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THE QUOTIENT SHAPES OF l_p AND L_p SPACES

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Abstract. All l_p spaces (over the same field), $p \neq \infty$, have the finite quotient shape type of the Hilbert space l_2. It is also the finite quotient shape type of all the subspaces $l_p(p')$, $p < p' \leq \infty$, as well as of all their direct sum subspaces $F_0^{\mathbb{N}}(p')$, $1 \leq p' \leq \infty$. Furthermore, their countable and finite quotient shape types coincide. Similarly, for a given positive integer, all L_p spaces (over the same field) have the finite quotient shape type of the Hilbert space L_2, and their countable and finite quotient shape types coincide. Quite analogous facts hold true for the (special type of) Sobolev spaces (of all appropriate real functions).

1. Introduction

The shape theory (for compacta in the Hilbert cube) was founded by K. Borsuk, [1]. The theory was rapidly developed and generalized by many authors. The main references are [2], [3], [5], [6] and, especially, [11]. Although, in general, founded purely categorically, a shape theory is mostly well known only as the (standard) shape theory of topological spaces with respect to spaces having homotopy types of polyhedra. The generalizations founded in [8] and [16] are, primarily, also on that line.

The quotient shape theory was recently introduced by the author, [13]. It is, of course, a kind of the general (abstract) shape theory, [I.2, 11]. However, it is possible and non-trivial, and can be straightforwardly developed for every concrete category $\mathcal{C}$ and for every infinite cardinal $\kappa \geq \aleph_0$. Concerning a shape of objects, in general, one has to decide which ones are "nice" absolutely and/or relatively (with respect to the chosen ones). In this approach, the main principle reads as follows: *An object is "nice" if it is isomorphic to a quotient object belonging to a special full subcategory and if it (its "basis") has cardinality less than (less than or equal to) a given infinite cardinal.* It leads to the basic idea: to approximate a $\mathcal{C}$-object X by a suitable inverse

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system consisting of its quotient objects X_λ (and the quotient morphisms) which have cardinalities, or dimensions, in the case of vectorial spaces, less than (less than or equal to) κ. Such an approximation exists in the form of any κ^--expansion (κ-expansion) of X,

$$\boldsymbol{p}_{\kappa^-} = (p_\lambda) : X \to \boldsymbol{X}_{\kappa^-} = (X_\lambda, p_{\lambda\lambda'}, \Lambda_{\kappa^-})$$
$$(\boldsymbol{p}_\kappa = (p_\lambda) : X \to \boldsymbol{X}_\kappa = (X_\lambda, p_{\lambda\lambda'}, \Lambda_\kappa)),$$

where $\boldsymbol{X}_{\kappa^-}$ ($\boldsymbol{X}_\kappa$) belongs to the subcategory pro-$\mathcal{D}_{\kappa^-}$ (pro-$\mathcal{D}_\kappa$) of pro-$\mathcal{D}$, and $\mathcal{D}_{\kappa^-}$ ($\mathcal{D}_\kappa$) is the subcategory of $\mathcal{D}$ determined by all the objects having cardinalities, or dimensions, for vectorial spaces, less than (less than or equal to) κ, while $\mathcal{D}$ is a full subcategory of $\mathcal{C}$. Clearly, if $X \in Ob\mathcal{D}$ and the cardinality $|X| < \kappa$ ($|X| \leq \kappa$), then the rudimentary pro-morphism $\lfloor 1_X \rfloor : X \to \lfloor X \rfloor$ is a κ^--expansion (κ-expansion) of X. The corresponding shape category $Sh_{\mathcal{D}_{\kappa^-}}(\mathcal{C})$ ($Sh_{\mathcal{D}_\kappa}(\mathcal{C})$) and shape functor $S_{\kappa^-} : \mathcal{C} \to Sh_{\mathcal{D}_{\kappa^-}}(\mathcal{C})$ ($S_\kappa : \mathcal{C} \to Sh_{\mathcal{D}_\kappa}(\mathcal{C})$) exist by the general (abstract) shape theory, and they have all the appropriate general properties. Moreover, there exist the relating functors $S_{\kappa^-\kappa} : Sh_{\mathcal{D}_\kappa}(\mathcal{C}) \to Sh_{\mathcal{D}_{\kappa^-}}(\mathcal{C})$ and $S_{\kappa\kappa'} : Sh_{\mathcal{D}_{\kappa'}}(\mathcal{C}) \to Sh_{\mathcal{D}_\kappa}(\mathcal{C})$, $\kappa \leq \kappa'$, such that $S_{\kappa^-\kappa} S_\kappa = S_{\kappa^-}$ and $S_{\kappa\kappa'} S_{\kappa'} = S_\kappa$. We should mention that the simplest and very often interesting case is even $\mathcal{D} = \mathcal{C}$. In such a case we simplify the notation $Sh_{\mathcal{D}_{\kappa^-}}(\mathcal{C})$ ($Sh_{\mathcal{D}_\kappa}(\mathcal{C})$) to $Sh_{\kappa^-}(\mathcal{C})$ ($Sh_\kappa(\mathcal{C})$) or to Sh_{κ^-} (Sh_κ) when $\mathcal{C}$ is fixed.

Especially, in the case of $\kappa = \aleph_0$, the κ^--shape is said to be the *finite (quotient) shape*, because all the objects in the expansions are of finite (bases) cardinalities, and the category is denoted by $Sh_{\mathcal{D}_{\underline{0}}}(\mathcal{C})$ or by $Sh_{\underline{0}}(\mathcal{C}) \equiv Sh_{\underline{0}}$ only, whenever $\mathcal{D} = \mathcal{C}$. The $\aleph_0$-shape is said to be the *countable (quotient) shape*, and the quotient shape category is denoted by $Sh_{\aleph_0}(\mathcal{C}) \equiv Sh_{\aleph_0}$ only, whenever $\mathcal{D} = \mathcal{C}$ is fixed.

In [13], several well known concrete categories were considered and many examples are given which show that the quotient shape theory yields classifications strictly coarser than those by isomorphisms. In [14] and [15] were considered the quotient shapes of (purely algebraic, topological and normed) vectorial spaces and topological spaces, respectively. In this paper we continue the studying of quotient shapes of normed vectorial spaces of [Section 4.1, 14], primarily focused to the well known l_p and L_p spaces. In the last section, the same technique is applied to the Sobolev spaces (a special kind, i.e., including only functions having *the usual* partial derivatives). The main results (the continuum hypothesis CH assumed) briefly read as follows:

- all l_p spaces, for $p \neq \infty$, i.e., $1 \leq p < \infty$, over the same field $F \in \{\mathbb{R}, \mathbb{C}\}$, are of the same finite quotient shape type and admit a common expansion-object consisting of Hilbert spaces;
- to that type belong all the (proper) subspaces $l_p(p') \trianglelefteq l_{p'}$, $p < p' \leq \infty$, too, where $l_p(p')$ is l_p (algebraically) carrying the norm $\|\cdot\|_{p'}$;

- to that type also belong all their direct sum subspaces $F_0^{\mathbb{N}}(p') \trianglelefteq l_{p'}$, where $F_0^{\mathbb{N}}(p')$ is $F_0^{\mathbb{N}}$ (algebraically) carrying the norm $\|\cdot\|_{p'}$;
- for each $n \in \mathbb{N}$, all $L_p(K_n)$ spaces ($K_n \subseteq \mathbb{R}^n$)$-$ n-cube), over the same field, are of the same finite quotient shape type and admit a common expansion-object consisting of Hilbert spaces;
- given an $n \in \mathbb{N}$, for each domain $\Omega_n \subseteq \mathbb{R}^n$ and each $k \in \mathbb{N}$, all Sobolev spaces $W_p^{(k)}(\Omega_n)$ (consisting of all real functions on Ω_n having their supports in Ω_n and all partial derivatives up to order k continuous), over the same field, are of the same finite quotient shape type and admit a common expansion-object consisting of Hilbert spaces;
- the countable shape types of all l_p spaces, $p \neq \infty$, (all $L_p(K_n)$ spaces, n fixed; all $W_p^{(k)}(\Omega_n)$ spaces, Ω_n and k fixed) over the same field reduce to the unique finite quotient shape type (respectively).

2. Preliminaries

For the sake of completeness, let us briefly repeat the construction of a quotient shape category and a quotient shape functor ([13], [14]). Given a category pair $(\mathcal{C}, \mathcal{D})$, where $\mathcal{D} \subseteq \mathcal{C}$ is full, and a cardinal κ, let $\mathcal{D}_{\kappa^-}$ ($\mathcal{D}_\kappa$) denote the full subcategory of $\mathcal{D}$ determined by all the objects having cardinalities or, in some special cases, the cardinalities of "bases" less than (less or equal to) κ. By following the main principle, let $(\mathcal{C}, \mathcal{D}_{\kappa^-})$ ($(\mathcal{C}, \mathcal{D}_\kappa)$) be such a pair of *concrete* categories. If

(a) every $\mathcal{C}$-object (X, σ), where σ indicates a structure on the set-object X, admits a directed set $R(X, \sigma, \kappa^-) \equiv \Lambda_{\kappa^-}$ ($R(X, \sigma, \kappa) \equiv \Lambda_\kappa$) of equivalence relations λ on X such that each quotient object $(X/\lambda, \sigma_\lambda)$ has to belong to $\mathcal{D}_{\kappa^-}$ ($\mathcal{D}_\kappa$), while each quotient morphism p_λ : $(X, \sigma) \to (X/\lambda, \sigma_\lambda)$ has to belong to $\mathcal{C}$;
(b) the induced morphisms between quotient objects belong to $\mathcal{D}_{\kappa^-}$ ($\mathcal{D}_\kappa$);
(c) every morphism $f : (X, \sigma) \to (Y, \tau)$ of $\mathcal{C}$, having the codomain in $\mathcal{D}_{\kappa^-}$ ($\mathcal{D}_\kappa$), factorizes uniquely through a quotient morphism $p_\lambda : (X, \sigma) \to (X/\lambda, \sigma_\lambda)$, $f = gp_\lambda$, with g belonging to $\mathcal{D}_{\kappa^-}$ ($\mathcal{D}_\kappa$),

then $\mathcal{D}_{\kappa^-}$ ($\mathcal{D}_\kappa$) is a pro-reflective subcategory of $\mathcal{C}$. Consequently, there exists a (nontrivial) *"quotient shape" category* $Sh_{(\mathcal{C}, \mathcal{D}_{\kappa^-})} \equiv Sh_{\mathcal{D}_{\kappa^-}}(\mathcal{C})$ ($Sh_{(\mathcal{C}, \mathcal{D}_\kappa)} \equiv Sh_{\mathcal{D}_\kappa}(\mathcal{C})$) obtained by the general construction.

Therefore, a κ^--shape morphism $F_{\kappa^-} : (X, \sigma) \to (Y, \tau)$ is represented by a diagram (in *pro*-$\mathcal{C}$)

$$\begin{array}{ccc} (\boldsymbol{X}, \boldsymbol{\sigma})_{\kappa^-} & \overset{p_{\kappa^-}}{\leftarrow} & (X, \sigma) \\ \boldsymbol{f}_{\kappa^-} \downarrow & & \\ (\boldsymbol{Y}, \boldsymbol{\tau})_{\kappa^-} & \overset{q_{\kappa^-}}{\leftarrow} & (Y, \tau) \end{array}$$

(with $\boldsymbol{p}_{\kappa^-}$ and $\boldsymbol{q}_{\kappa^-}$ - a pair of appropriate expansions), and similarly for a κ-shape morphism $F_\kappa : (X,\sigma) \to (Y,\tau)$. Since all $\mathcal{D}_{\kappa^-}$-expansions ($\mathcal{D}_\kappa$-expansions) of a $\mathcal{C}$-object are mutually isomorphic objects of *pro*-$\mathcal{D}_{\kappa^-}$ (*pro*-$\mathcal{D}_\kappa$), the composition and identities follow straightforwardly. Observe that every quotient morphism p_λ is an effective epimorphism (if $U : \mathcal{C} \to Set$ is the forgetful functor related to the concrete category $\mathcal{C}$, then $U(p_\lambda)$ is a surjection), and thus condition (E2) for an expansion follows trivially.

The corresponding *"quotient shape" functors* $S_{\kappa^-} : \mathcal{C} \to Sh_{\mathcal{D}_{\kappa^-}}(\mathcal{C})$ and $S_\kappa : \mathcal{C} \to Sh_{\mathcal{D}_\kappa}(\mathcal{C})$ are defined in the same general manner. That means,

- $S_{\kappa^-}(X,\sigma) = S_\kappa(X,\sigma) = (X,\sigma)$;
- if $f : (X,\sigma) \to (Y,\tau)$ is a $\mathcal{C}$-morphism, then, for every $\mu \in M_{\kappa^-}$, the composite $g_\mu f : (Y,\tau) \to (Y_\mu,\tau_\mu)$ factorizes (uniquely) through a $p_{\lambda(\mu)} : (X,\sigma) \to (X_{\lambda(\mu)},\sigma_{\lambda(\mu)})$, and thus, the correspondence $\mu \mapsto \lambda(\mu)$ yields a function $\varphi : M_{\kappa^-} \to \Lambda_{\kappa^-}$ and a family of $\mathcal{D}_{\kappa^-}$-morphisms $f_\mu : (X_{\varphi(\mu)},\sigma_{\varphi(\mu)}) \to (Y_\mu,\tau_\mu)$ such that $q_\mu f = f_\mu p_{\varphi(\mu)}$;
- one easily shows that $(\varphi, f_\mu) : (\boldsymbol{X},\boldsymbol{\sigma})_{\kappa^-} \to (\boldsymbol{Y},\boldsymbol{\tau})_{\kappa^-}$ is a morphism of *inv*-$\mathcal{D}_{\kappa^-}$, so the equivalence class $\boldsymbol{f}_{\kappa^-} = [(\varphi, f_\mu)] : (\boldsymbol{X},\boldsymbol{\sigma})_{\kappa^-} \to (\boldsymbol{Y},\boldsymbol{\tau})_{\kappa^-}$ is a morphism of *pro*-$\mathcal{D}_{\kappa^-}$;
- then we put $S_{\kappa^-}(f) = \langle \boldsymbol{f}_{\kappa^-} \rangle \equiv F_{\kappa^-} : (X,\sigma) \to (Y,\tau)$ in $Sh_{\mathcal{D}_{\kappa^-}}(\mathcal{C})$.

The identities and composition are obviously preserved. In the same way one defines the functor S_κ.

Furthermore, since $(\boldsymbol{X},\boldsymbol{\sigma})_{\kappa^-}$ is a subsystem of $(\boldsymbol{X},\boldsymbol{\sigma})_\kappa$ (more precisely, $(\boldsymbol{X},\boldsymbol{\sigma})_\kappa$ is a subobject of $(\boldsymbol{X},\boldsymbol{\sigma})_{\kappa^-}$ in *pro*-$\mathcal{D}$), one easily shows that there exists a functor $S_{\kappa^-\kappa} : Sh_{\mathcal{D}_\kappa}(\mathcal{C}) \to Sh_{\mathcal{D}_{\kappa^-}}(\mathcal{C})$ such that $S_{\kappa^-\kappa} S_\kappa = S_{\kappa^-}$, i.e., the diagram

$$\begin{array}{ccccc} & & \mathcal{C} & & \\ & \swarrow S_{\kappa^-} & & S_\kappa \searrow & \\ Sh_{\mathcal{D}_{\kappa^-}}(\mathcal{C}) & & \xleftarrow{S_{\kappa^-\kappa}} & & Sh_{\mathcal{D}_\kappa}(\mathcal{C}) \end{array}$$

commutes. Moreover, an analogous functor $S_{\kappa\kappa'} : Sh_{\mathcal{D}_{\kappa'}}(\mathcal{C}) \to Sh_{\mathcal{D}_\kappa}(\mathcal{C})$, satisfying $S_{\kappa\kappa'} S_{\kappa'} = S_\kappa$, exists for every pair of infinite cardinals $\kappa \leq \kappa'$.

3. On the quotient shapes of normed and Banach spaces

We shall now apply the quotient shape theory to the category $\mathcal{C} = NVect_F$ of all normed vectorial spaces over $F \in \{\mathbb{R},\mathbb{C}\}$ (with $\mathcal{D} \subseteq \mathcal{C}$ and, especially, $\kappa = \aleph_0$; see also [Section 4.1, 14]) and to the category $\mathcal{C} = BVect_F$ of all Banach spaces (with $\mathcal{D} = \mathcal{C}$). Clearly, the morphisms of $\mathcal{C}$ are all the corresponding continuous linear functions. Hereby, $\mathcal{D}_{\kappa^-}$ ($\mathcal{D}_\kappa$) denotes the full subcategory determined by all the objects having dimension (the cardinality of an *algebraic* base) less than (less or equal to) κ. Recall that, by the results of [14], the *algebraic* quotient shape type classifications and the isomorphism classification of vectorial spaces coincide, while those of the *normed* (and *topological* as well) vectorial spaces are, in general, strictly coarser than the isomorphism classification. Hereby we give much more attention to the Banach spaces, especially to the well known l_p and L_p spaces.

We shall frequently use and apply in the sequel several general or special well known facts without referring to any source. So we remind the readers that

- our general shape theory technique is that of [11];
- the needed set theoretic (especially, concerning cardinals) and topological facts can be found in [4];
- the facts concerning functional analysis are taken from [9], [10] or [12];
- our category theory language follows that of [7].

Since we are dealing with the quotient spaces, recall that the "quotient" norm on the quotient normed space X/Z (Z is a closed subspaces of X) is defined by

$$\|[x]\| = \inf\{\|x + z\| \mid z \in Z\}.$$

By Theorem 4.2 of [14], for every $\kappa \geq \aleph_0$, the subcategories

$$(NVect_F)_{\kappa^-}, (NVect_F)_\kappa \subseteq NVect_F$$

are pro-reflective, i.e., every normed vectorial space X admits an $(NVect_F)_{\kappa^-}$-expansion and an $(NVect_F)_\kappa$-expansion. The following needed fact is a consequence of that theorem and its proof (see also Remark (4.9) of [14]).

THEOREM 3.1. *For every cardinal $\kappa \geq \aleph_0$, the subcategories*

$$(BVect_F)_{\kappa^-}, (BVect_F)_\kappa \subseteq BVect_F$$

are pro-reflective.

PROOF. Recall the well known fact that each closed subspace Z of a Banach space X yields the quotient space X/Z that is a Banach space. □

The three following theorems bring the main facts for our purpose.

THEOREM 3.2. *For every cardinal $\kappa \geq \aleph_0$, the subcategories*

$$(BVect_F)_{\kappa^-}, (BVect_F)_\kappa \subseteq NVect_F$$

are pro-reflective.

PROOF. It is a well known fact that every normed vectorial space Y admits a dense isometric linear embedding into a Banach space X (over the same field $F \in \{\mathbb{R}, \mathbb{C}\}$). So we may assume, without loss of generality, that $Y \trianglelefteq X$ and $Cl(Y) = X$. If $\dim Y < \aleph_0$, then $Cl(Y) = X$ means that $Y = X$, and we may apply Theorem 3.1 to Y. Let $\dim Y \geq \aleph_0$. Then, clearly, $\dim X \geq \dim Y \geq \aleph_0$. Let $\kappa \geq \aleph_0$ and let

$$\boldsymbol{p}_{\kappa^-} = (p_\lambda) : X \to \boldsymbol{X}_{\kappa^-} = (X_\lambda, p_{\lambda\lambda'}, \Lambda_{\kappa^-})$$

be a $(BVect_F)_{\kappa^-}$-expansion of X, which exists by Theorem 3.1. Notice that the inclusion $i : Y \hookrightarrow X$ is a continuous linear function. We are to show that the composite

$$\boldsymbol{q}_{\kappa^-} \equiv \boldsymbol{p}_{\kappa^-} i = (q_\lambda = p_\lambda i) : Y \to \boldsymbol{X}_{\kappa^-} = (X_\lambda, p_{\lambda\lambda'}, \Lambda_{\kappa^-})$$

is a $(BVect_F)_{\kappa^-}$-expansion of Y. The commutativity condition $p_{\lambda\lambda'} q_{\lambda'} = q_\lambda$, $\lambda \leq \lambda'$, obviously holds. Let Z be a Banach space (over the same F) such that $\dim Z < \kappa$, and let $f : Y \to Z$ be a continuous linear function. Since Z is a Banach space, there exists a unique continuous linear extension $g : X \to Z$ of f, i.e., $gi = f$. Since $\boldsymbol{p}_{\kappa^-} : X \to \boldsymbol{X}_{\kappa^-}$ is a $(BVect_F)_{\kappa^-}$-expansion, there exist a $\lambda \in \Lambda_{\kappa^-}$ and a unique continuous linear function $g^\lambda : X_\lambda \to Z$ such that $g^\lambda p_\lambda = g$. Then $g^\lambda q_\lambda = g^\lambda p_\lambda i = gi = f$, which shows that $\boldsymbol{q}_{\kappa^-} \equiv \boldsymbol{p}_{\kappa^-} i : Y \to \boldsymbol{X}_{\kappa^-}$ is a $(BVect_F)_{\kappa^-}$-expansion of Y. The proof in the κ-case is quite similar. □

Denote by $(NVect_F)_{\kappa^-}$ and $(BVect_F)_{\kappa^-}$ ($(NVect_F)_\kappa$ and $(BVect_F)_\kappa$) the full subcategory of $NVect_F$ and $BVect_F$, respectively, determined by all the objects having dimension, i.e., the cardinality of an *algebraic* base, less than (less or equal to) κ. If $\kappa = \aleph_0$, we denote the κ^--case by $(NVect_F)_{\underline{0}}$ and $(BVect_F)_{\underline{0}}$. Since $BVect_F \subseteq NVect_F$, the "relative" case $(NVect_F, (BVect_F)_{\kappa^-})$ ($(NVect_F, (BVect_F)_\kappa)$) admits to consider the quotient shape of a normed space with respect to Banach spaces, i.e., via a $(BVect_F)_{\kappa^-}$-expansion ($(BVect_F)_\kappa$-expansion).

THEOREM 3.3. *Let X be a normed vectorial space and let $Y \trianglelefteq X$ be a dense subspace, $Cl(Y) = X$. Then*

(i) $Sh_{\underline{0}}(Y) = Sh_{\underline{0}}(X)$ with respect to $(NVect_F, (NVect_F)_{\underline{0}})$ as well as to $(NVect_F, (BVect_F)_{\underline{0}})$.

If, in addition, X is a Banach space and $\dim Y \geq \kappa \geq \aleph_0$ *(*$\dim Y > \kappa \geq \aleph_0$*), then*

(ii) $Sh_{\kappa^-}(Y) = Sh_{\kappa^-}(X)$ with respect to the category pair $(NVect_F, (BVect_F)_{\kappa^-})$;

(iii) $Sh_\kappa(Y) = Sh_\kappa(X)$ with respect to the category pair $(NVect_F, (BVect_F)_\kappa)$.

PROOF. Statement (i) can be proven likewise the proof of Theorem 3.2. If $\dim Y < \aleph_0$, then $Cl(Y) = X$ means $Y = X$. Thus (i) holds true trivially (as well as (ii) and (iii)). Let $\dim Y \geq \aleph_0$. Then, clearly, $\dim X \geq \dim Y \geq \aleph_0$. Notice that the inclusion $i : Y \to X$ is a continuous linear function. Let

$$\boldsymbol{p}_{\underline{0}} = (p_\lambda) : X \to \boldsymbol{X}_{\underline{0}} = (X_\lambda, p_{\lambda\lambda'}, \Lambda_{\underline{0}})$$

be an $(NVect_F)_{\underline{0}}$-expansion of X. In order to prove statement (i), it suffices to show that the composite

$$\boldsymbol{q}_{\underline{0}} \equiv \boldsymbol{p}_{\underline{0}} i = (q_\lambda = p_\lambda i) : Y \to \boldsymbol{X}_{\underline{0}} = (X_\lambda, p_{\lambda\lambda'}, \Lambda_{\underline{0}})$$

is an $(NVect_F)_{\underline{0}}$-expansion of Y. The commutativity condition $p_{\lambda\lambda'}q_{\lambda'} = q_\lambda$, $\lambda \leq \lambda'$, obviously holds. Let Z be a normed space (over the same $F \in \{\mathbb{R}, \mathbb{C}\}$) such that $\dim Z < \aleph_0$, and let $f : Y \to Z$ be a continuous linear function. Since Z is finite–dimensional, it is a Banach space. (The same argument implies that $\boldsymbol{p}_{\underline{0}} : X \to \boldsymbol{X}_{\underline{0}}$ is indeed a $(BVect_F)_{\underline{0}}$-expansion of X.) Then there exists a unique continuous linear extension $g : X \to Z$ of f, i.e., $gi = f$. Since $\boldsymbol{p}_{\underline{0}} : X \to \boldsymbol{X}_{\underline{0}}$ is an $(NVect_F)_{\underline{0}}$-expansion, there exist a $\lambda \in \Lambda_{\underline{0}}$ and a unique continuous linear function $g^\lambda : X_\lambda \to Z$ such that $g^\lambda p_\lambda = g$. Then $g^\lambda q_\lambda = g^\lambda p_\lambda i = gi = f$, and the first statement is proven. In light of Theorem 3.2, the proofs of statements (ii) and (iii) hold in the same way, because a "testing" space Z, $\dim Z < \kappa$ ($\dim Z \leq \kappa$), has to be a Banach space. □

Notice that statement (ii) of Theorem 3.3 does not hold for the category $NVect_F$ because the extension property fails when the codomain Z is not a Banach space. Further, if Y is a dense and closed subspace of a Banach space X, then $Y = X$, and statements (ii) and (iii) are trivial. However, if Y is a non-closed dense subspace in X, then Y is not a Banach space, and thus, if $\dim Y = \kappa \geq \aleph_0$, the identity rudimentary morphism $\lfloor 1_Y \rfloor : Y \to \lfloor Y \rfloor$ (which obviously is an $(NVect_F)_\kappa$-expansion of Y) is *not* any expansion with respect to $BVect_F$. This justifies the conditions $\aleph_0 \leq \kappa \leq \dim Y$ and $\aleph_0 \leq \kappa < \dim Y$ in (ii) and (iii) respectively. Nevertheless, in the most interesting application, the structure of separable or/and complete normed vectorial spaces yields a significant reduction of the non-trivial possibilities. In a way, it seems that hereby the completeness takes the role of compactness in the topological case.

If $(\mathcal{C}, \mathcal{D})$ and $(\mathcal{C}, \mathcal{D}')$ are pro-reflective category pairs, where $\mathcal{D}' \subseteq \mathcal{D}$, and the naturally induced functor $Sh_{(\mathcal{C},\mathcal{D})} \to Sh_{(\mathcal{C},\mathcal{D}')}$ is an equivalence of the categories (i.e., there is a canonical bijection between all the corresponding pairs of morphism sets), then we say that the quotient shape theory of $(\mathcal{C}, \mathcal{D})$ *reduces to* that of $(\mathcal{C}, \mathcal{D}')$.

Recall that every finite-dimensional normed space is a Banach space and that there is no Banach space having (algebraic) dimension countable infinite ([10], 7.2., Zad. 4., p. 338). Therefore, $(NVect_F)_{\underline{0}} = (BVect_F)_{\underline{0}} = (BVect_F)_{\aleph_0}$, and we have established the following facts.

THEOREM 3.4. *The quotient shape theory of*

(i) $(NVect_F, (NVect_F)_{\underline{0}})$ *reduces to* $(NVect_F, (BVect_F)_{\underline{0}})$,
(ii) $(NVect_F, (BVect_F)_{\aleph_0})$ *reduces to* $(NVect_F, (BVect_F)_{\underline{0}})$,
(iii) $(sNVect_F, (BVect_F)_{\aleph_0})$ *reduces to* $(sNVect_F, (BVect_F)_{\underline{0}})$;
(iv) $(BVect_F, (BVect_F)_{\aleph_0})$ *reduces to* $(BVect_F, (BVect_F)_{\underline{0}})$,

where $sNVect_F \subseteq NVect_F$ *denotes the full subcategory of all separable spaces. Consequently, the only non-trivial quotient shape theory of Banach spaces having the algebraic dimension less or equal to* $2^{\aleph_0}$ *(for instance, the separable ones) is the finite one.*

Remark 3.5. There are Banach (Hilbert) spaces having the algebraic dimension greater than $2^{\aleph_0}$ (see [10], Teorem 4 and Korolar 2 of paragraph 8, Section 7). For instance, given an $F \in \{\mathbb{R}, \mathbb{C}\}$ and a $p \in \mathbb{R}$, $1 \leq p < \infty$, the *direct sum of the family* $\mathcal{F} = (F_j = F, j \in J)$, where J is the well ordered set of all countable ordinals j (i.e., all $j < \omega_1$ - the first uncountable ordinal), endowed with the norm $\|\cdot\|_p$, is a Banach (Hilbert, if $p = 2$) space, denoted by $l_p(\mathcal{F})$. More precisely, the vectors of $l_p(\mathcal{F})$ are all the functions

$$x : J \to \cup_{j \in J} F_j$$

such that, for every $j \in J$, $x(j) \in F_j$ and

$$\sum_{j \in J} |x(j)|^p < \infty,$$

while

$$\|x\|_p = (\sum_{j \in J} |x(j)|^p)^{\frac{1}{p}}.$$

(Clearly, every x is a function having at most countably many non-zero values, $x(j) \neq 0$.) With the general continuum hypothesis assumed (GCH),

$$\dim(l_p(\mathcal{F})) \leq |l_p(\mathcal{F})| \leq (|J| \cdot |F|)^{|J|} = |J|^{|J|} = 2^{|J|} = 2^{\aleph_1}$$

holds. Further, it is evident that $\dim l_p(\mathcal{F}) > \aleph_0$, and thus (by the continuum hypothesis, CH),

$$|F| = 2^{\aleph_0} \leq \dim l_p(\mathcal{F}).$$

Hence, by Lemma 3.2. (iv) of [14] (and GCH),

$$\dim(l_p(\mathcal{F})) = |l_p(\mathcal{F})| \geq |2^J(\aleph_0)| = |2^J| = 2^{|J|} = 2^{\aleph_1},$$

where $2^J(\aleph_0)$ is the set of all countable subsets of (the uncountable set) J (and thus, $2^J(\aleph_0)$ is of the same cardinality as 2^J). Therefore,

$$\dim(l_p(\mathcal{F})) = 2^{\aleph_1} = 2^{2^{\aleph_0}} > 2^{\aleph_0}.$$

Notice that, generally, $|J| = \kappa$ implies $\dim(l_p(\mathcal{F})) = 2^\kappa$, confirming that there is no countable infinite-dimensional Banach (Hilbert) space.

In the remark below we point out several misprints and two **errors** in [14] (the basic article for this work).

Remark 3.6.

(a) In the first part of the proof of Lemma 3.2 should stay $|F| > \aleph_0$ instead of $|F| \geq \aleph_0$ (at three places);

(b) In the construction of the canonical quotient expansion of a vectorial space X (Section 3, Theorem 3.1; and in [13], Section 12, Theorem 12.1), the condition $\dim Z_\lambda = \dim X$ is accidently dropped;

(c) Statement (ii) of Corollary 4.3 in [14], is false. Consequently the necessity part of Corollary 4.16 in [14] is false;

(d) There is a gap in the proof of Corollary 4.4. in [14]. Namely, the projection $p_\lambda : X \to X_\lambda$ might not belong to $\boldsymbol{q}_{\kappa^-}$. Nevertheless, its statement in the case of κ^- when $\kappa = \aleph_0$ (the only one that is used in the sequel) is valid. The correct reformulation and a rather explicit proof of that case is given by Proposition 3.7 below.

PROPOSITION 3.7. *Let* $X = (V, \|\cdot\|)$ *and* $Y = (V, \|\cdot\|')$ *be normed vectorial spaces over the same field* $F \in \{\mathbb{R}, \mathbb{C}\}$. *If the identity function* $1_V : X \to Y$ *is continuous, then* $S_{\underline{0}}(1_V) : X \to Y$ *is an isomorphism of* $Sh_{\underline{0}}(NVect_F)$.

PROOF. In the finite-dimensional case, there is nothing to prove. So assume that X and Y are infinite-dimensional, i.e., $\dim X = \dim Y = \dim V \geq \aleph_0$. Let

$$\boldsymbol{p}_{\underline{0}} = (p_\lambda) : X \to \boldsymbol{X}_{\underline{0}} = (X_\lambda, p_{\lambda\lambda'}, \Lambda_{\underline{0}}),$$
$$\boldsymbol{q}_{\underline{0}} = (q_\mu) : Y \to \boldsymbol{Y}_{\underline{0}} = (Y_\mu, q_{\mu\mu'}, M_{\underline{0}}),$$

be the canonical $(NVect_F)_{\underline{0}}$-expansions of X, Y respectively. Recall that $X_\lambda = (X/Z_\lambda, \|\cdot\|_\lambda)$, where $Z_\lambda \trianglelefteq V$ is closed in X, $\dim Z_\lambda = \dim V$ and $\dim(V/Z_\lambda) < \aleph_0$, and similarly for Y_μ (by means of $Z_\mu \trianglelefteq V$ closed in Y). Since $1_V : X \to Y$ is continuous, $M_{\underline{0}}$ is a subset of $\Lambda_{\underline{0}}$, and 1_V yields a unique *inv*-$(NVect_F)_{\underline{0}}$-morphism

$$(i, i_\mu) : \boldsymbol{X}_{\underline{0}} \to \boldsymbol{Y}_{\underline{0}},$$

where $i : M_{\underline{0}} \to \Lambda_{\underline{0}}$, $i(\mu) = \lambda_\mu$ is the inclusion, and

$$i_\mu : X_{\lambda_\mu} = (V_{\lambda_\mu} = V_\mu, \|\cdot\|_{\lambda_\mu}) \to (V_\mu, \|\cdot\|'_\mu) = Y_\mu,\ i([x = v]_{\lambda_\mu}) = [v = y]_\mu,$$

is the induced isomorphism of $(NVect_F)_{\underline{0}}$. (Actually, i_μ is the identity on the same finite-dimensional quotient space V/Z_μ.) More precisely, $i_\mu p_{\lambda_\mu} = q_\mu 1_V$ means that

$$i_\mu p_{\lambda_\mu}(x = v) = i_\mu([v]_{\lambda_\mu}) = [v]_\mu = q_\mu(v = y) = q_\mu 1_V(v = x),\ v \in V,$$

while $i_\mu p_{\lambda_\mu \lambda_{\mu'}} = q_{\mu\mu'} i_{\mu'}$, $\mu \leq \mu'$, means that, for every $[x = v]_{\lambda_{\mu'}} \in X_{\lambda_{\mu'}}$,

$$i_\mu p_{\lambda_\mu \lambda_{\mu'}}([x = v]_{\lambda_{\mu'}}) = i_\mu([v]_{\lambda_\mu}) = [v]_\mu = q_{\mu\mu'}([v = y]_{\mu'}) = q_{\mu\mu'} i_{\mu'}([v = x]_{\lambda_{\mu'}}).$$

Denote by

$$\mathbf{1}_V = [(i, i_\mu)] : \boldsymbol{X}_{\underline{0}} \to \boldsymbol{Y}_{\underline{0}}$$

the equivalence class of (i, i_μ), i.e., the corresponding morphism of *pro*-$(NVect_F)_{\underline{0}}$. Notice that $Y = Z_\mu \dot{+} W_\mu$, $X = Z_{\lambda_\mu} \dot{+} W_\mu$ and $X_{\lambda_\mu} \cong W_\mu \cong Y_\mu$, where $W_\mu \trianglelefteq V$ is a finite-dimensional (hence, closed in Y and X) direct complement of the both $Z_\mu, Z_{\lambda_\mu} \trianglelefteq V$.

Conversely, given a $\lambda \in \Lambda_{\underline{0}}$, $X_\lambda = X/Z_\lambda$, where $Z_\lambda \trianglelefteq V$ is closed in X, $\dim Z_\lambda = \dim X$ and $\dim(X/Z_\lambda) < \aleph_0$. Since Z_λ is closed subspace of X and $\dim(X/Z_\lambda) < \aleph_0$, there exists a closed direct complement $W_\lambda \trianglelefteq X$ of Z_λ (see [10], Section 8.11,(b), p. 440, that holds true for a normed space as well, because W_λ is finite-dimensional). Thus $X = W_\lambda \dot{+} Z_\lambda$, and clearly, $W_\lambda \cong X/Z_\lambda = X_\lambda$. Then $V = W_\lambda \dot{+} Z_\lambda$, and $W_\lambda \trianglelefteq V$ is a finite-dimensional (and hence, closed) subspace of Y. It follows that there exists a closed direct complement Z_{μ_λ} of W_λ in Y, i.e., $Y = W_\lambda \dot{+} Z_{\mu_\lambda}$ (see [10], Section 8.11(c), p. 440, or Section 6.5, Zad. 4., p. 286). By the canonical construction of $\boldsymbol{Y}_{\underline{0}}$, $Y_{\mu_\lambda} = Y/Z_{\mu_\lambda}$, and $Y_{\mu_\lambda} \cong W_\lambda \cong X_\lambda$. Since W_λ is a common closed direct summand of X and Y, it follows that, for every $\lambda \in \Lambda_{\underline{0}}$ and each $v \in V$, the both equivalence classes $[x = v]_\lambda = v + Z_\lambda \in X_\lambda$ and $[y = v]_{\mu_\lambda} = v + Z_{\mu_\lambda} \in Y_{\mu_\lambda}$ canonically corresponds to a unique vector $w_{[v]} \in W_\lambda$. More precisely, for every $\lambda \in \Lambda_{\underline{0}}$, there exist two canonical linear bijections

$$\phi_\lambda : W_\lambda \to X_\lambda,\ \phi_\lambda(w_{[v]}) = [v]_\lambda = v + Z_\lambda$$
$$\psi_\lambda : W_\lambda \to Y_{\mu_\lambda},\ \psi_\lambda(w_{[v]}) = [v]_{\mu_\lambda} = v + Z_{\mu_\lambda}.$$

Since these spaces are finite-dimensional, ϕ_λ and ψ_λ are isomorphisms of the normed spaces. Put

$$g : \Lambda_{\underline{0}} \to M_{\underline{0}},\ g(\lambda) = \mu_\lambda$$

and

$$g_\lambda = \phi_\lambda \psi_\lambda^{-1} : Y_{g(\lambda)} = Y_{\mu_\lambda} \to X_\lambda,\ \lambda \in \Lambda_{\underline{0}}.$$

Then g_λ is an isomorphism of the Banach spaces and

$$g_\lambda([y = v]_{\mu_\lambda}) = \phi_\lambda \psi_\lambda^{-1}([y = v]_{\mu_\lambda}) = \phi_\lambda(w_{[v]}) = [v = x]_\lambda.$$

Further, for every related pair $\lambda \leq \lambda'$ in $\Lambda_{\underline{0}}$ and every $v = y \in Y$,

$$g_\lambda q_{\mu_\lambda}(y = v) = \phi_\lambda \psi_\lambda^{-1}([v]_{\mu_\lambda}) = [v = x]_\lambda$$

and

$$p_{\lambda\lambda'} g_{\lambda'} q_{\mu_{\lambda'}}(y = v) = p_{\lambda\lambda'} \phi_{\lambda'} \psi_{\lambda'}^{-1}([v]_{\mu_{\lambda'}}) = p_{\lambda\lambda'}([v]_{\lambda'}) = [v = x]_\lambda.$$

Therefore,

$$g_\lambda q_{\mu_\lambda} = p_{\lambda\lambda'} g_{\lambda'} q_{\mu_{\lambda'}} : Y \to X_\lambda.$$

Since $\boldsymbol{q}_{\underline{0}} : Y \to \boldsymbol{Y}_{\underline{0}}$ is an $(NVect_F)_{\underline{0}}$-expansions of Y and X_λ is finite-dimensional, there exist a $\mu \in M_{\underline{0}}$ and a unique continuous linear function $h^\mu : Y_\mu \to X_\lambda$ such that

$$g_\lambda q_{\mu_\lambda} = h^\mu q_\mu = p_{\lambda\lambda'} g_{\lambda'} q_{\mu_{\lambda'}} : Y \to X_\lambda.$$

We may assume, without loss of generality, that $\mu \geq \mu_\lambda, \mu_{\lambda'}$. Then

$$g_\lambda q_{\mu_\lambda \mu} q_\mu = g_\lambda q_{\mu_\lambda} = h^\mu q_\mu = p_{\lambda\lambda'} g_{\lambda'} q_{\mu_{\lambda'}} = p_{\lambda\lambda'} g_{\lambda'} q_{\mu_{\lambda'} \mu} q_\mu.$$

Since each q_μ is an epimorphism, it follows that

$$g_\lambda q_{\mu_\lambda \mu} = p_{\lambda\lambda'} g_{\lambda'} q_{\mu_{\lambda'} \mu}.$$

In this way we have proven that

$$(g, g_\lambda) : \boldsymbol{Y}_{\underline{0}} \to \boldsymbol{X}_{\underline{0}}$$

is a morphism of *inv*-$(NVect_F)_{\underline{0}}$. Denote by $\boldsymbol{g} = [(g, g_\lambda)] : \boldsymbol{Y}_{\underline{0}} \to \boldsymbol{X}_{\underline{0}}$ the induced morphism of *pro*-$(NVect_F)_{\underline{0}}$. We are to prove that $S_{\underline{0}}(1_V) : X \to Y$ is an isomorphism of $Sh_{\underline{0}}(NVect_F)$ by showing that $\boldsymbol{g} = (\mathbf{1}_V)^{-1}$ in *pro*-$(NVect_F)_{\underline{0}}$. Namely,

$$(g, g_\lambda)(i, i_\mu) = (ig, g_\lambda i_{\mu_\lambda}) \sim (1_{\Lambda_{\underline{0}}}, 1_\lambda)$$

in *inv*-$(NVect_F)_{\underline{0}}$. Indeed, given a $\lambda'' \geq \lambda, \lambda' \equiv \lambda_{\mu_\lambda}$, one readily verifies that

$$g_\lambda i_{\mu_\lambda} p_{\lambda'\lambda''} = p_{\lambda\lambda''}$$

holds true. Similarly,

$$(i, i_\mu)(g, g_\lambda) = (gi, i_\mu g_{\lambda_\mu}) \sim (1_{M_{\underline{0}}}, 1_\mu)$$

in *inv*-$(NVect_F)_{\underline{0}}$. Indeed, given a $\mu'' \geq \mu, \mu' \equiv \mu_{\lambda_\mu}$, one easily sees that

$$i_\mu g_{\lambda_\mu} q_{\mu'\mu''} = q_{\mu\mu''}.$$

holds true. Therefore, $\mathbf{1}_V : \boldsymbol{X}_{\underline{0}} \to \boldsymbol{Y}_{\underline{0}}$ is an isomorphism of *pro*-$(NVect_F)_{\underline{0}}$, which completes the proof. □

4. The quotient shape classification of l_p spaces

Let us consider, for all $1 \leq p \leq \infty$, the well known normed vectorial spaces l_p (over $F \in \{\mathbb{R}, \mathbb{C}\}$). Recall that, for every $1 \leq p < \infty$,

$$l_p = (\{x = (\xi^i) \in F^{\mathbb{N}} \mid \Sigma_{i\in\mathbb{N}} \left|\xi^i\right|^p < \infty\}, \|\cdot\|_p),$$
$$\|x\|_p = (\Sigma_{i\in\mathbb{N}} \left|\xi^i\right|^p)^{\frac{1}{p}},$$

while

$$l_\infty = (\{x = (\xi^i) \in F^{\mathbb{N}} \mid (|\xi^i|) \text{ bounded}\}, \|\cdot\|_\infty),$$
$$\|x\|_\infty = \sup\{|\xi^i| \mid i \in \mathbb{N}\}.$$

The algebraic operations are coordinatewise. Of course, no pair l_p, $l_{p'}$, $p \neq p'$, is mutually isomorphic in $NVect_F$. Namely, they are not homeomorphic as the topological (metric) spaces. However, algebraically, for all $1 \leq p \leq p' \leq \infty$, it holds

$$F_0^{\mathbb{N}} \trianglelefteq l_1 \trianglelefteq l_p \trianglelefteq l_{p'} \trianglelefteq l_\infty \trianglelefteq F^{\mathbb{N}}$$

(in $Vect_F$; $F_0^{\mathbb{N}}$ is the direct sum space). Furthermore, one readily sees that $\dim F_0^{\mathbb{N}} = \aleph_0$, while, for every p, $\dim l_p > \aleph_0$. Since $|F| \leq 2^{\aleph_0}$, Lemma 3.3 (ii) of [14] implies that, for every p, $\dim l_p = \dim F^{\mathbb{N}} = 2^{\aleph_0}$. (In general, with CH assumed, for a *separable* Banach space X, $\dim X = \infty$ is equivalent to $\dim X = 2^{\aleph_0}$.) Therefore, for every p, $F_0^{\mathbb{N}} \not\cong l_p \cong F^{\mathbb{N}}$ algebraically (in $Vect_F$). Now, for every related pair $p \leq p'$, denote by

$$F_0^{\mathbb{N}}(p') \trianglelefteq l_p(p') \trianglelefteq l_{p'}$$

the corresponding normed vectorial subspaces of $l_{p'}$ (the *vectorial* spaces $F_0^{\mathbb{N}}$ and l_p carrying the *norm* $\|\cdot\|_{p'}$). Clearly, $l_p(p) = l_p$, while $F_0^{\mathbb{N}}(p)$ is not isomorphic (in $NVect_F$) to l_p nor to any $l_p(p')$, neither $l_p(p')$ is isomorphic to l_p or to $l_{p'}$, whenever $p < p'$. Notice that, in general, the normed spaces $F_0^{\mathbb{N}}(p')$ and $l_p(p')$ are not closed in $l_{p'}$, and therefore, they are not Banach spaces.

Although all the considered spaces, but l_∞, are separable, and all but $F_0^{\mathbb{N}}(p)$ and $l_p(p')$ are complete, our common framework will be the category $NVcet_F$ with respect to Banach spaces. Then, according to Theorem 3.4, the quotient shape classifications of all $F_0^{\mathbb{N}}(p)$, l_p and $l_p(p')$ spaces reduce to their finite shape classification.

By Example 4.8 of [14] (followed now by Proposition 3.7), for all $1 \leq p \leq p' \leq \infty$, the normed vectorial spaces l_p and $l_p(p')$ have the same finite quotient shape, i.e.,

$$Sh_{\underline{0}}(l_p(p')) = Sh_{\underline{0}}(l_p).$$

Further, by Example 4.10 of [14] (followed now by Proposition 3.7), all the normed vectorial spaces $F_0^{\mathbb{N}}(p)$ are of the same finite quotient shape, i.e., for all $1 \leq p, p' \leq \infty$,

$$Sh_{\underline{0}}(F_0^{\mathbb{N}}(p)) = Sh_{\underline{0}}(F_0^{\mathbb{N}}(p')).$$

An appropriate shape relationship between $l_p(p')$ and $l_{p'}$. $p < p'$, i.e., between l_p and $l_{p'}$, $p \neq p'$, remained as an open problem (see Remark 4.9 of [14]). In solving the problem, the following topological facts are crucial.

LEMMA 4.1.

(i) *For each p, $1 \leq p < \infty$, $l_p = Cl(F_0^{\mathbb{N}}(p))$. Consequently, for all $1 \leq p \leq p' < \infty$, $Cl(l_p(p')) = l_{p'}$.*

(ii) *For each p, $1 \leq p < \infty$, $Cl(l_p(\infty)) = Cl(F_0^{\mathbb{N}}(\infty))$ (in l_∞). Consequently, for all $1 \leq p, p' < \infty$, $Cl(l_p(\infty)) = Cl(l_{p'}(\infty))$, which are proper (and Banach) subspaces of l_∞.*

PROOF. Clearly, for every $1 \leq p \leq \infty$, $Cl(F_0^{\mathbb{N}}(p)) \trianglelefteq l_p$ and $F_o^{\mathbb{N}}(\infty) \triangleleft l_p(\infty)$, and thus, $Cl(F_0^{\mathbb{N}}(\infty)) \trianglelefteq Cl(l_p(\infty))$ in l_∞. Further, for all $1 \leq p \leq p' \leq \infty$, $Cl(l_p(p')) \trianglelefteq l_{p'}$, and $Cl(F_0^{\mathbb{N}}(\infty)) \triangleleft l_\infty$ since l_∞ is not separable. We need to prove the appropriate converses.

(i) Let $1 \leq p < \infty$ and let $x \in l_p$. Since all these spaces are metric, we have to find a sequence (x_n) in $F_0^{\mathbb{N}}(p)$ such that $\lim(x_n) = x$ in l_p. Recall that $x = (\xi^i)$, $\xi^i \in F \in \{\mathbb{R}, \mathbb{C}\}$, $i \in \mathbb{N}$, such that $\Sigma_{i \in \mathbb{N}} \left|\xi^i\right|^p < \infty$. Put, for each $n \in \mathbb{N}$,

$$x_n = (\xi^1, \ldots, \xi^n, 0, 0, \ldots) \in F_0^{\mathbb{N}}.$$

Then

$$\|x - x_n\|_p = (\Sigma_{i>n} \left|\xi^i\right|^p)^{\frac{1}{p}},$$

and thus,

$$(\|x - x_n\|_p)^p = \Sigma_{i>n} \left|\xi^i\right|^p.$$

Since the series $\Sigma_{i\in\mathbb{N}} \left|\xi^i\right|^p$ converges in $\mathbb{R}$, the rest $\Sigma_{i>n} \left|\xi^i\right|^p$ is arbitrarily small when n is large enough. This means that

$$(\forall\varepsilon > 0)(\exists n_0(\varepsilon) \in \mathbb{N})(\forall n \geq n_0)(\|x - x_n\|_p)^p < \varepsilon.$$

Since $p \geq 1$, it implies that $\|x - x_n\|_p < \varepsilon^p < \varepsilon$, whenever $\varepsilon < 1$, and hence $\lim(x_n) = x$ in l_p.

(ii) It suffices to prove that, for each p, $1 \leq p < \infty$, $Cl(l_p(\infty)) \subseteq Cl(F_0^{\mathbb{N}}(\infty))$ holds true. Let us firstly prove that

$$Cl(F_0^{\mathbb{N}}(\infty)) = \{y = (\eta^i) \in F^{\mathbb{N}} \mid \lim(\eta^i) = 0 \text{ in } F\} \equiv c_0 \lhd l_\infty.$$

Notice that $F_0^{\mathbb{N}}(\infty) \subseteq c_0$ holds trivially and that c_0 is complete ([10], 2.6., Zad. 1. p. 86) and hence closed in l_∞. We are to prove that $c_0 \subseteq Cl(F_0^{\mathbb{N}}(\infty))$. Let $y = (\eta^i) \in c_0$. It suffices to find a sequence (y_n) in $F_0^{\mathbb{N}}(\infty)$ such that $\lim(y_n) = y$ in l_∞. This means that

$$(\forall\varepsilon > 0)(\exists n_0(\varepsilon) \in \mathbb{N})(\forall n \geq n_0)$$
$$\|y - y_n\|_\infty = \sup\{|\eta^i - \eta_n^i| \mid i \in \mathbb{N}\} < \varepsilon.$$

must hold. Let us put, for each n,

$$y_n = (\eta^1, \ldots, \eta^n, 0, 0, \ldots) \in F_0^{\mathbb{N}}.$$

Then

$$\|y - y_n\|_\infty = \sup\{|\eta^i - \eta_n^i| \mid i \in \mathbb{N}\} = \sup\{|\eta^i| \mid i > n\}.$$

Since $\lim(\eta^i) = 0$ in F is equivalent to $\lim(|\eta^i|) = 0$ in $\mathbb{R}$, it follows that $\|y - y_n\|_\infty$ becomes arbitrarily small when $n \to \infty$. Therefore, $\lim(y_n) = y$ in l_∞, that completes the proof of the statement.

It remains to prove that, for each p, $1 \leq p < \infty$, $Cl(l_p(\infty)) \subseteq c_0$. Let $x \in Cl(l_p(\infty))$. Then there exists a sequence (x_n) in $l_p(\infty)$ such that $x = \lim(x_n)$ in l_∞. Recall that, for every $n \in \mathbb{N}$, $x_n = (\xi_n^i)$, $\Sigma_{i\in\mathbb{N}} \left|\xi_n^i\right|^p < \infty$ (while $\|x_n\|_\infty = \sup\{|\xi_n^i| \mid i \in \mathbb{N}\}$). Since $\Sigma_{i\in\mathbb{N}} \left|\xi_n^i\right|^p$ converges in $\mathbb{R}$, the rest $\Sigma_{i>i_0(n)} \left|\xi_n^i\right|^p$ becomes arbitrarily small when $i_0(n) \to \infty$. Consequently, $\lim_i(\left|\xi_n^i\right|^p) = 0$ in $\mathbb{R}$. Since $p \geq 1$, $\lim_i(\left|\xi_n^i\right|) = 0$ in $\mathbb{R}$ as well, which is equivalent to $\lim_i(\xi_n^i) = 0$ in F. Therefore, for every $n \in \mathbb{N}$, $x_n \in c_0$. Since c_0 is closed in l_∞, $x = \lim(x_n) \in c_0$ holds true, that completes the proof of the lemma.

□

COROLLARY 4.2. *All the spaces l_p, $1 \leq p < \infty$, $l_p(p')$, $1 \leq p < p' \leq \infty$, $F_0^{\mathbb{N}}(s)$, $1 \leq s \leq \infty$, and c_0 have the same finite quotient shape, that is also their countable quotient shape type (with respect to $BVect_F$). Explicitly, if*

$1 \leq p_i < \infty$, $i = 1, 2$, *then, for all* p_i', p_i'', $1 \leq p_i \leq p_i', p_i'' \leq \infty$,

$$\begin{aligned} Sh_{\underline{0}}(l_{p_1}) &= Sh_{\underline{0}}(l_{p_1}(p_1')) = Sh_{\underline{0}}(F_0^{\mathbb{N}}(p_1'')) = Sh_{\underline{0}}(c_0) \\ &= Sh_{\aleph_0}(l_{p_2}) = Sh_{\aleph_0}(l_{p_2}(p_2')) = Sh_{\aleph_0}(F_0^{\mathbb{N}}(p_2'')) = Sh_{\aleph_0}(c_0). \end{aligned}$$

PROOF. The first equality follows by Example 4.8 of [14], the second and third by Example 4.10 of [14], Lemma 4.1 and Theorem 3.3, while the rest follows then by Theorem 3.4. ◻

Since every space $Cl(l_p(\infty))$, $p \neq \infty$, is a proper closed subspace of (non-separable) l_∞, it seems that $Sh_{\underline{0}}(l_\infty)$ might differ from (the unique) type $Sh_{\underline{0}}(l_p)$. We now have only the following fact.

COROLLARY 4.3.

$$Sh_{\aleph_0}(l_\infty) = Sh_{\underline{0}}(l_\infty) = Sh_{\underline{0}}(F_0^{\mathbb{N}}(\infty))^{**}) = Sh_{\aleph_0}(F_0^{\mathbb{N}}(\infty))^{**}),$$

*where " ** " indicates the second dual (normed) space.*

PROOF. The first and third equality follow by Theorem 3.4. Let us prove the second one. Firstly, if Y is a dense subspace of a normed vectorial space X, i.e., $Cl(Y) = X$ (over $F \in \{\mathbb{R}, \mathbb{C}\}$) then, by the Hahn-Banach theorem and by the uniqueness of extension of an continuous functional on Y onto $Cl(Y) = X$, one can straightforwardly prove that $Y^* \cong X^*$ (the first dual spaces). Then, clearly, $Y^{**} \cong X^{**}$. Recall that $Cl(F_0^{\mathbb{N}}(\infty)) = c_0$ (see the proof of Lemma 4.1) and $c_0^* \cong l_1$ ([10], 2.6., Zad. 6., p. 86). Therefore,

$$F_0^{\mathbb{N}}(\infty)^{**} \cong Cl(F_0^{\mathbb{N}}(\infty))^{**} = c_0^{**} \cong l_1^* \cong l_\infty,$$

proves the second equality. ◻

REMARK 4.4. Recall that the subspace $c \trianglelefteq l_\infty$ (of all convergent sequences in F) is isomorphic to its subspace c_0 (in $BVect_F$). Therefore, all the above proven quotient shape facts relating c_0 to l_p, $l_p(p')$ and $F_0^{\mathbb{N}}(p')$ spaces hold true for c as well.

Recall that, in general, if $\boldsymbol{p} : X \to \boldsymbol{X}$, $\boldsymbol{p}' : X' \to \boldsymbol{X}'$ are expansion of X, X' respectively, and $\boldsymbol{f} : \boldsymbol{X} \to \boldsymbol{X}'$ is an isomorphism, then $\boldsymbol{fp} : X \to \boldsymbol{X}'$ and $\boldsymbol{f}^{-1}\boldsymbol{p}' : X' \to \boldsymbol{X}$ are also expansions of X and X' respectively. Hence, X and X' admit the same expansion systems. By Corollary 4.2, all l_p and $l_p(p')$ spaces, $1 \leq p < p' \leq \infty$, and all direct sum spaces $F_0^{\mathbb{N}}(s)$, $1 \leq s \leq \infty$, admit $(BVect_F)_{\underline{0}}$-expansions, all having a common expansion systems. Let us determine one of such common systems. It is much easier to consider a unitary or the Hilbert case ($F_0^{\mathbb{N}}(2)$ or l_2) than a general one. Although $\dim(F_0^{\mathbb{N}}(2)) = \aleph_0 < 2^{\aleph_0} = \dim l_2$, we choose the Hilbert space l_2 because there are orthogonal complements in it. So, let us construct the *canonical* $(BVect_F)_{\underline{0}}$-expansion of l_2.

Let $\{Z_\lambda \mid \lambda \in \Lambda\}$ be the set of all closed subspaces Z_λ of l_2. Then, for every $\lambda \in \Lambda$, $l_2 = Z_\lambda \oplus Z_\lambda^\perp$ (the orthogonal sum), where $Z_\lambda^\perp$ is the orthogonal complement of Z_λ in l_2. Define

$$\lambda \leq \lambda' \Leftrightarrow Z_{\lambda'} \trianglelefteq Z_\lambda.$$

Then, obviously, $(\Lambda, \leq)$ is a partially ordered set. Furthermore, it is directed because the intersection of two closed subspaces is a closed subspace which is a subspace of both of them. Notice that the quotient functions

$$q_\lambda : L_2 \to l_2/Z_\lambda,\ \lambda \in \Lambda,$$
$$q_{\lambda\lambda'} : l_2/Z_{\lambda'} \to l_2/Z_\lambda,\ \lambda \leq \lambda',$$

are continuous open linear surjections satisfying $q_{\lambda\lambda'}q_{\lambda'} = q_\lambda$ and $q_{\lambda\lambda'}q_{\lambda'\lambda''} = q_{\lambda\lambda''}$, $\lambda \leq \lambda' \leq \lambda''$. Notice that, for each λ, $l_2/Z_\lambda \cong Z_\lambda^\perp$. Put $Y_\lambda = l_2/Z_\lambda$, $\lambda \in \Lambda$. Then

$$\boldsymbol{q} = (q_\lambda) : l_2 \to \boldsymbol{Y} = (Y_\lambda, q_{\lambda\lambda'}, \Lambda)$$

is a morphism of *pro-*$BVect_F$. Let

$$\Lambda_{\underline{0}} = \{\lambda \in \Lambda \mid \dim(l_2/Z_\lambda) < \infty\} \subseteq \Lambda,$$

carrying the partial order of Λ. Then $\Lambda_{\underline{0}}$ is a directed partially ordered set as well. Namely, $\dim(l_2/Z_\lambda) < \dim Z_\lambda^\perp = \dim Y_\lambda < \infty$, $\lambda \in \Lambda_{\underline{0}}$, implies that, for every pair $\lambda, \lambda' \in \Lambda_{\underline{0}}$, the intersection subspace $Z_\lambda \cap Z_{\lambda'}$ is of the same kind, i.e.,

$$\dim(l_2/(Z_\lambda \cap Z_{\lambda'})) < \infty,$$

and thus there exists a $\lambda'' \geq \lambda, \lambda'$, $\lambda'' \in \Lambda_{\underline{0}}$ (with $Y_{\lambda''} \equiv l_2/(Z_\lambda \cap Z_{\lambda'})$ and $\max\{\dim Y_\lambda, \dim X_{\lambda'}\} \leq \dim Y_{\lambda''} < \infty$). Let

$$\boldsymbol{q}_{\underline{0}} = (q_\lambda) : l_2 \to \boldsymbol{Y}_{\underline{0}} = (Y_\lambda, q_{\lambda\lambda'}, \Lambda_{\underline{0}})$$

to be the restriction of $\boldsymbol{q} : l_2 \to \boldsymbol{Y}$. It remains to verify that $\boldsymbol{q}_{\underline{0}}$ has the factorization property (E1) with respect to every finite–dimensional Banach space W. We may assume, without loss of generality, that $W = F^n$, for some $n \in \mathbb{N}$. Let $f : l_2 \to F^n$ be a continuous linear function. Then the kernel $N(f) = f^{-1}[\{\theta\}]$ is a closed subspace of l_2 implying that there exists a $\lambda \in \Lambda$ such that $Z_\lambda = N(f)$. Since $l_2/N(f) \cong \mathrm{Im}(f)$ and $\dim(\mathrm{Im}(f)) \leq n < \infty$, it follows that $\lambda \in \Lambda_{\underline{0}}$, i.e., $l_2/N(f) = Y_\lambda$ is a term of $\boldsymbol{Y}_{\underline{0}}$. Now the desired factorization of f through q_λ and a unique continuous linear $f^\lambda : Y_\lambda \to F^n$, $f = f^\lambda q_\lambda$, is the well known fact. Observe that the expansion system $\boldsymbol{Y}_{\underline{0}}$, as an object, belongs to the pro-category *pro-*$(HVect_F)_{\underline{0}}$ of all finite-dimensional *Hilbert* spaces. Hence, $\boldsymbol{q}_{\underline{0}} : l_2 \to \boldsymbol{Y}_{\underline{0}}$ is also the canonical $(HVect_F)_{\underline{0}}$-expansion of l_2. (Caution! Hereby $HVect_F \subseteq UVect_F$ denotes the *full* subcategories of $BVect_F \subseteq NVect_F$ respectively, i.e., all the continuous linear functions are included. Especially, for every pair $X, Y \in Ob(UVect_F)$, $X \cong Y$ in $UVect_F$ if and only if $X \cong Y$ in $NVect_F$.) Further, the cardinal $|\Lambda_{\underline{0}}| = 2^{\aleph_0} = |l_2| = \dim l_2$, and it is minimal for all $\underline{0}$-expansions of the considered spaces. Namely, $|\Lambda_{\underline{0}}| \geq 2^{\aleph_0}$ obviously holds. On the other hand, the cardinality of the set of all closed subspaces of l_2, each providing the finite-dimensional quotient space, and the cardinality of the set of all finite-dimensional subspaces of l_2 equals (via orthogonal complements). Since each finite-dimensional vectorial space is determined by a finite subset of a chosen basis B (infinite) and by a finite subset of the field F (infinite), the cardinality of the set of all finite-dimensional subspaces of l_2 is less than or equal to

$$\begin{aligned} |\mathcal{F}(\mathcal{F}(B) \times \mathcal{F}(F))| &= |\mathcal{F}(B) \times \mathcal{F}(F)| = |\mathcal{F}(B)| \cdot |\mathcal{F}(F)| = |B| \cdot |F| \\ &= 2^{\aleph_0} \cdot 2^{\aleph_0} = 2^{\aleph_0} \end{aligned}$$

($\mathcal{F}$ indicates the set of all finite subsets), and the conclusion follows. Notice that, though $F_0^{\mathbb{N}}(2)$ admits a *countable orthonormal* basis, $\dim(F_0^{\mathbb{N}}(2)) = \aleph_0$, the analogous canonical construction for $F_0^{\mathbb{N}}(2)$ cannot yield the countable cardinality of the index set. Namely, in this unitary non-Hilbert case, one has to take into account *all* the *direct* complements of a Y_λ. Or, in a more general way, the countable cardinality of an index set implies that the system is isomorphic to an inverse sequence. Then every such a candidate, for this case, should be isomorphic to $(F^n, q_{nn'}, \mathbb{N})$, which cannot be an expansion of any normed space on the direct sum vectorial space $F_0^{\mathbb{N}}$ (see also Lemma 3.4 of [14]) Finally, by Theorem 3.4, $\boldsymbol{q}_{\aleph_0} = \boldsymbol{q}_{\underline{0}} : l_2 \to \boldsymbol{Y}_{\underline{0}} = \boldsymbol{Y}_{\aleph_0}$ is also the canonical $(BVect_F)_{\aleph_0}$-expansion of l_2, and consequently, the canonical $(HVect_F)_{\aleph_0}$-expansion of l_2 too.

We summarize the obtained results in the following theorem.

THEOREM 4.5. *For all p and all ordered pairs (p,p'), $1 \leq p < p' \leq \infty$, and all s, $1 \leq s \leq \infty$, there exist $(BVect_F)_{\underline{0}}$-expansions (the case κ^-, when $\kappa = \aleph_0$)*

$$\begin{aligned} \boldsymbol{q}(p)_{\underline{0}} &= (q(p)_\lambda) : l_p \to \boldsymbol{Y}_{\underline{0}} = (Y_\lambda, q_{\lambda\lambda'}, \Lambda_{\underline{0}}), \\ \boldsymbol{q}(p,p')_{\underline{0}} &= (q(p,p')_\lambda) : l_p(p') \to \boldsymbol{Y}_{\underline{0}} \ \text{ and} \\ \boldsymbol{q}'(s)_{\underline{0}} &= (q'(s)_\lambda) : F_0^{\mathbb{N}}(s) \to \boldsymbol{Y}_{\underline{0}}, \end{aligned}$$

such that all of them share the same inverse system $\boldsymbol{Y}_{\underline{0}}$ of $(HVect_F)_{\underline{0}} \subseteq (BVect_F)_{\underline{0}}$, which has for its terms Y_λ all the finite-dimensional quotient spaces by all appropriate closed subspaces of l_2, and for the bonds $q_{\lambda\lambda'}$ the corresponding quotient functions, and the index set $\Lambda_{\underline{0}}$ is of the minimal cardinality $|\Lambda_{\underline{0}}| = 2^{\aleph_0}$ among all their expansions. Those expansions are also their $(HVect_F)_{\underline{0}}$-expansions as well as their expansions in the countable case (the case κ, when $\kappa = \aleph_0$, with respect to $(BVect_F)_{\aleph_0}$ and to $(HVect_F)_{\aleph_0}$).

REMARK 4.6. The finite quotient shape classifications obtained in Examples 4.8 and 4.10 of [14] are valid for every $F \in \{\mathbb{Q}, \mathbb{R}, \mathbb{C}\}$ However, the improvements obtained hereby for $F \in \{\mathbb{R}, \mathbb{C}\}$ are not valid for $F = \mathbb{Q}$. Namely, no (metric) completion of a normed vectorial space over $\mathbb{Q}$ is a *normed* space over $\mathbb{Q}$ (but a complete metric space only!).

5. The quotient shape classification of L_p spaces

We shall firstly recall and briefly consider the needed (algebraic) objects. Given an $n \in \mathbb{N}$, let K_n denote the n-cube

$$[a_1, b_1] \times \cdots \times [a_n, b_n] \subseteq \mathbb{R}^n,$$

where $a_i, b_i \in \mathbb{R}$ and $a_i < b_i$, $i = 1, \dots, n$. Denote by $C(K_n)$ the vectorial space of all mappings (i.e., continuous functions) $x : K \to F$ (over $F \in \{\mathbb{R}, \mathbb{C}\}$, with the usual addition and multiplication by scalars). Then $C(K_n)$ is a proper subspace of the vectorial space F^{K_n} of all functions of K_n to F. Notice that (CH assumed), $|F| = 2^{\aleph_0}$ and $\dim(F^{K_n}) > \aleph_0$, i.e., $\dim(F^{K_n}) \geq 2^{\aleph_0}$. and hence, $|F| \leq \dim(F^{K_n})$. Then, by Lemma 3.2. (iv) of [14] (and the general continuum hypothesis GCH), for every $n \in \mathbb{N}$,

$$\dim(F^{K_n}) = |F^{K_n}| = |F|^{|K_n|} = (2^{\aleph_0})^{2^{\aleph_0}} = 2^{2^{\aleph_0}} = 2^{\aleph_1}.$$

Further, it is obvious that, for every $n \in \mathbb{N}$, $\dim C(K_n) \geq 2^{\aleph_0} = |F|$. Therefore, by Lemma 3.2. (iv) of [14], $\dim C(K_n) = |C(K_n)|$. Finally, since $|C(K_n)| < |F^{K_n}|$, it follows (by GCH) that, for every $n \in \mathbb{N}$, $\dim C(K_n) = 2^{\aleph_0}$. Consequently, for every $n \in \mathbb{N}$ (and every n-cube $K_n \subseteq \mathbb{R}^n$), $C(K_n) \cong F^{\mathbb{N}}$ (algebraically).

For each $p \in \mathbb{R}$, $p \geq 1$, let, for every $x \in C(K_n)$,

$$\|x\|_p = (\int_{K_n} |x(t)|^p \, dt)^{\frac{1}{p}}.$$

Then $(C(K_n), \|\cdot\|_p) \equiv C_p(K_n)$ is a normed vectorial space (separable, non-complete) and, for every pair $p \neq p'$, the spaces $C_p(K_n)$ and $C_{p'}(K_n)$ are *not* mutually isomorphic in $NVect_F$. (One may think that $C_p(K_n)$ is a generalized analogue of $l_1(p)$: K_n, mapping, definite integral versus $\mathbb{N}$, absolutely summable sequence, series.) Our first goal is to prove that, for a given $n \in \mathbb{N}$, all the normed spaces $C_p(K_n)$ are of the same finite quotient shape type. In the simplest case of $n = 1$ and $F = \mathbb{R}$, we already know (Example 4.7 of [14]) that, for a given $[a, b] \subseteq \mathbb{R}$, all the normed spaces $C_p([a, b])$ have the same finite quotient shape type, i.e.,

$$(\forall p, p' \in \mathbb{R},\ p, p' \geq 1) Sh_{\underline{0}}(C_p([a, b])) = Sh_{\underline{0}}(C_{p'}([a, b])).$$

In the proof we have applied Jensen's inequality

$$\varphi(\int_a^b f(t)dt) \leq \int_a^b \varphi(f(t))dt,$$

($f \geq 0$ and φ convex). Namely, by means of $\varphi(t) = t^{\frac{p'}{p}}$, $p \leq p'$, we have shown that there exists an $\alpha > 0$ such that, for every $x \in C([a,b])$,

$$\|x\|_p \leq \alpha \|x\|_{p'}$$

holds. This has implied that the identity function on $C([a,b])$ becomes the continuous

$$1_p^{p'} : C_{p'}([a,b]) \to C_p([a,b]),$$

whenever $p \leq p'$. Then the conclusion has followed by the special $\underline{0}$-case of Corollary 4.4 of [14], which is correctly proven by Proposition 3.7 hereby. One readily sees that the same result holds for the complex functions $x : [a,b] \to \mathbb{C}$, in the case $F = \mathbb{C}$, as well.

We shall hereby apply the same technique for every $n \in \mathbb{N}$. Firstly, let us make a small technical simplification by reducing an n-cube K_n to the unit n-cube

$$I^n = [0,1]^n \subseteq \mathbb{R}^n.$$

Recall that there exists a unique linear homeomorphism $h_n : I^n \to K_n$ (given, in the canonical basis of $\mathbb{R}^n$, by its diagonal matrix H_n having all the diagonal elements $\alpha_{ii} \neq 0$). It induces an (algebraic, linear) isomorphism

$$h_n^F : C(K_n) \to C(I^n),\ h_n^F(x) = xh_n,$$

of the vectorial spaces (having the inverse $(h_n^F)^{-1} = (h_n^{-1})^F$).

Lemma 5.1. *For every $n \in \mathbb{N}$ and each $p \in \mathbb{R}$, $p \geq 1$, $h_n^F : C_p(K_n) \to C_p(I^n)$ is an isomorphism of $NVect_F$.*

Proof. It suffices to prove that h_n^F and $(h_n^F)^{-1}$ are continuous. The continuity of h_n^F immediately follows by the next fact:

$$(\exists M > 0)(\forall x \in C_p(K_n))\ \|xh_n\|_p = M\|x\|_p .$$

This fact is a consequence of Change of variables theorem (the change by h_n). Namely,

$$\int_{K_n} |x(t)|^p\, dt = \int_{I^n} |x(h_n(\tau))|^p \cdot |\det(h_n)|\, d\tau = |\det(dh_n)| \int_{I^n} |(xh_n)(\tau))|^p\, d\tau,$$

because $\det(dh_n) \in \mathbb{R}$ (the product of the diagonal elements of the matrix H_n). Thus, $M = |\det(dh_n)|^{-\frac{1}{p}} > 0$ is the desired constant. Since $(h_n^F)^{-1} = (h_n^{-1})^F$, the continuity of $h_n^F)^{-1}$ follows in the same way. □

By Lemma 5.1, for every $n \in \mathbb{N}$ and every pair of n-cubes $K_n, K_n' \subseteq \mathbb{R}^n$, and for each $p \in \mathbb{R}$, $p \geq 1$, the normed spaces $C_p(K_n)$ and $C_p(K_n')$ are mutually isomorphic, i.e.,

$$C_p(K_n) \cong C_p(I^n) \cong C_p(K_n')$$

in $NVect_F$ holds true. Therefore, without loss of generality, we may consider the normed space $C_p(I^n)$ only, shortly denoted in the sequel by $C_p(n)$. We also include in our consideration the well known Banach space

$$C_\infty(K_n) \equiv (C(K_n), \|\cdot\|_\infty), \ \|x\|_\infty = \max\{|x(t)| \, t \in K_n\}.$$

As before, the unique linear homeomorphism $h_n : I^n \to K_n$ induces an (algebraic, linear) isomorphism

$$h_n^F : C_\infty(K_n) \to C_\infty(I^n) \equiv C_\infty(n), \ h_n^F(x) = xh_n,$$

of the vectorial spaces, and $(h_n^F)^{-1} = (h_n^{-1})^F$. Hereby, for every $n \in \mathbb{N}$, h_n^F is an isomorphism of Banach spaces. Indeed, one straightforwardly verifies that

$$\|xh_n\|_\infty \leq M \|x\|_\infty ,$$

where $M = \max\{|\alpha_{ii}| \mid i = 1, \dots, n\}$ (α_{ii} - diagonal elements of the matrix H_n).

THEOREM 5.2. *Given $n \in \mathbb{N}$, for every $x \in C(I^n)$ and each related pair $p \leq p'$, the inequalities*

$$\|x\|_p \leq \|x\|_{p'} \leq \|x\|_\infty$$

hold true. Consequently, the identity functions $1_p^{p'}(n) : C_{p'}(n) \to C_p(n)$, $p \leq p'$, and $1_p^\infty(n) : C_\infty(n) \to C_p(n)$ are continuous.

PROOF. We firstly need Jensen's inequality for real multivariable mappings, i.e., for every mapping

$$f : K_n \to \mathbb{R}, \ f(t) = f(t_1, \dots, t_n) \geq 0, \ n \in \mathbb{N}$$

(though I^n would do). The basic case $n = 1$ holds true because it is Jensen's original inequality. Notice that, in a proof by induction, one may reduce (by Fubini's theorem) the proving of the inductive step $n \mapsto n + 1$, to the verification of the first step $1 \mapsto 2$. Denote

$$g(t_1) \equiv \int_{a_2}^{b_2} f(t_1, t_2) dt_2, \ t_1 \in [a_1, b_1].$$

Then $g(t_1) \geq 0$, and thus for every convex function φ,

$$\begin{aligned}
\varphi(\int_{K_2} f(t)dt) &= \varphi(\int_{a_1}^{b_1} (\int_{a_2}^{b_2} f(t_1, t_2)dt_2)dt_1) = \varphi(\int_{a_1}^{b_1} g(t_1)dt_1) \\
&\leq \int_{a_1}^{b_1} \varphi(g(t_1))dt_1 = \int_{a_1}^{b_1} \varphi(\int_{a_2}^{b_2} f(t_1, t_2)dt_2)dt_1 \\
&\leq \int_{a_1}^{b_1} (\int_{a_2}^{b_2} \varphi(f(t_1, t_2))dt_2)dt_1 \\
&= \int_{a_1}^{b_1} (\int_{a_2}^{b_2} (\varphi f)(t_1, t_2))dt_2)dt_1 = \int_{K_2} (\varphi f)(t)dt,
\end{aligned}$$

that verifies the first step, and proves the inequality by induction. Now observe that the function

$$\varphi : \mathbb{R}^+ \cup \{0\} \to \mathbb{R},\ \varphi(\tau) = \tau^{\frac{p'}{p}},$$

is convex, whenever $p \leq p'$. By Jensen's inequality from above, it follows that

$$\begin{aligned}(\|x\|_p)^{p'} &= \Big(\int_{I^n} |x(t)|^p\, dt\Big)^{\frac{p'}{p}} = \varphi(\int_{I^n} |x(t)|^p\, dt)\\ &\leq \int_{I^n} \varphi(|x(t)|^p)dt = \int_{I^n} |(x(t)|^p)^{\frac{p'}{p}}\, dt = \int_{I^n} |x(t)|^{p'}\, dt = (\|x\|_{p'})^{p'}.\end{aligned}$$

Hence, $\|x\|_p \leq \|x\|_{p'}$.Further, for every $1 \leq p < \infty$,

$$\begin{aligned}\|x\|_p &= \Big(\int_{I^n} |x(t)|^p\, dt\Big)^{\frac{1}{p}} \leq \Big(\int_{I^n} (\max\{|x(t)|\}^p dt\Big)^{\frac{1}{p}}\\ &= (\max\{|x(t)|\}^p \int_{I^n} dt)^{\frac{1}{p}} = (\max\{|x(t)|\}^p \cdot 1)^{\frac{1}{p}} = \|x\|_\infty .\end{aligned}$$

□

We have achieved our first goal by the following

Corollary 5.3. *Given an $n \in \mathbb{N}$, all the normed vectorial spaces $C_p(K_n)$ and $C_\infty(K'_n)$ are of the same finite quotient shape type. More precisely,*

$$(\forall n \in \mathbb{N})(\forall K_n, K'_n \subseteq \mathbb{R}^n,\ n\text{-cubes})(\forall p, p' \in \mathbb{R},\ p, p' \geq 1)$$
$$Sh_{\underline{\emptyset}}(C_p(K_n)) = Sh_{\underline{\emptyset}}(C_{p'}(K'_n)) = Sh_{\underline{\emptyset}}(C_\infty(K_n)) = Sh_{\underline{\emptyset}}(C_\infty(K'_n)).$$

Proof. Recall that $C_p(K_n) \cong C_p(n) \cong C_p(K'_n)$ and $C_\infty(K_n) \cong C_\infty(n) \cong C_\infty(K'_n)$, and then apply Theorem 5.2 and Proposition 3.7. □

We can now pass to the L_p spaces. Recall that $L_p(K_n)$, $1 \leq p < \infty$, is the completion of $C_p(K_n)$ in its second dual space $C_p(K_n)^{**}$, i.e.,

$$L_p(K_n) = Cl(j[C_p(K_n)]),$$

where $j : C_p(K_n) \to C_p(K_n)^{**}$ is the canonical embedding via the first dual space $C_p(K_n)^*$. The normed space $L_p(I^n) = Cl(j[C_p(n)])$ is denoted by $L_p(n)$. Each $L_p(K_n)$ is separable Banach space, $L_2(K_n)$ is a Hilbert space, and the (algebraic) dimension $\dim(L_p(K_n)) = 2^{\aleph_0}$.

Lemma 5.4. *For every $n \in \mathbb{N}$, and every pair of n-cubes $K_n, K'_n \subseteq \mathbb{R}^n$ and each $p \in \mathbb{R}$, $p \geq 1$, the normed vectorial spaces $L_p(K_n)$ and $L_p(K'_n)$ are mutually isomorphic, $L_p(K_n) \cong L_p(K'_n)$.*

Proof. Notice that there exists the completion functor

$$Cl : NVect_F \to BVectlF$$

determined by $X \mapsto Cl(X) \subseteq X^{**}$ and $(f : X \to Y) \mapsto (\bar{f} : Cl(X) : Cl(Y))$ (the unique continuous linear extension of $jf : X \to Cl(Y)$). By Lemma

5.1, $h_n^F : C_p(K_n) \to C_p(n)$, $h_n^F(x) = xh_n$, is an isomorphism of $NVect_F$. By applying functor Cl to h_n^F, it follows that

$$\overline{h_n^F} : L_p(K_n) \to L_p(n)$$

is an isomorphism of $BVect_F$. Then the conclusion

$$L_p(K_n) \cong L_p(n) \cong L_p(K'_n)$$

follows obviously. □

In order to establish the final results about the quotient shapes of L_p spaces, we only need include within some general facts obtained in Section 3.

THEOREM 5.5. *For every $n \in \mathbb{N}$, all the normed vectorial spaces $L_p(K_n)$, $C_{p'}(K'_n)$ and $C_\infty(K''_n)$, where $K_n, K'_n, K''_n \subseteq \mathbb{R}^n$ are n-cubes and $1 \leq p, p' < \infty$, have the same finite quotient shape type, that is also their countable quotient shape type (with respect to $BVect_F$). Explicitly, for all all n-cubes $K_n, K'_n, K''_n \subseteq \mathbb{R}^n$ and all $1 \leq p_i, p'_i, p''_i, p'''_i < \infty$, $i = 1, 2$,*

$$\begin{aligned} Sh_{\underline{0}}(L_{p_1}(K_n)) &= Sh_{\underline{0}}(L_{p'_1}(n)) = Sh_{\underline{0}}(C_{p''_1}(K'_n)) = Sh_{\underline{0}}(C_{p'''_1}(n)) \\ &= Sh_{\underline{0}}(C_\infty(n)) = Sh_{\underline{0}}(C_\infty(K''_n)) = Sh_{\aleph_0}(C_\infty(n)) \\ &= Sh_{\aleph_0}(L_{p_2}(K_n)) = Sh_{\aleph_0}(L_{p'_2}(n)) = Sh_{\aleph_0}(C_{p''_2}(K'_n)) \\ &= Sh_{\aleph_0}(C_{p'''_2}(n)). \end{aligned}$$

PROOF. By Theorem 3.3 (index $i = 1$ is omitted), $Sh_{\underline{0}}(L_p(K_n)) = Sh_{\underline{0}}(C_p(K_n))$, while by Corollary 5.3, $Sh_{\underline{0}}(C_p(K_n)) = Sh_{\underline{0}}(C_{p'}(n)) = Sh_{\underline{0}}(C_\infty(n))$. Then, by Theorem 3.3 again, $Sh_{\underline{0}}(L_p(K_n)) = Sh_{\underline{0}}(L_{p'}(n))$. Further, in the same way (by applying Theorem 3.3 and Corollary 5.3), $Sh_{\underline{0}}(L_{p'}(n)) = Sh_{\underline{0}}(C_{p'}(n)) = Sh_{\underline{0}}(C_{p''}(K'_n)) = Sh_{\underline{0}}(C_{p'''}(n)) = Sh_{\underline{0}}(C_\infty(n)) = Sh_{\underline{0}}(C_\infty(K''_n))$. The rest follows then by Theorem 3.4. □

Since every $L_2(K_n)$ is a Hilbert space, we can easily establish an analogue of Theorem 4.5 for L_p spaces. Let

$$\boldsymbol{u}(n)_{\underline{0}} = (u(n)_\mu) : L_2(n) \to \boldsymbol{L}(n)_{\underline{0}} = (L(n)_\mu, u(n)_{\mu\mu'}, M(n)_{\underline{0}})$$

be the canonical $(BVect_F)_{\underline{0}}$-expansion of $L_2(n)$. Then $\boldsymbol{L}(n)_{\underline{0}}$ is actually an object of $(HVect_F)_{\underline{0}} \subseteq (BVect_F)_{\underline{0}}$, and $\boldsymbol{u}(n) : L_2(n) \to \boldsymbol{L}(n)_{\underline{0}}$ is an $(HVect_F)_{\underline{0}}$-expansion of $L_2(n)$ as well. Then Theorem 5.5 yields the following consequence.

COROLLARY 5.6. *Let $n \in \mathbb{N}$. For all n-cubes $K_n, K'_n, K''_n \subseteq \mathbb{R}^n$ and all $1 \leq p, p' < \infty$, there exist $(BVect_F)_{\underline{0}}$-expansions (the case κ^-, when $\kappa = \aleph_0$)*

$$\begin{aligned} &\boldsymbol{u}(n,p)_{\underline{0}} = (u(n,p)_\mu) : L_p(K_n) \to \boldsymbol{L}(n)_{\underline{0}} = (L(n)_\mu, u_{\mu\mu'}(n), M(n)_{\underline{0}}), \\ &\boldsymbol{u}(n,p')_{\underline{0}} = (u(n,p')_\mu) : C_{p'}(K'_n) \to \boldsymbol{L}(n)_{\underline{0}} \text{ and} \\ &\boldsymbol{u}'(n)_{\underline{0}} = (u'(n)_\mu) : C_\infty(K''_n) \to \boldsymbol{L}(n)_{\underline{0}}, \end{aligned}$$

such that all of them share the same inverse system $\boldsymbol{L}(n)_{\underline{0}}$ *of* $(HVect_F)_{\underline{0}} \subseteq (BVect_F)_{\underline{0}}$*, which has for its terms* $L(n)_\mu$ *all the finite-dimensional quotient spaces by all appropriate closed subspaces of* $L_2(n)$*, and for the bonds* $u(n)_{\mu\mu'}$ *the corresponding quotient functions, and the index set* $M(n)_{\underline{0}}$ *is of the minimal cardinality* $|M(n)_{\underline{0}}| = 2^{\aleph_0}$ *among all their expansions. Those expansions are also their* $(HVect_F)_{\underline{0}}$*-expansions as well as their expansions in the countable case (the case* κ*, when* $\kappa = \aleph_0$*, with respect to* $(BVect_F)_{\aleph_0}$ *and to* $(HVect_F)_{\aleph_0}$*).*

Remark 5.7.

(a) In the same manner of L_p spaces, one can establish the quite analogous finite (equals countable) quotient shape classification of (the completeness' $L_p(\mathbb{R}^n)$ of) the normed vectorial spaces $C_{0p}(\mathbb{R}^n)$, $1 \leq p < \infty$, of all mappings with compact support.

(b) It seems that the quotient shape studying of Banach algebras could yield very useful results as well.

6. The quotient shapes of Sobolev spaces

We shall now apply the same technique to the more general class of normed vectorial spaces - the Sobolev spaces (yet, in the special case of real functions having the continuous partial derivatives up to a given order ([10], 2.8.5., p. 102).

Given an $n \in \mathbb{N}$, let $\Omega_n \subseteq \mathbb{R}^n$ be a domain (connected open subspace). Further, given a $k \in \mathbb{N}$, let $C_0^{(k)}(\Omega_n)$ be the vectorial space of all functions $x : \Omega_n \to \mathbb{R}$ (over $F = \mathbb{R}$) such that the support

$$\mathrm{supp} f = Cl(\{t \in \Omega_n \mid x(t) \neq 0\}) \subseteq \Omega_n$$

and that x have all (usual) partial derivatives up to order k continuous on Ω_n. Then, for each p, $1 \leq p < \infty$, by

$$\|x\|_p = \Big(\int_{\Omega_n} |x(t)|^p \, dt \Big)^{\frac{1}{p}} + \Big(\int_{\Omega_n} \Big(\sum_{k_1 + \cdots + k_n} \Big(\frac{\partial^k x(t)}{\partial t_1^{k_1} \cdots \partial t_n^{k_n}} \Big)^2 \Big)^{\frac{p}{2}} dt \Big)^{\frac{1}{p}}$$

it is defined a norm on $C_0^{(k)}(\Omega_n)$. The normed space $(C_0^{(k)}(\Omega_n), \|\cdot\|_p)$ is denoted by $C_{0p}^{(k)}(\Omega_n)$.

Lemma 6.1. *Let an* $n \in \mathbb{N}$*, a domain* $\Omega_n \subseteq \mathbb{R}^n$ *and a* $k \in \mathbb{N}$ *be given. Then, for every* $x \in C_0^{(k)}(\Omega_n)$ *and each ordered pair* $p \leq p'$*,* $1 \leq p, p' < \infty$*, the inequality*

$$\|x\|_p \leq \|x\|_{p'}$$

holds true. Consequently, the identity function $1_p^{p'} : C_{0p'}^{(k)}(\Omega_n) \to C_{0p}^{(k)}(\Omega_n)$ *is continuous, whenever* $p \leq p'$*.*

PROOF. According to the definition, $\|x\|_p$ can be written down as the sum $\alpha(x,p)+\beta(x,p)$, where

$$\alpha(x,p) \equiv \Big(\int_{\Omega_n} |x(t)|^p \, dt\Big)^{\frac{1}{p}} \geq 0, \quad \text{and}$$

$$\beta(x,p) \equiv \Big(\int_{\Omega_n} \Big(\sum_{k_1+\cdots+k_n} \Big(\frac{\partial^k x(t)}{\partial t_1^{k_1} \cdots \partial t_n^{k_n}}\Big)^2\Big)^{\frac{p}{2}} dt\Big)^{\frac{1}{p}} \geq 0.$$

Further, as in the proof of Theorem 5.2, for each pair $p \leq p'$,

$$\Big(\int_{\Omega_n} |x(t)|^p \, dt\Big)^{\frac{p'}{p}} \leq \int_{\Omega_n} |x(t)|^{p'} dt \quad \text{and}$$

$$\Big(\int_{\Omega_n} \Big(\sum_{k_1+\cdots+k_n} \Big(\frac{\partial^k x(t)}{\partial t_1^{k_1} \cdots \partial t_n^{k_n}}\Big)^2\Big)^{\frac{p}{2}} dt\Big)^{\frac{p'}{p}}$$

$$\leq \int_{\Omega_n} \Big(\sum_{k_1+\cdots+k_n} \Big(\frac{\partial^k x(t)}{\partial t_1^{k_1} \cdots \partial t_n^{k_n}}\Big)^2\Big)^{\frac{p'}{2}} dt$$

hold true by Jensen's inequality. Therefore,

$$\alpha(x,p)^{p'} \leq \alpha(x,p')^{p'} \quad \text{and} \quad \beta(x,p)^{p'} \leq \beta(x,p')^{p'},$$

and thus,

$$\alpha(x,p) \leq \alpha(x,p') \quad \text{and} \quad \beta(x,p) \leq \beta(x,p'),$$

and finally,

$$\|x\|_p = \alpha(x,p)+\beta(x,p) \leq \alpha(x,p')+\beta(x,p') = \|x\|_{p'} \, .$$

The conclusion follows immediately. □

The closure $Cl(j[C_{0p}^{(k)}(\Omega_n)])$ of $C_{0p}^{(k)}(\Omega_n)$, isometrically embedded in the second dual space $C_{0p}^{(k)}(\Omega_n)^{**}$, is called the Sobolev space, denoted by $W_p^{(k)}(\Omega_n)$. It is a Banach space, and $W_{02}^{(k)}(\Omega_n)$ is a Hilbert space.

THEOREM 6.2. *For every $n \in \mathbb{N}$, every domain $\Omega_n \subseteq \mathbb{R}^n$ and every $k \in \mathbb{N}$, all the normed vectorial spaces $C_{0p}^{(k)}(\Omega_n)$ and all Sobolev spaces $W_{p'}^{(k)}(\Omega_n)$, $1 \leq p, p' < \infty$, have the same finite quotient shape type, that is also their countable quotient shape type (with respect to $BVect_F$). Explicitly, for all $1 \leq p_i, p_i', p_i'', p_i''' < \infty$, $i = 1,2$,*

$$\begin{aligned} Sh_{\underline{0}}(W_{p_1}^{(k)}(\Omega_n)) &= Sh_{\underline{0}}(\Omega_{p_1'}^{(k)}(\Omega_n)) = Sh_{\underline{0}}(C_{0p_1''}^{(k)}(\Omega_n)) = Sh_{\underline{0}}(C_{0p_1'''}^{(k)}(\Omega_n)) \\ &= Sh_{\aleph_0}(W_{p_2}^{(k)}(\Omega_n))) = Sh_{\aleph_0}(W_{p_2'}^{(k)}(\Omega_n)) = Sh_{\aleph_0}(C_{0p_2''}^{(k)}(\Omega_n)) \\ &= Sh_{\aleph_0}(C_{0p_2'''}^{(k)}(\Omega_n)). \end{aligned}$$

Proof. By Theorem 3.3 (index $i = 1$ is omitted), $Sh_{\underline{0}}(W_p^{(k)}(\Omega_n)) = Sh_{\underline{0}}(C_{0p}^{(k)}(\Omega_n))$, while by Lemma 6.1 and Proposition 3.7, $Sh_{\underline{0}}(C_{0p}^{(k)}(\Omega_n))) = Sh_{\underline{0}}(C_{0p'}^{(k)}(\Omega_n))$. Then, by Theorem 3.3 again,

$$Sh_{\underline{0}}(C_{0p'}^{(k)}(\Omega_n))) = Sh_{\underline{0}}(W_{p'}^{(k)}(\Omega_n)),$$

and so on for the finite shape. The statements concerning the countable shape follow then by Theorem 3.4. □

Since every $W_2^{(k)}(\Omega_n)$ is a Hilbert space, we can establish an analogue of Theorem 4.5 and Corollary 5.6 for the Sobolev spaces $W_p^{(k)}(\Omega_n)$. Let

$$\begin{aligned}\boldsymbol{w}(k,\Omega_n) &= (w(k,\Omega_n)_\nu) : W_2^{(k)}(\Omega_n) \to \boldsymbol{W}(k,\Omega_n)_{\underline{0}} \\ &= (W(k,\Omega_n)_\nu, r(k,\Omega_n)_{\nu\nu'}, N(k,\Omega_n)_{\underline{0}})\end{aligned}$$

be the canonical $(BVect_F)_{\underline{0}}$-expansion of $W_2^{(k)}(\Omega_n)$. Then $\boldsymbol{W}(k,\Omega_n)_{\underline{0}}$ is an object of $(HVect_F)_{\underline{0}} \subseteq (BVect_F)_{\underline{0}}$, and $\boldsymbol{w}(k,\Omega_n) : W_2^{(k)}(\Omega_n) \to \boldsymbol{W}(k,\Omega_n)_{\underline{0}}$ is an $(HVect_F)_{\underline{0}}$-expansion of $W_2^{(k)}(\Omega_n)$ as well. Then Theorem 6.2 gives the following corollary.

Corollary 6.3. *Let $n \in \mathbb{N}$, $\Omega_n \subseteq \mathbb{R}^n$ a domain and $k \in \mathbb{N}$. For all $1 \le p, p' < \infty$, there exist $(BVect_F)_{\underline{0}}$-expansions (the case κ^-, when $\kappa = \aleph_0$)*

$$\begin{aligned}\boldsymbol{w}(k,\Omega_n,p)_{\underline{0}} &= (w(k,\Omega_n,p)_\nu) : W_p^{(k)}(\Omega_n) \to \boldsymbol{W}(k,\Omega_n)_{\underline{0}} \\ &= (W(k,\Omega_n)_\nu, r(k,\Omega_n)_{\nu\nu'}, N(k,\Omega_n)_{\underline{0}})\end{aligned}$$

and

$$\boldsymbol{w}'(k,\Omega_n,p')_{\underline{0}} = (w'(k,\Omega_n,p')_\nu) : C_{0p'}^{(k)}(\Omega_n) \to \boldsymbol{W}(k,\Omega_n)_{\underline{0}}$$

such that all of them share the same inverse system $\boldsymbol{W}(k,\Omega_n)_{\underline{0}}$ of $(HVect_F)_{\underline{0}} \subseteq (BVect_F)_{\underline{0}}$, which has for its terms $W(k,\Omega_n)_\nu$ all the finite-dimensional quotient spaces by all appropriate closed subspaces of $W_2^{(k)}(\Omega_n)$, and for the bonds $r(k,\Omega_n)_{\nu\nu'}$ the corresponding quotient functions, and the index set $N(k,\Omega_n)_{\underline{0}}$ is of the minimal cardinality $|N(k,\Omega_n)_{\underline{0}}| = 2^{\aleph_0}$ among all their expansions. Those expansions are also their $(HVect_F)_{\underline{0}}$-expansions as well as their expansions in the countable case (the case κ, when $\kappa = \aleph_0$, with respect to $(BVect_F)_{\aleph_0}$ and to $(HVect_F)_{\aleph_0}$).

Remark 6.4. In light of Corollary 6.3, for some classes of partial differential equations, it might exist satisfactory *approximate* solutions "within the spectra" of finite-dimensional Hilbert spaces (i.e., Euclidean spaces) coming from their $(HVect_F)_{\underline{0}}$-expansions. Much more interesting and very useful quotient shape classification could be that of the general Sobolev spaces H^s (of all appropriate generalized functions - distributions; [12], Part II, Sections 6 and 8.8). We believe that it is analogous to that of the special Sobolev spaces $W_p^{(k)}(\Omega_n)$ established in Theorem 6.2.

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References

[1] K. Borsuk, *Concerning homotopy properties of compacta*, Fund. Math. **62** (1968), 223–254.

[2] K. Borsuk, Theory of Shape, Monografie Matematyczne **59**, Polish Scientific Publishers, Warszawa, 1975.

[3] J.-M. Cordier and T. Porter, Shape Theory: Categorical Methods of Approximation, Ellis Horwood Ltd., Chichester, 1989. (Dover edition, 2008.)

[4] J. Dugundji, Topology, Allyn and Bacon, Inc. Boston, 1978.

[5] J. Dydak and J. Segal, Shape theory: An introduction, Lecture Notes in Math. **688**, Springer-Verlag, Berlin, 1978.

[6] D.A. Edwards and H.M. Hastings, Čech and Steenrod homotopy theories with applications to geometric topology, Lecture Notes in Math. **542**, Springer-Verlag, Berlin, 1976.

[7] H. Herlich and G. E. Strecker, Category Theory: An Introduction, Allyn and Bacon Inc., Boston, 1973.

[8] N. Koceić Bilan and N. Uglešić, *The coarse shape*, Glas. Mat. Ser. III **42(62)** (2007), 145–187.

[9] E. Kreyszig, Introductory Functional Analysis with Applications, John Wiley & Sons, New York, 1989.

[10] S. Kurepa, Funkcionalna analiza : elementi teorije operatora, Školska knjiga, Zagreb, 1990.

[11] S. Mardešić and J. Segal, Shape Theory, North-Holland, Amsterdam, 1982.

[12] W. Rudin, Functional Analysis, Second Edition, McGraw-Hill, Inc., New York, 1991.

[13] N. Uglešić, *The shapes in a concrete category*, Glas. Mat. Ser. III **51(71)** (2016), 255–306.

[14] N. Uglešić, *On the quotient shape of vectorial spaces*, Rad Hrvat. Akad. Znan. Umjet. Mat. Znan. **21** (2017), 179–203.

[15] N. Uglešić, *On the quotient shape of topological spaces*, Topology Appl. **239** (2018), 142–151.

[16] N. Uglešić and B. Červar, *The concept of a weak shape type*, International J. of Pure and Applied Math. **39** (2007), 363–428.

Kvocientni oblici l_p i L_p prostora

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SAŽETAK. Svi l_p prostori (nad istim poljem F, $p \neq \infty$) imaju konačni kvocientni oblikovni tip Hilbertova prostora l_2. To je ujedno konačni kvocientni oblikovni tip i svih njihovih podprostora $l_p(p')$, $p < p' \leq \infty$, kao i svih direktnih suma $F_0^{\mathbb{N}}(p')$, $1 \leq p' \leq \infty$. Nadalje, njihovi prebrojivi kvocientni oblikovni tipovi svode se na onaj konačni. Slično, za dani $n \in \mathbb{N}$, svi L_p prostori (nad istim poljem, $p \neq \infty$) imaju konačni kvocientni oblikovni tip Hilbertova prostora L_2, a i njihovi prebrojivi kvocientni oblikovni tipovi se svode na onaj konačni. Analogne tvrdnje vriede i za Soboljevljeve prostore realnih funkcija s odgovarajućim neprekidnim parcijalnim derivacijama.

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