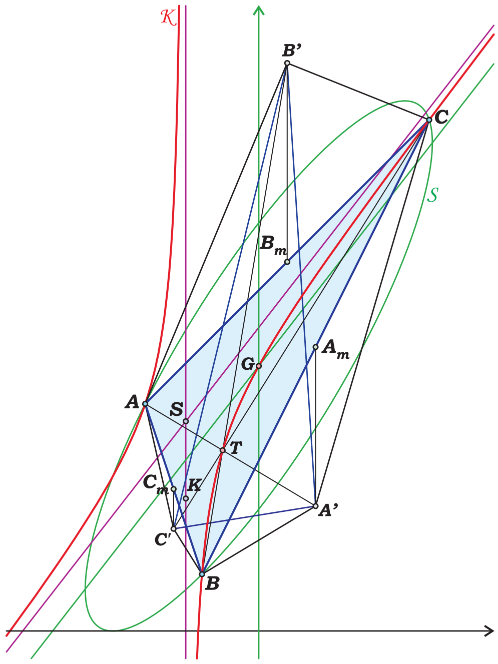


FIGURE 2. Kiepert hyperbola of the standard triangle $ABC$ in an isotropic plane

and the hyperbola $\mathcal{K}$ has the center

$$S = \left( \frac{3p}{2q}, -q - \frac{9p^2}{2q^2} \right). \tag{2.7}$$

The previously mentioned facts can be resumed as the following theorem.

THEOREM 2.2. *The Kiepert hyperbola of the allowable triangle $ABC$ is the isogonal image of its Brocard diameter and this line is its isotropic asymptote, while its nonisotropic asymptote is parallel to the Steiner axis of this triangle. The Kiepert hyperbola of the standard triangle $ABC$ has equation (2.2) and parametric equation (2.3). Its nonisotropic asymptote and its center $S$ are given by equations (2.5) and (2.7), respectively (see Figure 2).*

The first statement of Theorem 2.2 is proved in [10].

As according to [10] the circumscribed circle of the triangle $ABC$ is the isogonal image of the absolute line, it immediately follows:

COROLLARY 2.3. *A nonisotropic asymptote of the Kiepert hyperbola of the allowable triangle $ABC$ is parallel to the lines which are symmetrical to the lines $AO$, $BO$, $CO$ with respect to the angle bisectors of the angles $A$,*

*$B$, $C$, where $O$ is the intersection point of the Brocard diameter with the circumscribed circle of the triangle $ABC$.*

With $x = 0$ from (2.2) we get $y = -\frac{2}{3}\,q$, which can also be obtained from (2.3) with $t = 0$. Hence, the hyperbola $\mathcal{K}$ passes through the centroid $G = (0, -\frac{2}{3}\,q)$ of the triangle $ABC$. It is in accordance with the fact that the symmedian center $K$ of the triangle $ABC$, as the isogonal point to the centroid $G$, lies on its Brocard diameter.

With $y = x^2$ from (2.2), after dividing by $2\,q$, we get the equation $x^3 + qx - p = 0$ for the abscissas of the intersection points of the Kipert hyperbola $\mathcal{K}$ with the circumscribed circle of the triangle $ABC$. The abscissas $a$, $b$, $c$ of the points $A$, $B$, $C$ satisfy this equation because e.g.

$$a^3 + qa = a(a^2 + q) = a \cdot bc = p.$$

Consequently, we have the statement:

Corollary 2.4. *The Kiepert hyperbola of an allowable triangle is circumscribed to this triangle (see Figure 2).*

Homotheties with centers $A$, $B$, $C$ and coefficient 2 map the points $A'$, $B'$, $C'$ to the points on the altitudes of the anticomplementary triangle $A_nB_nC_n$ of the triangle $ABC$, whose spans to the vertices of that triangle are proportional to the lengths of its sides. Therefore it follows:

Corollary 2.5. *Choosing the points on the altitudes of the anticomplementary triangle of the allowable triangle $ABC$ such that its spans to the vertices of the triangle are proportional to the lengths of its sides, we obtain the vertices of a triangle which is homological with the triangle $ABC$, and the center of homology is the point $T$, which determines the Kiepert hyperbola of the triangle $ABC$.*

An analogous statement of Corollary 2.5 in the Euclidean case can be found in [5].

If we state the assertion of Corollary 2.5 for the complementary triangle $A_mB_mC_m$ of the triangle $ABC$, then it can be formulated in this way:

Corollary 2.6. *Choosing the points on the altitudes of the allowable triangle $ABC$ such that its spans to the vertices of the triangle are proportional to the lengths of its sides, we obtain the vertices of a triangle which is homological with the triangle $ABC$, and the center of this homology determines the Kiepert hyperbola of the complementary triangle of the triangle $ABC$.*

## 3. Kiepert hyperbola and some other significant elements of a triangle

Now we are going to study the relationships between the Kiepert hyperbola and some other significant elements of the triangle $ABC$.

The point $S'$ is anticomplementary to the point $S$ from (2.7) and because of the equality $S' = 3G - 2S$, with $G = (0, -\frac{2}{3}q)$, we get

$$S' = \left(-3\frac{p}{q},\ 9\left(\frac{p}{q}\right)^2\right).$$

According to [20], the obtained point $S'$ is the Steiner point of the triangle $ABC$, the fourth common point (with the exception of the points $A$, $B$, $C$) of the circumscribed circle $\mathcal{K}_c$ and the circumscribed Steiner ellipse $\mathcal{S}$ of that triangle (see Figure 3). This consideration enables us to establish the next theorem.

Theorem 3.1. *The center of the Kiepert hyperbola of an allowable triangle is the complementary point to its Steiner point (see Figure 3).*

An analogous statement of Theorem 3.1 in the Euclidean case is given in [3] and [16]
The Euler circle $\mathcal{K}_e$ and the inscribed Steiner ellipse $\mathcal{S}'$ of a triangle are complementary to its circumscribed circle $\mathcal{K}_c$ and the circumscribed Steiner ellipse $\mathcal{S}$, so we have

Corollary 3.2. *The center of the Kiepert hyperbola of an allowable triangle is the fourth intersection (with the exception of the midpoints of its sides) of its Euler circle and its inscribed Steiner ellipse (see Figure 3).*

Theorem 3.3. *If the points $D$, $E$, $F$ are the intersection points of the corresponding sides of the orthic triangle $A_hB_hC_h$ and the complementary triangle $A_mB_mC_m$ of the allowable triangle $ABC$, then the lines $A_mD$, $B_mE$, $C_mF$ pass through the center of the Kiepert hyperbola of that triangle (see Figure 4).*

In the Euclidean case the analogous statement can be found in [5].

Proof. Owing to [9], the lines $B_hC_h$ and $B_mC_m$ are given by the equations of the form $y = 2ax + 2bc - q$, $y = -ax + \frac{1}{2}bc - q$, respectively and due to $bc - q = a^2$ they pass through the point

$$D = \left(-\frac{1}{2}\frac{bc}{a}, a^2\right). \tag{3.1}$$

By [9], the point $A_m$ is of the form

$$A_m = \left(-\frac{1}{2}a, -\frac{1}{2}q - \frac{1}{2}bc\right). \tag{3.2}$$

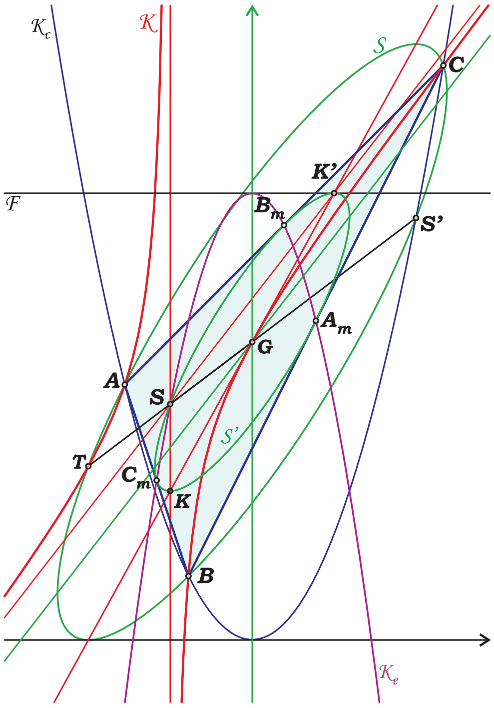


FIGURE 3. The center of the Kiepert hyperbola, the Steiner point, the Euler circle and the Steiner ellipses of the standard triangle $ABC$ in an isotropic plane

The line

$$y = \frac{a}{q}(q - 3bc)x - q - \frac{3ap}{2q}$$

passes through the points $D$ and $A_m$ because of

$$\begin{aligned} \frac{a}{q}\,(q - 3bc)\left(-\frac{bc}{2a}\right) - q - \frac{3ap}{2q} &= -\frac{1}{2}\,bc + \frac{3bc}{2q}(bc - a^2) - q \\ &= -\frac{1}{2}\,bc + \frac{3bc}{2q}\cdot q - (bc - a^2) = a^2, \\ \frac{a}{q}\,(q - 3bc)\left(-\frac{1}{2}\,a\right) - q - \frac{3ap}{2q} &= -\frac{1}{2}\,a^2 + \frac{3a^2bc}{2q} - \frac{3ap}{2q} - q \\ &= -\frac{1}{2}(bc - q) - q = -\frac{1}{2}q - \frac{1}{2}bc, \end{aligned}$$

and owing to

$$\frac{a}{q}(q - 3bc)\frac{3p}{2q} - q - \frac{3ap}{2q} = -\frac{9p^2}{2q^2} - q,$$

it also passes through the point $S$ from (2.7). ∎

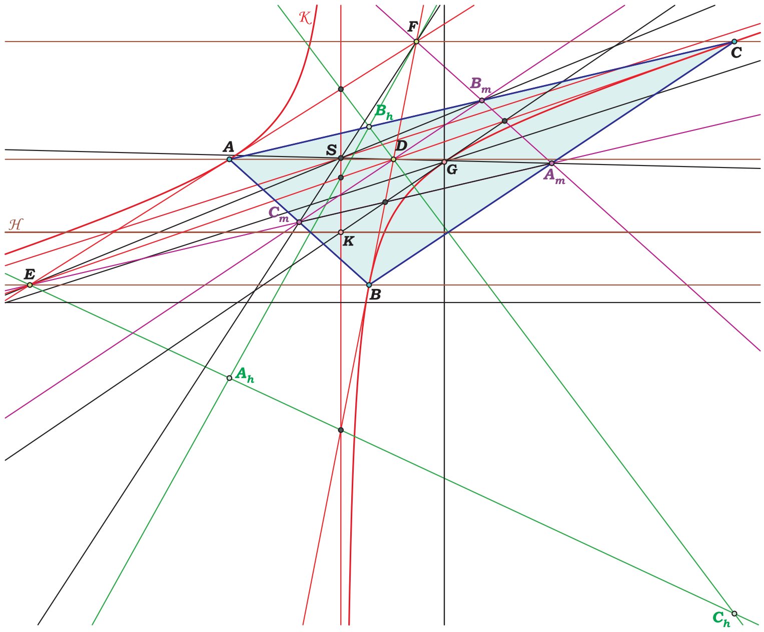


FIGURE 4. The tangent lines of the Kiepert hyperbola and some important lines of the triangle in an isotropic plane

By (3.1), the line $AD$ has the slope zero as well as the orthic line of the triangle $ABC$ (see [9]). Therefore, we obtain the following statement.

COROLLARY 3.4. *With the labels from Theorem 3.3, the lines $AD$, $BE$, $CF$ are parallel to the orthic line of the triangle $ABC$ (see Figure 4).*

Analogous statements in the Euclidean case can be found in [13].

According to Theorem 7 in [21], two points $T_1$ and $T_2$, which in formulas (2.1) correspond to values $t_1$ and $t_2$ of the parameter $t$, have the joint line with the equation

$$y = -\frac{q}{9p}(2t_1t_2 + 3\omega t_1 + 3\omega t_2 - 2q)x + \frac{1}{3}(t_1t_2 - 2q), \tag{3.3}$$

wherefrom by $t_1 = t_2 = t$ the following statement follows immediately.

COROLLARY 3.5. *The tangent line of the Kiepert hyperbola of the standard triangle $ABC$ at the point $T$ from (2.1) is given by the equation*

$$y = -\frac{2q}{9p}(t^2 + 3\omega t - q)x + \frac{1}{3}(t^2 - 2q). \tag{3.4}$$

By $t = 0$, from (3.4) we obtain the equation

$$y = \frac{2q^2}{9p}x - \frac{2}{3}q \tag{3.5}$$

of the line $GK$ and it is the tangent line of the Kiepert hyperbola of the triangle $ABC$ at the centroid $G$ of that triangle.

The line $GK$ and the Brocard diameter $\Omega K$ are the tangent lines of the Kiepert hyperbola of the triangle $ABC$, therefore the line $\Omega G$, the Euler line of that triangle, is the polar line of the point $K$ with respect to this hyperbola. This implies the following statement.

Corollary 3.6. *The Euler line of an allowable triangle is the polar line of its symmedian center with respect to its Kiepert hyperbola.*

One can assign the values $b-c$, $c-a$, $a-b$ of the parameter $t$ to the points $A$, $B$, $C$ on the Kiepert hyperbola of the standard triangle $ABC$ (see Theorem 4 in [21]). However, with $t = b - c$ the right-hand side of equation (3.4) can be written in the form

$$\begin{aligned}
&-\frac{2q}{9p}[(b-c)^2 + 3\omega(b-c) - q]x + \frac{1}{3}[(b-c)^2 - 2q] \\
&= -\frac{2q}{9p}[-(q+3bc) - \frac{1}{q}(b-c)^2(c-a)(a-b) - q]x + \frac{1}{3}[-(q+3bc) - 2q] \\
&= -\frac{2}{9p}[-q(2q+3bc) + (q+3bc)(2q-3bc)]x - q - bc \\
&= \frac{2}{9p} \cdot 9b^2c^2x - q - bc = \frac{2bc}{a}x - q - bc,
\end{aligned}$$

so we obtain:

Theorem 3.7. *The tangent lines $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$ of the Kiepert hyperbola of the standard triangle $ABC$ at its vertices $A$, $B$, $C$ are given by the equations*

$$y = \frac{2bc}{a}x - q - bc, \qquad y = \frac{2ca}{b}x - q - ca, \qquad y = \frac{2ab}{c}x - q - ab. \tag{3.6}$$

In the Euclidean case the analogous statement can be found in [15].

Let us multiply the first equation in (3.6) by $3bc - 2q$, and the equation $y = -ax + \frac{1}{2}bc - q$ of the midline $B_mC_m$ by $2(q+3bc)$. By adding the obtained equations we obtain the equation where the coefficient next to $y$ is $9bc$, the coefficient next to $x$ is

$$\frac{2bc}{a}(3bc-2q) - 2a(q+3bc) = \frac{1}{a}[2bc(3bc-2q) - 2(bc-q)(q+3bc)] = \frac{1}{a} \cdot 2q^2,$$

and the free coefficient is of the form

$$-(q+bc)(3bc-2q) + (bc-2q)(q+3bc) = -6bcq,$$

which can be reduced to $9bcy = \frac{2}{a}\,q^2x - 6bcq$. This equation determines equation (3.5) of the line $GK$. Therefore, we have proved:

THEOREM 3.8. *The tangent lines of the Kiepert hyperbola of an allowable triangle at its vertices meet the corresponding midlines at the points which lie on a tangent line of this hyperbola at the centroid of the considered triangle (see Figure 4).*

THEOREM 3.9. *The tangent lines of the Kiepert hyperbola of an allowable triangle at its vertices meet the corresponding sides of its orthic triangle at the points which lie on the Brocard diameter of the considered triangle, i.e., on the isotropic asymptote of this hyperbola (see Figure 4).*

PROOF. For example, we can get the abscissae of the point $\mathcal{A} \cap B_hC_h$ . Namely, out of the first equation in (3.6) and the equation of the line $B_hC_h$ that can be found in the proof of Theorem 3.3 we get $x = \frac{3p}{2q}$ that proves the claim of theorem. □

In the Euclidean case the analogous statements of Theorems 3.8 and 3.9 can be found in [13].

THEOREM 3.10. *The line $GK$ meets a nonisotropic asymptote of the Kiepert hyperbola of the triangle $ABC$ at the point $K'$, which is symetrical to the point $K$ with respect to the point $G$ and which lies on the Feuerbach line of the triangle $ABC$ (see Figure 3).*

PROOF. The point

$$K' = \left(-\frac{3p}{2q}, -q\right) \tag{3.7}$$

lies on lines (3.5) and (2.5) because each of two numbers

$$\frac{2q^2}{9p}\left(-\frac{3p}{2q}\right) - \frac{2}{3}q \quad \text{and} \quad \frac{-3p}{2q}\left(-\frac{3p}{2q}\right) - q - \frac{9p^2}{4q^2}$$

is equal to $-q$. The midpoint of the point $K'$ from (3.7) and the symmedian center $K$, which is by [8] of the form $K = \left(\frac{3p}{2q}, -\frac{1}{3}q\right)$, is the centroid $G = (0, -\frac{2}{3}\,q)$ of the triangle $ABC$. The point $K'$ also lies on the Feuerbach line of that triangle, which by [2] has the equation $y = -q$. □

According to [20], the inscribed Steiner ellipse $\mathcal{S}'$ of the triangle $ABC$ passes through the point $K$ and its center is the point $G$. Therefore, we have the following statement.

COROLLARY 3.11. *The point $K'$ from Theorem 3.10 lies on the inscribed Steiner ellipse $\mathcal{S}'$ of the triangle $ABC$ (see Figure 3).*

The line through the point $K$, parallel to line (2.5), is defined by

$$y = -\frac{3p}{2q}x + \frac{9p^2}{2q^2} - \frac{1}{3}q. \tag{3.8}$$

By adding (2.5) and (3.8) and dividing the result by 2 we get (2.6). Therefore, line (2.6) is the angle bisector of lines (2.5) and (3.8) and it passes through the point $K$. This consideration enables us to establish the next theorem.

THEOREM 3.12. *If $K$ is the symmedian center of an allowable triangle $ABC$, then the homothety $(K, 2)$ maps its Steiner axis to the nonisotropic asymptote of its Kiepert hyperbola.*

An analogous statement of Theorem 3.12 in the Euclidean case states that the asymptotes of the Kiepert hyperbola of a triangle are parallel to its Steiner axes and can be found in [11] and [16].

Let $T$ be any point on the hyperbola $\mathcal{K}$. Then a quadrangle $ABCT$ is inscribed in this hyperbola and the triangle with vertices $L = BC \cap AT$, $M = CA \cap BT$, $N = AB \cap CT$ is the autopolar triangle with respect to the hyperbola $\mathcal{K}$. Therefore, e.g. the line $MN$ is the polar line of the point $L$ with respect to this hyperbola. Due to collinearity of the points $B$, $C$, $L$, polar lines of these points pass through one point $D$. Polar lines of the points $B$, $C$ are the tangent lines at these points, and the point $D$ is the pole of the line $BC$, and the polar line $MN$ of the point $L$ passes through this point $D$. The absolute point $\Omega$ and the centroid $G$ of the triangle lie on the hyperbola $\mathcal{K}$. With $T = G$, the triangle $LMN$ is a complementary triangle $A_mB_mC_m$, and with $T = \Omega$, the triangle $LMN$ is an orthic triangle $A_hB_hC_h$ of the triangle $ABC$. Therefore, the point $D$ also lies on the lines $B_mC_m$ and $B_hC_h$, and the point $D$ is in fact the point $D$ from Theorem 3.3. Analogously, it is valid for the points $E$ and $F$, i.e. we have:

THEOREM 3.13. *The intersection points $D$, $E$, $F$ of the corresponding sides of the orthic triangle $A_hB_hC_h$ and the complementary triangle $A_mB_mC_m$ of an allowable triangle $ABC$ are the poles of the lines $BC$, $CA$, $AB$ with respect to the Kiepert hyperbola of the triangle $ABC$. If $T$ is any point of that hyperbola then with $L = BC \cap AT$, $M = CA \cap BT$, $N = AB \cap CT$ the lines $MN$, $NL$, $LM$ pass through the points $D$, $E$, $F$, respectively.*

COROLLARY 3.14. *The complementary triangle $A_mB_mC_m$ and the orthic triangle $A_hB_hC_h$ of an allowable triangle $ABC$ are autopolar triangles with respect to the Kiepert hyperbola of the triangle $ABC$.*

The fact that $D$ is the pole of the line $BC$ can easily be verified directly and analytically. The point $D$ from (3.1) lies on the tangent lines $\mathcal{B}$, $\mathcal{C}$ of the hyperbola $\mathcal{K}$ at the points $B$, $C$ because, for example for the tangent line $\mathcal{B}$, out of the second equation in (3.6), we get

$$\frac{2ca}{b}\left(-\frac{bc}{2a}\right) - q - ca = -c^2 - ca - q = bc - q = a^2.$$

Therefore, the following is valid.

THEOREM 3.15. *Tangent lines of the Kiepert hyperbola of an allowable triangle $ABC$ at its vertices $A$, $B$, $C$ determine the triangle $DEF$, whose vertices $D$, $E$, $F$ are the intersection points of the corresponding sides of the complementary triangle and the orthic triangle of the triangle $ABC$ (see Figure 4).*

In the Euclidean case the analogous statements of Theorems 3.13 and 3.15 are in [1], p.137, and in [14].

THEOREM 3.16. *The Kiepert hyperbola of an allowable triangle $ABC$ passes through the points $U'$, $V'$, $W'$ which are anticomplementary to the midpoints $U$, $V$, $W$ of the segments $AS'$, $BS'$, $CS'$, where $S'$ is the Steiner point of the triangle $ABC$, while $AU'$, $BV'$, $CW'$ are the diameters of this hyperbola.*

PROOF. By Theorem 3.1, the point $S$, complementary to the point $S'$, is the center of the Kiepert hyperbola. We obtain the equalities $A + S' = 2U$, $S' + 2S = 3G$, $2U + U' = 3G$, where $G$ is the centroid of the triangle $ABC$, wherefrom

$$A + U' = 2U - S' + 3G - 2U = 3G - S' = 2S,$$

i.e., $S$ is the midpoint of the segment $AU'$. □

According to Theorem 8 in [21], two points $T_1$ and $T_2$ given by parameters $t = t_1$ and $t = t_2$ from (2.1) and the point $T_3' = (\frac{3p}{2q}, \frac{1}{2}\,\omega t_3 - \frac{1}{3}\,q)$ are collinear if and only if $t_1 + t_2 + t_3 = 0$. With $t_3 = 3\,\omega$, the point $T_3'$ has the ordinate

$$\frac{3}{2}\omega^2 - \frac{1}{3}q = -\frac{1}{6q^2}(27p^2 + 4q^3) - \frac{1}{3}q = -q - \frac{9p^2}{2q^2}$$

and it coincides with the point $S$ from (2.7), the center of the Kiepert hyperbola of the triangle $ABC$. This proves the following theorem.

THEOREM 3.17. *The points $T_1$ and $T_2$ of the Kiepert hyperbola of the standard triangle $ABC$ with the parametric equations (2.3), which correspond to the values $t_1$ and $t_2$ of the parameter $t$, are diametrically opposite points of that hyperbola if and only if*

$$t_1 + t_2 = -3\omega. \tag{3.9}$$

By $t_2 = 0$ from (3.9), we find that $t_1 = -3\omega$, and by $t = -3\omega$, according to Theorem 5 in [21], from (2.1) we get $T = \left(\frac{3p}{q}, 3\omega^2\right)$ for a diametrically opposite point of the centroid $G$ on the considered hyperbola $\mathcal{K}$. The obtained point $T$ lies on the circumscribed Steiner ellipse of the triangle $ABC$ and therefore it is the fourth intersection point (with the exception of the points $A$, $B$, $C$) of this ellipse with the hyperbola $\mathcal{K}$. So we have proved the statement.

Theorem 3.18. *The fourth intersection point of the circumscribed Steiner ellipse with the Kiepert hyperbola of an allowable triangle $ABC$ and the centroid of this triangle are the endpoints of one diameter of this hyperbola (see Figure 3).*

The previous result implies the equality $G+T=2S$, which together with the equality $2S+S'=3G$ gives $T+S'=2G$. As by [20] the centroid $G$ is the center of the circumscribed Steiner ellipse of the triangle $ABC$, we get:

Theorem 3.19. *The fourth intersection $T$ of the circumscribed Steiner ellipse $\mathcal{S}$ with the Kiepert hyperbola of an allowable triangle $ABC$ is symmetrical to the Steiner point $S'$ of that triangle with respect to its centroid, i.e. $TS'$ is a diameter of the ellipse $\mathcal{S}$ (see Figure 3).*

In the Euclidean case the analogous statement is given in [12].

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## Kiepertova hiperbola u izotropnoj ravnini

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Sažetak. Pojam Kiepertove hiperbole dopustivog trokuta u izotropnoj ravnini je uveden u ovom članku. Značajna svojstva Kiepertove hiperbole su istražena u slučaju standardnog trokuta. Relacije između uvednih pojmova i nekih dobro poznatih elemenata trokuta su također proučavane.

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