Interdisciplinary Description of Complex Systems 16(3-B), 465-484, 2018

BORDA AND PLURALITY COMPARISON WITH REGARD TO COMPROMISE AS A SORITES PARADOX

Aleksandar Hatzivelkos*

University of Applied Sciences Velika Gorica, Department of Mathematics
Velika Gorica, Croatia

DOI: 10.7906/indecs.16.3.18
Regular article

Received: 16th March 2018.
Accepted: 27th July 2018.

ABSTRACT

Social choice decision aggregation is a form of complex system modelling which is based upon voters' rankings over a set of candidates. Different social choice functions, such as Borda count, plurality count or Condorcet methods models different aspects of social choice decision criteria. One of such criteria which has not been fully described or modelled, is the notion of compromise. This article aims to define a measure which would capture the notion of compromise on a given profile of voter preferences, about certain candidate being appointed to a certain position by a certain social welfare function. The goal is to define what compromise should mean, and proposes the so-called "d-measure of divergence" as a measure of divergence for some candidate to be positioned to a certain position. This study compares the results of two well-established social welfare functions, Borda and plurality count *d*-measure of divergence.

KEY WORDS

Borda count, plurality count, compromise

CLASSIFICATION

JEL: D72

*Corresponding author, η: hatzi@vvg.hr; +385 1 6222 501;
University of Applied Sciences Velika Gorica, Zagebačka 5, HR – 10 410 Velika Gorica, Croatia

INTRODUCTION

Basis of this article is the mathematical description of the notion of compromise. The need to formally determine how we should interpret the notion of compromise comes from the following example. Let there is an election in which one hundred voters should choose between three candidates: A, B and C. Each voter places its vote by ordering those candidates. That ordering we will call a preference, and denote it α_i. Set of all preferences for those hundred voters, a profile is given in Table 1, in which fifty one voters have preference A ├ B ├ C, while forty nine voters have preference C ├ B ├ A.

Table 1. Basic motivation profile.

51	49
A	C
B	B
C	A

Given the profile α, which candidate should win? Most of the classical social choice functions would say – candidate A. Borda count would produce A ├ B ├ C linear ordering, result of plurality count would be A ├ C ├ B (with A winning the most first places, and B the least). Condorcet method would duel all candidates, and those duels would yield A ├ B ├ C ordering. All those classical methods have one thing in common: winner should be candidate A. Nevertheless, that is a candidate that 49% of voters see as the worst choice. Should A then be a winner? What should be result if we approach to a profile α looking for compromise? If we want from social choice function to address notion of compromise, would it be better if candidate B is declared as a winner? This leads us to the main topic of this article: finding a way for determining a value which should capture notion of compromise on a given profile, for placing a candidate on a certain position in linear ordering.

Let us concentrate on the example in Table 1. If we take a look at candidate A, in a given profile he/she is placed first in 51 preferences, and placed third in 49 preferences. Therefore, in 51 preferences, distance between his position and the first place is 0 (places), and in 49 preferences that distance equals 2 (places). If we simply sum all those distances (for each candidate) over profile, we would get a measure of distance between profile placements of a candidate and a first place. But, for such a measure, one can easily prove that ranking based on it gives result equivalent to Borda count.

In the core of the notion of compromise, however, lays a need to *punish* of discourage bigger distances; this means that when we are looking for a way to describe compromise about a candidate being placed at the winning position, each position should contribute to a sum (of distances) with more than its linear contribution. Therefore, we will take a look at a sum of *weighted* distances, that is, distances to the power of d, d being a real number greater than 1.

If we sum such weighted distances from the first place over profile α for a candidates A, B and C, we get the following expression:

$$\beta_1^d(A) = 51 \cdot 0^d + 49 \cdot 2^d = 49 \cdot 2^d,$$
$$\beta_1^d(B) = 51 \cdot 1^d + 49 \cdot 1^d = 100,$$
$$\beta_1^d(C) = 51 \cdot 2^d + 49 \cdot 0^d = 51 \cdot 2^d.$$

We introduced notion $\beta_1{}^d(M_i)$ for some candidate M_i, which we will call a *d-measure of divergence* from the first position. The idea is that smaller value of $\beta_1{}^d(M_i)$ captures notion of the greater level of compromise on a given profile for a candidate to be placed on a first place of linear ordering. Unlike distance function from works of Seiford and Cook [1], we do not form measure of distance between preferences. Rather than that, we establish measure divergence from compromise (or consensus) that certain candidate should be positioned on certain position. But the goal is similar: it is in interest of society to minimize that measure. This leads us to the basic definitions, as done in [2].

BASIC DEFINITIONS

Definition 1.1

(d-Measure of divergence from the first position.) Let $M = \{M_1; \ldots; M_m\}$ be set of m candidates, and let $\alpha \in L(M)^n$ be a profile of n voters over those candidates. We define a d-measure of divergence from the first position for a candidate M_k, $\beta_1^d(M_i)$, as a

$$\beta_1^d(M_k) = \sum_{i=1}^{n} \left|\alpha_i^k - 1\right|^d, \tag{1}$$

where α_i^k stands for a position of the candidate M_k in a preference of i-th voter α_i, and for some real value $d > 1$.

Now we can easily extend definition to a d-measure of divergence from a j-th position of the k-th candidate.

Definition 1.2

(d-Measure of divergence from the j-th position.) Let $M = \{M_1; \ldots; M_m\}$ be set of m candidates, and let $\alpha \in L(M)^n$ be a profile of n voters over those candidates. We define a d-measure of divergence from a j-th position for a candidate M_k, $\beta_j^d(M_k)$, as a

$$\beta_j^d(M_k) = \sum_{i=1}^{n} \left|\alpha_i^k - j\right|^d, \tag{2}$$

where α_i^k stands for a position of the candidate M_k in a preference of i-th voter α_i, and for some real value $d > 1$.

Given this definition, it is only natural to gather $\beta_j^d(M_k)$, values in a form of a matrix.

Definition 1.3

(d-Measure of divergence matrix.) Let $M = \{M_1; \ldots; M_m\}$ be set of m candidates, and let $\alpha \in L(M)^n$ be a profile of n voters over those candidates. We define a d-measure of divergence matrix:

$$\boldsymbol{M}^d = \begin{bmatrix} \beta_1^d(M_1) & \beta_2^d(M_1) & \ldots & \beta_m^d(M_1) \\ \beta_1^d(M_2) & \beta_2^d(M_2) & \ldots & \beta_m^d(M_2) \\ \vdots & \vdots & \ddots & \vdots \\ \beta_1^d(M_m) & \beta_2^d(M_m) & \ldots & \beta_m^d(M_m) \end{bmatrix}, \tag{3}$$

for some real value $d > 1$.

As we can see, in j-th column of a matrix $\boldsymbol{M}^d$ we have d-measures of divergence from j-th position for all candidates, while in i-th row of matrix $\boldsymbol{M}^d$, we have d-measures of divergence from all positions for a candidate M_i.

COMPROMISE AS A SORITES PARADOX

Before proceeding, let us say something about the value of parameter d. In the example given in the Table 1 we see that, if we want candidate B to have smaller d-measure of divergence from a first position than candidate A (which means, if we want candidate B to be declared a compromise winner on a given profile), it should be

$$\beta_1^d(B) < \beta_1^d(A) \ \Rightarrow d > \ \log_2 100 - \log_2 49. \tag{2}$$

But, what is the value of d that should be used generally? Answering to that question requires finding an answer to the following version of a Sorites paradox [3]: Let us say that n voters are voting through strict linear ordering over the set of three candidates, {A, B, C}. For some $k \in \mathbf{N}$, $\lceil n/2 \rceil \leq k \leq n$, they form a profile α_{basic} given in Table 2.

Table 2. Basic definition profile.

k	$n-k$
A	C
B	B
C	A

Let us accept reasoning that candidate B should be a compromise winner on profile α_{basic} given in Table 2 for $k = \lceil n/2 \rceil$, and that candidate A should be a compromise winner on profile α_{basic} for $k = n$. Sorites paradox arises from question: what is the value of k for which we should consider candidate B compromise winner on a given profile α_{basic}, while for value $k + 1$ a compromise winner should be candidate A?

This question can be seen as a version of a classical Sorites paradox given by Megarian logician Eubulides of Miletus about a number of grains which are (not) forming a heap. Phenomenon that lies at a heart of the paradox is recognized as the phenomenon of vagueness; the concept of heap appears to lack sharp boundaries, just as the concept of compromise winner does in our case. Nevertheless, we will approach to the issue not as to a paradox, but (same as Eubulides did) as a puzzle.

From statement that for a value of $k = \lceil n/2 \rceil$ we have one candidate as a compromise winner on a given profile, while for a value of $k = n$ we have another candidate as a compromise winner, using the Least-number principle, we shall conclude that there is some k_0 between those two values, such that for k_0 compromise winner on a profile basic is a candidate B, and for $k_0 + 1$ compromise winner is candidate A. Value k_0 should be result of an *a priori* social choice of a group which is about to use this model.

There are numerous situations in which similar social choices are being made. For instance, in a number of states two-third parliament majority is needed to make constitution changes, as opposed to simple majority needed for other type of decisions. So, why 2/3? Why not 3/4 or 4/7? Measure of majority needed for such constitutional changes represents similar social decision, as one presented in this article – society decided where to *draw the line*. Similarly, social decision should be made about value of k_0. When value for k_0 is determined, for d we have:

$$\log_2 \frac{n}{n-k_0} < d < \log_2 \frac{n}{n-k_0-1}. \tag{2}$$

d-MEASURE OF DIVERGENCE FROM THE FIRST POSITION

If we interpret d-measure of divergence from the first position as a measure of compromise for a social function choice winner selection, we can compare results of the classical social choice function. For instance Borda count is usually considered as a social choice function that emphasize compromise candidate as a winner, especially when compared to the plurality winner. Does this thesis hold if we use d-measure of divergence from the first position as a measure for selection of the compromise candidate for a winner?

In this section we will provide an answer to that question. To do that, we will first consider three candidate scenario, followed by scenarios with more candidates. Let us consider a Theorem 2.1.

2.1. Theorem. Let be a profile over the set of candidates $M = \{A, B, C\}$. Let W_{BC} stands for a unique Borda count winner candidate and W_{PC} for a unique plurality winner candidate (if there is such) over some profile α. For every d > 1 there is

$$\beta_1^d(W_{\mathrm{BC}}) \leq \beta_1^d(W_{\mathrm{PC}}).$$

Equality holds if and only if $W_{\mathrm{BC}} = W_{\mathrm{PC}}$. There is a combinatorial proof of this theorem. Although there are six different preferences over the set of three candidates, number of all possible combinations of preferences that can form a profile can be reduced.

Two profiles, α_1^C and α_2^C (Table 3) we will call Condorcet triples. Those profiles consist of three preferences on which neutral and anonymous social choice function should form a tie (or a cycle) as a result. As Saari showed [5], all scoring point functions are invariant in regard of Condorcet triplet removal, which includes both Borda and plurality count. Furthermore, it is easy to prove that a d-measure of divergence from the j-th position preserves ordering among candidates when profile is reduced for the Condorcet triple:

2.2. Lemma. Let A and B be any two candidates from the set of candidates M. For a d-measure of divergence from the j-th position β_i^d, and profiles α and α', α' being the profile derived from α by removal of one Condorcet triple, we have

$$\beta_i^d(A, \alpha) < \beta_i^d(B, \alpha) \Leftrightarrow \beta_i^d(A, \alpha') < \beta_i^d(B, \alpha').$$

With removal of one Condorcet triple from the profile α, every of three candidates loses exactly one first placement, one second placement and one third placement. Therefore all candidates lose same amount of points in Borda count, as well as same amount of first places count in plurality count. That means that their placements remain the same in both Borda and plurality count. Same applies to the d-measure of divergence from the j-th place: all three candidates lose same values in calculation of β_j^d, which means that their relative position in regard to d-measure of divergence from the first position remains the same.

Table 3. Condorcet triples: profiles of α_1^C (left) and α_2^C (right).

1	1	1		1	1	1
A	C	B		C	A	B
B	A	C		B	C	A
C	B	A		A	B	C

Since both Borda and plurality count, as well as d-measure of divergence from the j-th position relations among candidates, are Condorcet triple invariant, we can reduce a set of all possible combinations of voters preferences to a set of preferences without Condorcet triples. This means that largest profile we should analyse consists of (some number of) two preferences from α_1^C and two from α_2^C. Beside profiles that reduce to one-preference profiles, Table 4 shows all possible two-preference profiles without Condorcet triples.

Profiles in Table 4 are same up to the permutation of candidates; each of profiles in one group of profiles can be obtained from the other by some permutations of the candidates. There are 15 different profiles, which equals to $\binom{6}{2}$, the number of possible ways to choose two from six preferences. Same way, three and four preference reduced profiles are grouped up to the permutation of candidates in Tables 5 and 6.

In Table 5 we have all together 18 profiles which equals to $\binom{6}{3} - 2$, that is all three preference combinations except for Condorcet triples. In Table 6 there are 9 four-preference profiles which equals to $\binom{3}{2} \cdot \binom{3}{2}$, number of way we can select two preferences out of each Condorcet triple.

Proof of the Theorem 2.1 now follows from Lemmas 1-11 given in the Appendix.

Table 4. Two-preference reduced profiles.

1.

n	*m*
A	A
B	C
C	B

~

n	*m*
B	B
A	C
C	A

~

n	*m*
C	C
A	B
B	A

2.

n	*m*
A	C
B	B
C	A

~

n	*m*
B	A
C	C
A	B

~

n	*m*
B	C
A	A
C	B

3.

n	*m*
A	B
B	A
C	C

~

n	*m*
A	C
C	A
B	B

~

n	*m*
B	C
C	B
A	A

4.

n	*m*
A	C
B	A
C	B

~

n	*m*
A	B
C	A
B	C

~

n	*m*
B	C
A	B
C	A

~

~

n	*m*
B	A
C	B
A	C

~

n	*m*
C	B
A	C
B	A

~

n	*m*
C	A
B	C
A	B

In case with more than three candidates, similar claim cannot be proven. If there are four candidates, there are profiles on which plurality count produces different winner than Borda count, and with smaller d-measure of divergence form the first position. Consider the following theorem.

2.3. Theorem. Let $M = \{A, B, C, D\}$ be the set of candidates, and let W_{BC} stands for a unique Borda count winner candidate and W_{PC} for a unique plurality winner candidate over some profile α. For every $d > 1$ there is a profile α such that $W_{\mathrm{BC}} \neq W_{\mathrm{BC}}$, and for d-measures of divergence from the first position there is

$$\beta_1^d(W_{\mathrm{PC}}) \leq \beta_1^d(W_{\mathrm{BC}}).$$

Proof of the Theorem 2.3. For given d > 1, we will construct profile α such that it holds $\beta_1^d(W_{\mathrm{PC}}) \leq \beta_1^d(W_{\mathrm{BC}})$. Let us analyse following profile:

m	*i*	*j*
A	B	D
B	C	C
C	A	A
D	D	B

.

We will show that there are natural numbers m, i and j such that Theorem 2.3. holds for given $d > 1$. First, we will set

$$m > i, \;\; m > i. \tag{4}$$

This makes candidate A plurality winner on a profile. Now, let candidate B be Borda winner on the same profile. In that case we have

Table 5. Three-preference reduced profiles.

1.

k	*l*	*m*
A	C	C
B	A	B
C	B	A

~

k	*l*	*m*
B	C	C
A	B	A
C	A	B

~

k	*l*	*m*
C	A	A
B	C	B
A	B	C

~

~

k	*l*	*m*
B	A	A
C	C	B
A	B	C

~

k	*l*	*m*
A	B	B
C	A	C
B	C	A

~

k	*l*	*m*
C	B	B
A	C	A
B	A	C

2.

k	*l*	*m*
A	C	A
B	A	C
C	B	B

~

k	*l*	*m*
A	B	A
B	A	C
C	C	B

~

k	*l*	*m*
B	C	B
A	B	C
C	A	A

~

~

k	*l*	*m*
A	C	A
B	A	C
C	B	B

~

k	*l*	*m*
A	B	A
B	A	C
C	C	B

~

k	*l*	*m*
B	C	B
A	B	C
C	A	A

3.

k	*l*	*m*
A	C	B
B	A	A
C	B	C

~

k	*l*	*m*
A	C	B
C	A	A
B	B	C

~

k	*l*	*m*
B	A	C
C	B	B
A	C	A

~

~

k	*l*	*m*
B	C	A
A	B	B
C	A	C

~

k	*l*	*m*
C	A	B
B	C	C
A	B	A

~

k	*l*	*m*
C	B	A
A	C	C
B	A	B

Table 6. Four-preference reduced profiles.

1.

k	*l*	*m*	*n*
A	C	C	A
B	A	B	C
C	B	A	B

~

k	*l*	*m*	*n*
A	B	B	A
C	A	C	B
B	C	A	C

~

k	*l*	*m*	*n*
B	C	C	B
A	B	A	C
C	A	B	A

2.

k	*l*	*m*	*n*
A	C	C	B
B	A	B	A
C	B	A	C

~

k	*l*	*m*	*n*
A	B	B	C
C	A	C	A
B	C	A	B

~

k	*l*	*m*	*n*
B	A	A	C
A	B	C	B
C	C	B	A

3.

k	*l*	*m*	*n*
A	C	A	B
B	A	C	A
C	B	B	C

~

k	*l*	*m*	*n*
B	A	B	C
C	B	A	B
A	C	C	A

~

k	*l*	*m*	*n*
C	A	C	B
B	C	A	C
A	B	B	A

$$2m + 3i > 3m + i + j \Leftrightarrow 2i > m + j, \tag{5}$$

$$2m + 3i > m + 2i + 2j \Leftrightarrow m + i > 2j, \tag{6}$$

$$2m + 3i > 3j \Leftrightarrow \frac{2}{3}m + i > j. \tag{7}$$

From condition (4) follows

$$m > i \Rightarrow m + i > 2i \overset{(5)}{\Rightarrow} m + i > m + j \Rightarrow i > j,$$

so we have

$$m + i > j + j = 2j,\ \frac{2}{3}m + i > \frac{2}{3}m + j > j.$$

Therefore, if conditions (4) and (5) are satisfied, then conditions (6) and (7) are met. Let us set a value for m which satisfies condition (5):

$$2i = m + j + 1 \Leftrightarrow m = 2i - j - 1.$$

Because of condition (4) we have

$$2i - j - 1 > i \Leftrightarrow i > j + 1. \tag{8}$$

$$2i - j - 1 > j \Leftrightarrow i > j + 1/2. \tag{9}$$

Let us just point out, that from condition (8) follows fulfilment of condition (9). We want to construct profile α, such that for a given $d > 1$ we have

$$\beta_1^d(A) < \beta_1^d(B) \Leftrightarrow (i + j)\cdot 2^{\mathrm{d}} < m + j\cdot 3^{\mathrm{d}}.$$

with m, i and j satisfying (4) and (5). Now we have

$$i\cdot 2^{\mathrm{d}} + j\cdot 2^{\mathrm{d}} < m + j\cdot 3^{\mathrm{d}},$$
$$i\cdot 2^{\mathrm{d}} < m + j\cdot(3^{\mathrm{d}} - 2^{\mathrm{d}}),$$
$$i\cdot 2^{\mathrm{d}} < 2i - j - 1 + j\cdot(3^{\mathrm{d}} - 2^{\mathrm{d}}),$$
$$i\cdot(2^{\mathrm{d}} - 2) + 1 < j\cdot(3^{\mathrm{d}} - 2^{\mathrm{d}} - 1).$$

It is easy to show that $2^d - 2 > 0$ and $3^d - 2^d - 1 > 0$ for all $d > 1$ (see proof of the Lemma 12 in the Appendix), so it follows:

$$j > \frac{2^d - 2}{3^d - 2^d - 1}\,i + \frac{1}{3^d - 2^d - 1}.$$

Shortly, we are looking for i, j and m such that $m = 2i - j - 1$ and

$$\frac{2^d - 2}{3^d - 2^d - 1}\,i + \frac{1}{3^d - 2^d - 1} < j < i - 1. \tag{10}$$

On the left side of inequality (10) we have linear expression (with regard to i), which can be interpreted as a line. Lemma 12 proofs that coefficient of that straight line is less than 1 for all $d > 1$ (with coefficient of the straight line on the right side of the inequality (10) being equal to 1). Therefore, there must exist solutions i and j, for i being large enough. To make sure that in the solution span for j there is at least one integer value, we will set a condition that right side of inequality must be greater than the left side for at least 1:

$$\frac{2^d - 2}{3^d - 2^d - 1}\,i + \frac{1}{3^d - 2^d - 1} + 1 < i - 1.$$

$$i > \frac{2\cdot 3^d - 2\cdot 2^d - 1}{3^d - 2\cdot 2^d + 1}. \tag{11}$$

Finally, we pick i such that it satisfies (11), j such it satisfies (10), while for m we have $m = 2i - j - 1$, which proves the Theorem 2.3.

Let us demonstrate construction of the profile on which Borda winner has a greater d-measure of divergence from the first position than a plurality winner with an example.

2.4. Example. Let M = {A, B, C, D} be set of candidates, and let d = 1,05. According to Theorem 2.3, values m, i and j from a profile α are equal to m = 44, i = 43 and j = 44. On this profile plurality winner is candidate A, while Borda winner is candidate B (with Borda score 217, while Borda scores of candidate A and C equals to 216 and 212). d-Measures of divergence from the first place are equal to:

$$\beta_1^{1,05}(A) = (43 + 41)\cdot 2^{1,05} = 173,9245,$$

$$\beta_1^{1,05}(B) = 44 + 41\cdot 3^{1,05} = 173,9455.$$

Construction from the Theorem 2.3 can be expanded to the arbitrary large set of candidates. With first four positions of the candidates that remain the same, all other candidates can be arbitrary placed below 4^{th} position in preferences of the constructed profile.

2.5. Theorem. Let $M = \{M_1, M_1, \ldots, M_k\}$ be set of candidates, and let W_{BC} stands for a unique Borda count winner candidate and W_{PC} for a unique plurality winner candidate over some profile α. For every $d > 1$ there is a profile α such that $W_{BC} \neq W_{PC}$, and for d-measures of divergence from the first position there is

$$\beta_1^d(W_{PC}) < \beta_1^d(W_{BC}).$$

Proof of the Theorem 2.5. follows the construction of the proof of Theorem 2.3, leading to the same conditions for construction of the profile which satisfy Theorem.

CONCLUSIONS

This article presents one possible mathematical look at the notion of compromise in social choice theory. Notion of a d-measure of divergence is introduced as a measure of divergence from (predetermined level of) compromise. If we accept concept of compromise as a Sorites paradox, with level of compromise as a social decision, we can compare results of established social choice function in a new light.

As shown in this article, Borda count always produces a winner with smaller d-measure of divergence from the first place than plurality count winner – but only in three candidates scenarios. We also proved that, if there are four or more candidates, there are profiles on which Borda winner can have greater d-measure of divergence from the first place than plurality winner. This result makes us to think again about wildly accepted opinion of Borda count as a more compromise social function of those two. Furthermore, introduced concept of a compromise as a Sorites paradox offers more room for further research. Object of further study should not be only d-measure of divergence from the first place, since concept offers much richer data to analyse and interpret. One of such goal could be utilization of d-measure matrix for construction social choice function(s) which would (in some sense) minimise d-measure of divergence.

APPENDIX

LEMMA 1.

Let α_1 be a profile over the set of candidates M = {A, B, C}, without any Condorcet triples. Furthermore, let α_1 be a profile of a form

n
ξ(A)
ξ(B)
ξ(C)

,

with ξ being some permutation over the set M. Then, for every profile α obtained as a union of α_1 and some number of copies of Condorcet triples α_1^C and α_2^C it follows

$$\beta_1^d(W_{BC}) < \beta_1^d(W_{PC}),$$

for some W_{BC}, $W_{PC} \in M$, winners by Borda and plurality count respectively over the profile α, where $W_{BC}, \neq W_{PC}$, if such W_{PC} exists.

PROOF OF THE LEMMA 1.

Proof of the Lemma is trivial, since Borda and plurality winner on the profile α_1 are the same. It follows that both methods have same winner on all profiles obtained by addition of some number of Condorcet triples.

LEMMA 2.

Let α_1 be a profile over the set of candidates M = {A, B, C}, without any Condorcet triples. Furthermore, let α_1 be a profile of a form

m	n
ξ(A)	ξ(A)
ξ(B)	ξ(C)
ξ(C)	ξ(B)

,

with ξ being some permutation over the set M. Then, for every profile α obtained as a union of α_1 and some number of copies of Condorcet triples α_1^C and α_2^C it follows

$$\beta_1^d(W_{BC}) < \beta_1^d(W_{PC}),$$

for some W_{BC}, $W_{PC} \in M$, winners by Borda and plurality count respectively over the profile α, where $W_{BC}, \neq W_{PC}$, if such W_{PC} exists.

PROOF OF THE LEMMA 2.

Same as for Lemma 1., proof is trivial, since both the Borda and the plurality count produce same winner on a given profile for all $n, m \in \mathbf{N}$.

LEMMA 3.

Let α_1 be a profile over the set of candidates M = {A, B, C}, without any Condorcet triples. Furthermore, let α_1 be a profile of a form

m	n
ξ(A)	ξ(C)
ξ(B)	ξ(B)
ξ(C)	ξ(A)

,

with ξ being some permutation over the set M. Then, for every profile α obtained as a union of α_1 and some number of copies of Condorcet triples α_1^C and α_2^C it follows

$$\beta_1^d(W_{BC}) < \beta_1^d(W_{PC}),$$

for some W_{BC}, $W_{PC} \in M$, winners by Borda and plurality count respectively over the profile α, where $W_{BC}, \neq W_{PC}$, if such W_{PC} exists.

PROOF OF THE LEMMA 3.

Because of symmetry of the positions of the candidates ξ(A) and ξ(C) in a profile α_1, without loss of generality we can assume that $m > n$. It follows that candidate ξ(A) is Borda winner, since because of $2m > m + n$ he/she has better Borda score than candidate $\xi(B)$, and because of $2m > 2n$ he/she has better Borda score than candidate $\xi(C)$. Plurality count under the same assumption produces $\xi(A)$ as a plurality winner, which proves the Lemma 3.

LEMMA 4.

Let α_1 be a profile over the set of candidates $M = \{A, B, C\}$, without any Condorcet triples. Furthermore, let α_1 be a profile of a form

m	n
$\xi(A)$	$\xi(B)$
$\xi(B)$	$\xi(A)$
$\xi(C)$	$\xi(C)$

,

with ξ being some permutation over the set M. Then, for every profile α obtained as a union of α_1 and some number of copies of Condorcet triples α_1^C and α_2^C it follows

$$\beta_1^d(W_{BC}) < \beta_1^d(W_{PC}),$$

for some W_{BC}, $W_{PC} \in M$, winners by Borda and plurality count respectively over the profile α, where $W_{BC}, \neq W_{PC}$, if such W_{PC} exists.

PROOF OF THE LEMMA 4.

Because of symmetry of the positions of the candidates $\xi(A)$ and $\xi(B)$ in a profile α_1, without loss of generality we can assume that $m > n$. For Borda count we have $\xi(A) \vdash \xi(B) \Leftrightarrow 2m+n > 2n + m$, because of $m > n$. Since $\xi(C)$ cannot be Borda winner, it follows that candidate $\xi(A)$ is both the Borda and the plurality winner on the profile α_1, which proves the Lemma 4.

LEMMA 5.

Let α_1 be a profile over the set of candidates $M = \{A, B, C\}$, without any Condorcet triples. Furthermore, let α_1 be a profile of a form

m	n
$\xi(A)$	$\xi(B)$
$\xi(B)$	$\xi(C)$
$\xi(C)$	$\xi(A)$

,

with ξ being some permutation over the set M. Then, for every profile α obtained as a union of α_1 and some number of copies of Condorcet triples α_1^C and α_2^C it follows

$$\beta_1^d(W_{BC}) < \beta_1^d(W_{PC}),$$

for some W_{BC}, $W_{PC} \in M$, winners by Borda and plurality count respectively over the profile α, where $W_{BC}, \neq W_{PC}$, if such W_{PC} exists.

PROOF OF THE LEMMA 5.

Without loss of generality, Lemma 5 is proven for the profile

m	n
A	B
B	C
C	A

.

First, let us assume that $m \leq n$. Then plurality winner is candidate B. Candidate C cannot be Borda winner, since B is better ranked candidate in on preferences. For candidate A to be a

Borda winner, we should have $2m > m+2n \Rightarrow m > 2n$, which contradicts $m \leq n$. Therefore, plurality winner is in that case also a Borda winner, which proves the Lemma 5.

In case of $m > n$, plurality winner is candidate A. If in that case holds $m + 2n > 2m \Rightarrow n > m/2$, we have a candidate B winning by Borda score (otherwise, candidate A is also a Borda winner, which proves the Lemma 5). Does then exist $d > 1$ for which candidate A has smaller d-measure of divergence from the first position than candidate B? Let us analyse corresponding d-measures of divergence from the first position:

$$\beta_1^d(A) < n \cdot 2^d > n \cdot 2 > m = \beta_1^d(B),$$

which holds for all $d > 1$. This means that Borda winner has smaller d-measure of divergence from the first position than plurality winner, for all $d > 1$, which proves the Lemma 5.

LEMMA 6.

Let α_1 be a profile over the set of candidates $M = \{A, B, C\}$, without any Condorcet triples. Furthermore, let α_1 be a profile of a form

k	l	m
ξ(A)	ξ(C)	ξ(C)
ξ(B)	ξ(A)	ξ(B)
ξ(C)	ξ(B)	ξ(A)

,

with ξ being some permutation over the set M. Then, for every profile α obtained as a union of α_1 and some number of copies of Condorcet triples α_1^C and α_2^C it follows

$$\beta_1^d(W_{BC}) < \beta_1^d(W_{PC}),$$

for some W_{BC}, $W_{PC} \in M$, winners by Borda and plurality count respectively over the profile α, where $W_{BC}, \neq W_{PC}$, if such W_{PC} exists.

PROOF OF THE LEMMA 6.

Without loss of generality, we will prove Lemma 6. for the profile

k	l	m
A	C	C
B	A	B
C	B	A

.

On that profile, plurality winner can be candidate A or candidate C. First, let us assume that plurality winner is candidate A. It follows $k > l + m$. From there we have:

$$k > l + m \Rightarrow 2k > 2l + 2m \Rightarrow 2k + l > 3l + 2m.$$

Since Borda score of candidate A equals to $2k + l$, and Borda score of candidate C equals to $2l + 2m$, it follows $2k + l > 3l + 2m > 2l + 2m$, which means that candidate A has a higher Borda score than candidate B. In the same time, we have $2k + l > k + m$, $k + l > m$, because of $k > l + m$, so candidate A has higher Borda score than candidate B, which makes candidate

A both plurality and Borda winner, and proves the Lemma 6.

If, however, we assume that candidate C is plurality winner on a given profile, then we have

$$k < m + l. \tag{12}$$

In this case, if candidate A is Borda winner, it should have higher Borda score than candidate B, which yields

$$2k + l > k + m \Rightarrow l + k > m, \tag{13}$$

and higher Borda score than candidate C

$$2k + l > 2l + 2m \Rightarrow k > m + l/2. \quad (14)$$

If this condition is met, candidate C is plurality, and candidate A Borda winner. Let us examine whether in that case the candidate C can have lower d-measure of divergence from the first place than candidate A. In other words, whether there is $d > 1$ such that $\beta_1^d(C) < \beta_1^d(A)$,? In that case we have (with $k > m$)

$$k \cdot 2^d < l + m \cdot 2^d \Leftrightarrow 2^d < \frac{l}{k-m}.$$

But, from condition (14) one has:

$$k > m + \frac{l}{2} \Leftrightarrow \frac{l}{k-m} < 2,$$

so we conclude that there is no such $d > 1$, which proves the Lemma 6.

LEMMA 7.

Let α_1 be a profile over the set of candidates $M = \{A, B, C\}$, without any Condorcet triples. Furthermore, let α_1 be a profile of a form

k	l	m
ξ(A)	ξ(C)	ξ(A)
ξ(B)	ξ(A)	ξ(C)
ξ(C)	ξ(B)	ξ(B)

,

with ξ being some permutation over the set M. Then, for every profile α obtained as a union of α_1 and some number of copies of Condorcet triples α_1^C and α_2^C it follows

$$\beta_1^d(W_{BC}) < \beta_1^d(W_{PC}),$$

for some W_{BC}, $W_{PC} \in M$, winners by Borda and plurality count respectively over the profile α, where $W_{BC}, \neq W_{PC}$, if such W_{PC} exists.

PROOF OF THE LEMMA 7.

Without loss of generality, we can drop permutation function ξ. On a given profile, plurality winner can be candidate A or C. If A is plurality winner, it follows $k + m > l$. For A to be Borda winner it should have greater Borda score than C, so $2k + l + 2m > 2l + m$, i.e. $2k + m > l$ which holds because of plurality winning condition. Therefore, if A is plurality winner, then it is also Borda winner, which proves the Lemma 7.

On the other hand, if C is plurality winner, then we have:

$$k + m < l. \quad (15)$$

For A to be Borda winner (compared to C), we have

$$2k + l + 2m < 2l + m \Rightarrow 2k + m > l. \quad (16)$$

Conditions (15) and (16) result in $m + k < l < m + 2k$. For a profile α_1 and $d > 1$ for which d-measure of divergence form the first place for a candidate C is smaller than the one for the candidate A, must hold:

$$\beta_1^d(C) < \beta_1^d(A) \Leftrightarrow m + k \cdot 2^d < l.$$

But from the condition (16) it follows that $m + k \cdot 2^d > m + 2k > l$, which leads to the conclusion that C cannot have smaller d-measure of divergence from the first place than candidate A, which proves the Lemma 7.

LEMMA 8.

Let α_1 be a profile over the set of candidates $M = \{A, B, C\}$, without any Condorcet triples. Furthermore, let α_1 be a profile of a form

k	l	m
ξ(A)	ξ(C)	ξ(B)
ξ(B)	ξ(A)	ξ(A)
ξ(C)	ξ(B)	ξ(C)

,

with ξ being some permutation over the set M. Then, for every profile α obtained as a union of α_1 and some number of copies of Condorcet triples α_1^C and α_2^C it follows

$$\beta_1^d(W_{BC}) < \beta_1^d(W_{PC}),$$

for some W_{BC}, $W_{PC} \in M$, winners by Borda and plurality count respectively over the profile α, where $W_{BC}, \neq W_{PC}$, if such W_{PC} exists.

PROOF OF THE LEMMA 8.

Without loss of generality, we will prove Lemma 8. for the profile

k	L	m
A	C	B
B	A	A
C	B	C

.

On that profile, every candidate can be a plurality winner. First, let us assume that plurality winner is candidate A. If so, then next conditions hold:

$$k > l, \tag{17}$$

$$k > m. \tag{18}$$

Let us prove that in this case, candidate A is also the Borda winner. He/she will have higher Borda score than candidate B if and only if

$$2k + l + m > k + 2m \Leftrightarrow k + l > m,$$

which holds because of condition (18). On the other hand, candidate A will have higher Borda score than candidate C if and only if

$$2k + l + m > 2l \Leftrightarrow 2k + m > l,$$

which holds because of condition (17). We conclude that in that case candidate A is both Borda and plurality winner, which proves the Lemma 8.

Secondly, let us now assume that candidate B is plurality winner. It follows:

$$m > k, \tag{19}$$

$$m > l. \tag{20}$$

Candidate B has higher Borda score than candidate C, that is, $k + 2m > l$ because of condition (18). But for candidate B does not have to have higher Borda score than candidate A. For A to have higher Borda score than B, it must hold

$$2k + l + m > \mathrm{k} + 2m \Rightarrow k + l > m. \tag{21}$$

There are integer solutions for conditions (19)-(21), for instance $k = 3$, $l = 4$ and $m = 5$ is one of such solutions. We will show, that in this case (with B being plurality, and A being Borda winner), candidate A has smaller d-measure of divergence from the first position for all $d > 1$. One has:

$$l + m < \{(21)\} < \mathrm{l} + (k + l) = k + l \cdot 2 < k + l \cdot 2^d,$$

which proves the Lemma 8.

Thirdly, let us now assume that candidate C is plurality winner. It follows:

$$l > k, \tag{22}$$

$$l > m. \tag{23}$$

Candidate C does not have to be Borda winner; both A and B can have higher Borda score. Candidate A has higher Borda score than candidate C if:

$$2k + m > l. \tag{24}$$

That is, together with conditions (22) and (23) fulfilled for $k = 2$, $m = 3$ and $l = 4$). Conversely, candidate B has higher Borda score than C if $k/2 + m > l$ (which is, together with conditions (22) and (23) fulfilled for $k = 4$, $m = 5$ and $l = 6$). Yet, candidate A will always have higher Borda score than B. It follows from $2k + l + m > k + 2m \Leftrightarrow k + l > m$, which holds because of condition (23). Therefore, between A and B, only candidate A can be Borda winner. We will prove that in this case, candidate A as a Borda winner, has smaller d-measure of divergence from the first place than candidate C for all $d > 1$, that is

$$\beta_1^d(A) < \beta_1^d(C) \Leftrightarrow l + m < (k + m) \cdot 2^d.$$

Now we have:

$$l + m < \{(24)\} < (k + m) \cdot 2 = (k + m) \cdot 2^d,$$

for all $d > 1$, which proves the Lemma.

LEMMA 9.

Let α_1 be a profile over the set of candidates $M = \{A, B, C\}$, without any Condorcet triples. Furthermore, let α_1 be a profile of a form

k	l	m	n
ξ(A)	ξ(C)	ξ(C)	ξ(A)
ξ(B)	ξ(A)	ξ(B)	ξ(C)
ξ(C)	ξ(B)	ξ(A)	ξ(B)

,

with ξ being some permutation over the set M. Then, for every profile α obtained as a union of α_1 and some number of copies of Condorcet triples α_1^C and α_2^C it follows

$$\beta_1^d(W_{BC}) < \beta_1^d(W_{PC}),$$

for some W_{BC}, $W_{PC} \in M$, winners by Borda and plurality count respectively over the profile α, where $W_{BC}, \neq W_{PC}$, if such W_{PC} exists.

PROOF OF THE LEMMA 9.

As before, without loss of generality we will prove Lemma 9. for the profile

k	l	m	n
A	C	C	A
B	A	B	C
C	B	A	B

.

In this case we have symmetry between positions of candidates A and C. Therefore, it is sufficient to analyse situation in which is candidate A plurality winner. In that case we have:

$$k + n > l + m. \tag{25}$$

Plurality winner A has higher Borda score than candidate B, because of

$$k + 2n + l > (k + n) + n + l > \{(25)\} > (l + m) + n + l = 2l + n + m > m.$$

But candidate A does not have to have higher Borda score than C. Namely, C is Borda winner if

$$2l + 2m + n > 2k + 2n + l,\ \ l + 2m > 2k + n, \tag{26}$$

which can be fulfilled (together with condition (25)) for $m > k$. For instance $k = 1$, $l = 3$, $m = 4$ and $n = 7$ meet conditions (25) and (26). In such profiles A is the plurality and C the Borda winner. We will prove that in this case, candidate C has lower d-measure of divergence from the first place than candidate A, for all $d > 1$. Let $R \in \mathbf{N}$ be a natural number such that

$$k + n = l + m + R \Leftrightarrow n = l + m - k + R.$$

Such a number exists because of condition (25). From (26) it follows

$$l + 2m > 2k + n \Leftrightarrow l + 2m > 2k + l + m - k + R,\ \ m > k + R.$$

If we compare d-measures of divergence from the first place for A and C, we have:

$$\beta_1^d(C) < \beta_1^d(A) \Leftrightarrow n + k \cdot 2^d < l + m \cdot 2^d.$$

From condition (27) one has $m > k$, and therefore

$$n < l + (m - k)\cdot 2^d \Leftrightarrow l + m - k + R < l + (m - k)\cdot 2^d \Leftrightarrow m - k + R < (m - k)\cdot 2^d \Leftrightarrow 1 + \frac{R}{m-k} < 2^d. \quad (27)$$

Since from condition (27) it follows that $R/(m - k) < 1$, Lemma 9. is proven for all $d > 1$.

LEMMA 10.

Let α_1 be a profile over the set of candidates $M = \{A, B, C\}$, without any Condorcet triples. Furthermore, let α_1 be a profile of a form

k	l	m	n
ξ(A)	ξ(C)	ξ(C)	ξ(B)
ξ(B)	ξ(A)	ξ(B)	ξ(A)
ξ(C)	ξ(B)	ξ(A)	ξ(C)

,

with ξ being some permutation over the set M. Then, for every profile α obtained as a union of α_1 and some number of copies of Condorcet triples α_1^C and α_2^C it follows

$$\beta_1^d(W_{BC}) < \beta_1^d(W_{PC}),$$

for some W_{BC}, $W_{PC} \in M$, winners by Borda and plurality count respectively over the profile α, where $W_{BC}, \neq W_{PC}$, if such W_{PC} exists.

PROOF OF THE LEMMA 10.

Without loss of generality we will prove Lemma 10. for the profile

k	l	m	n
A	C	C	B
B	A	B	A
C	B	A	C

.

Here, all three candidates can be plurality winner. Since there is symmetry between positions of candidates A and B, we will analyse two situations: one in which A is plurality winner, and other in which C is plurality winner. Furthermore, all claims proven for candidate A can be easily transformed into claims for candidate B.

First let us assume that candidate A is plurality winner. In that case we have:

$$k > l + m, \quad (28)$$
$$k > n. \quad (29)$$

Let us show that C has lower Borda score than A:

$$2k + l + n > 2l + 2m \Leftrightarrow 2k + n = l + 2m.$$

Now we have

$$2k + n > \{(28)\} > 2\cdot(l + m) + n = 2l + 2m + n > l + 2m.$$

But candidate B can have higher Borda score than plurality winner A. For that, it must hold

$$2n + k + m > 2k + l + n \Leftrightarrow n + m > k + l. \quad (30)$$

Conditions (28)-(30) can be met in the same profile. For instance $l = 2$, $m = 5$, $n = 6$ and $k = 8$ satisfy these conditions. In such profiles candidate A is plurality, and candidate B is Borda winner. We will prove that for all $d > 1$ we have $\beta_1^d(B) < \beta_1^d(A)$. With addition of conditions (29) and (30) we get

$$k + n + m > n + k + l \Leftrightarrow m > l. \tag{31}$$

Now we have

$$\beta_1^d(B) < \beta_1^d(A) \Leftrightarrow k + m + l\cdot 2^d > l + n + m\cdot 2^d.$$

If $k + m \leq l + n$, then because of (31) follows $l\cdot 2^d < m\cdot 2^d$, which proves the claim. Let us therefore assume that $k + m > l + n$. Thus, we have

$$k + m - l - n < (m - l)\cdot 2^d \Leftrightarrow \frac{k+m-l-n}{m-l} < 2^d.$$

This claim holds for all $d > 1$ if

$$\frac{k+m-l-n}{m-l} < 2 \Leftrightarrow k + m - l - n < (m - l)\cdot 2 \Leftrightarrow k + l < m + n,$$

which follows from condition (30). This proves that Borda winner B has lower d-measure of divergence from the first place for all $d > 1$ than plurality winner A (because of symmetry same claim holds for Borda winner A and plurality winner B).

Last thing to do, is to analyse profiles α_1 for which candidate C is plurality winner. In these cases we have:

$$l + m > k, \tag{32}$$

$$l + m > n. \tag{33}$$

But C does not have to be Borda winner. For A to be a Borda winner (similar analysis can be made for B because of symmetry), we should have:

$$2k + l + n > 2n + k + m \Leftrightarrow k + l > n + m, \tag{34}$$

for A to have higher Borda score than B, and

$$2k + l + n > 2l + 2m \Leftrightarrow 2k + n > l + 2m, \tag{35}$$

for A to have higher Borda score than C. Existence of such profile (with A Borda, and C plurality winner) demonstrates example of profile α_1 in which we have $k = 2$, $l = 3$, $m = 1$ and $n = 3$. These values satisfy conditions (32)-(35). We will prove that for profiles that satisfy these four conditions, for all $d > 1$ we have:

$$\beta_1^d(A) < \beta_1^d(C) \Leftrightarrow l + n + m\cdot 2^d > (k + n)\cdot 2^d.$$

Let us first show that $k + n > m$. Because of condition (35) we have $2k + n > l + 2m$. If we subtract k from both sides of that inequality (which is something we are allowed to do, because from (32) follows $k < l + m$), we have $k + n > m + (l + m - k)$. Since $l + m - k$ is a positive number (because of (32)), it follows that $k + n > m \Leftrightarrow (k + n)\cdot 2^d > m\cdot 2^d$.

Now we have

$$l + n < (k + n - m)\cdot 2^d,\ \frac{l+n}{k+n-m} < 2^d.$$

These inequalities hold for all $d > 1$ if there is

$$\frac{l+n}{k+n-m} < 2, \Leftrightarrow l + 2m < 2k + n,$$

which follows from condition (35). We conclude that, in this case also, Borda winner has lower d-measure of divergence from the first place than plurality winner, which proves the Lemma 10.

LEMMA 11.

Let α_1 be a profile over the set of candidates M = {A, B, C}, without any Condorcet triples. Furthermore, let α_1 be a profile of a form

k	l	m	n
ξ(A)	ξ(C)	ξ(A)	ξ(B)
ξ(B)	ξ(A)	ξ(C)	ξ(A)
ξ(C)	ξ(B)	ξ(B)	ξ(C)

,

with ξ being some permutation over the set M. Then, for every profile α obtained as a union of α_1 and some number of copies of Condorcet triples α_1^C and α_2^C it follows

$$\beta_1^d(W_{BC}) < \beta_1^d(W_{PC}),$$

for some W_{BC}, $W_{PC} \in M$, winners by Borda and plurality count respectively over the profile α, where $W_{BC}, \neq W_{PC}$, if such W_{PC} exists.

PROOF OF THE LEMMA 11.

Without loss of generality we will prove Lemma 11. for the profile

k	l	m	n
A	C	A	B
B	A	C	A
C	B	B	C

.

Here we also have symmetry between positions of two candidates – this time, candidates B and C. Therefore, there are two situations we must analyse: first, in which candidate A is plurality winner, and second in which candidate B is plurality winner. For A to be a plurality winner, we must have:

$$k + m > l, \tag{36}$$

$$k + m > n. \tag{37}$$

Let us prove, that in this case candidate A is also a Borda winner. For A to have higher Borda score than B, it should hold:

$$2k + 2m + l + n < 2n + k \Leftrightarrow k + 2m + l > n,$$

which follows from condition (37). On the other hand, for A to have higher Borda score than C, it should hold

$$2k + 2m + l + n < 2l + m \Leftrightarrow 2k + m + n > l,$$

which follows from condition (36). Since plurality winner A is also a Borda winner, Lemma 10. is proven for such profiles.

Second, if candidate B is plurality winner, we have;

$$n > k + m, \tag{38}$$

$$n > l. \tag{39}$$

In this case, candidate A can be Borda winner. For A to have higher Borda scores than B and C, it should hold:

$$2k + 2m + l + n > 2n + k,\ k + 2m + l > n, \tag{40}$$

$$2k + 2m + l + n > 2l + m,\ 2k + m + n > l. \tag{41}$$

Condition (41) is always fulfilled because of condition (39). Conditions (38)-(40) can be satisfied if we choose n which satisfy $\max\{l, k + m\} < n < (k + m) + m + l$. For instance, these conditions are fulfilled for $k = 2$, $l = 1$, $m = 3$ and some n which satisfies $5 < n < 9$. On such a profile the Borda winner is candidate A and plurality winner is candidate C. We will prove that for all $d > 1$ and d-measure of divergence from the first place holds:

$$\beta_1^d(A) < \beta_1^d(B) \Leftrightarrow l + n < k + (l + m)\cdot 2^d.$$

From condition (38) follows $n > k$, so we have:

$$l + n - k < (l + m)\cdot 2^d \Leftrightarrow \frac{l+n-k}{l+m} < 2^d.$$

Lemma is proven for all $d > 1$ if we have

$$\frac{l+n-k}{l+m} < 2 \Leftrightarrow n < k + l + 2m,$$

which follows from condition (40). This proves that Borda winner, candidate A has lower d-measure of divergence from the first place (for all $d > 1$) than plurality winner, candidate B.

Last thing we should consider is whether, with B being plurality winner, can Borda winner be candidate C. Answer to this question is – no. In such a scenario, Borda score of candidate C should be higher than Borda score of candidate A, so we should have:

$$2l + m > 2k + 2m + l + n \Leftrightarrow l > 2k + m + n,$$

which contradicts the condition (39). Therefore, we conclude that if B is plurality winner, only other candidate that can be Borda winner is candidate A, and in that case, we shoved that $\beta_1^d(A) < \beta_1^d(B)$, which proves the Lemma 10.

LEMMA 12.

Let $d \in \mathbf{R}$, $d > 1$. It follows

$$\frac{2^d - 2}{3^d - 2^d - 1} < 1.$$

PROOF OF THE LEMMA 12.

Let us first show that real function $f(d) = 3^d - 2^d - 1$ is a monotonically increasing for all $d > 1$. Since we have $f'(d) = 3^d \cdot \ln 3 - 2^d \cdot \ln 2$, it follows $f'(d) > 0$ if and only if $3^d \cdot \ln 3 - 2^d \cdot \ln 2 > 0$, which holds for

$$d > \frac{\ln \frac{\ln 2}{\ln 3}}{\ln \frac{3}{2}} = -1{,}1358.$$

Since $f(1) = 0$, it follows $3^d - 2^d - 1 > 0$ for all $d > 1$, so the inequality in Lemma 12. can be multiplied with this expression. Now we have:

$$\frac{2^d - 2}{3^d - 2^d - 1} < 1 \Leftrightarrow 2^d - 2 < 3^d - 2^d - 1 \Leftrightarrow 3^d - 2 \cdot 2^d + 1 > 0.$$

Let us show that $3^d - 2 \cdot 2^d + 1 > 0$ for all $d > 1$. If we analyse function $g(d) = 3^d - 2 \cdot 2^d + 1$, it follows $g'(d) > 0$ (meaning that $g(\cdot)$ increases monotonically) if and only if $3^d \cdot \ln 3 - 2 \cdot 2^d \cdot \ln 2 + 1 > 0$. Solving that inequality for d we have:

$$d > \frac{\ln \frac{2 \ln 2}{\ln 3}}{\ln \frac{3}{2}} = 0{,}5736.$$

so we conclude that $g(d)$ increases monotonically for all $d \geq 1$. Since $g(1) = 0$, the Lemma 12. follows.

ACKNOWLEDGMENTS

This work has been supported in part by Croatian Science Foundation under the project UIP-05-2017-9219 FORMALS, and by the University of Applied Sciences Velika Gorica.

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