UDC 629.5.015.5:629.544

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Coupled Horizontal and Torsional Vibrations of Container Ships

Original scientific paper

Necessity of performing hydroelasticity analysis of large container ships is pointed out. For this purpose theory of coupled horizontal and torsional vibration is described. Beam finite element for vibration analysis, with eight degrees of freedom, which includes bending, shear, torsional and warping stiffness, is developed. Coupling between horizontal and torsional vibration is realised through inertia forces. Application of the theory is illustrated in case of a large container ship.

Keywords: *container ship, horizontal and torsional vibration, finite element method*

Spregnute horizontalne i torzijske vibracije kontejnerskih brodova

Izvorni znanstveni rad

Naglašena je potreba provođenja hidroelastične analize velikih kontejnerskih brodova. Radi toga prikazana je teorija spregnutih horizontalnih i torzijskih vibracija brodova sa širokim palubnim otvorima. Za analizu vibracija razvijen je gredni konačni element s osam stupnjeva slobode gibanja, koji uključuje krutost na savijanje, smicanje, uvijanje i iskrivljavanje. Sprega horizontalnih i torzijskih vibracija ostvaruje se putem inercijskih sila. Primjena teorije ilustrirana je na primjeru velikoga kontejnerskog broda.

Ključne riječi: *kontejnerski brod, horizontalne i torzijske vibracije, metoda konačnih elemenata*

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Received (Primljeno): 2007-05-31
Accepted (Prihvaćeno): 2007-07-11
Open for discussion (Otvoreno za raspravu): 2008-12-24

1 Introduction

Increased sea transport requires building of ultra large container ships which are quite flexible [1]. Therefore, their strength has to be checked by hydroelastic analysis [2]. Methodology of hydroelastic analysis is described in [3]. It includes definition of the structural model, ship and cargo mass distributions, and geometrical model of ship wetted surface. Hydroelastic analysis is based on the modal superposition method. First, dry natural vibrations of ship hull are calculated. Then modal hydrostatic stiffness, modal added mass, modal damping and modal wave load are determined. Finally, calculation of wet natural vibrations is performed and transfer functions for determining ship structural response to wave excitation are obtained [4], [5].

This paper deals with dry natural vibrations of container ships as an important step in their hydroelastic analysis. A ship hull, as an elastic nonprismatic thin-walled girder, performs longitudinal, vertical, horizontal and torsional vibrations. Since the cross-sectional centre of gravity and deformation centre positions are not identical, coupled longitudinal and vertical, as well as horizontal and torsional vibrations occur, respectively. Thus, vibration coupling is realised through the mass only.

Distance between the centre of gravity and the deformation centre is negligible for longitudinal and vertical vibration as well as for horizontal and torsional vibration of conventional ships. Therefore, in the above cases ship hull vibrations are usually analysed separately. However, the deformation centre in ships with large hatch openings is located outside the cross-section, i.e. below the keel, and therefore the coupling of horizontal and torsional vibrations is extremely high.

The above problem is rather complicated due to geometrical discontinuity of the hull cross-section. Solution accuracy depends on the reliability of stiffness parameters determination, i.e. of bending, shear, torsional and warping moduli. The finite element method is a powerful tool to solve the above problem in a successful way.

One of the first solutions for coupled horizontal and torsional hull vibrations, dealing with the finite element technique, is given in [6] and [7]. Generalised and improved solutions are presented in [8] and [9] respectively. In all these references determination of hull stiffness is based on the classical thin-walled girder theory, which does not give a satisfactory value for the warping modulus of the open cross-section. Besides that, the initial values of stiffness moduli are determined, so that the application of the beam theory for hull vibration analysis is limited to a few lowest natural modes.

The developed higher-order theory of thin-walled girders offers the opportunity to overcome the aforementioned deficiencies [10]. Within this theory the energy approach is used to define hull stiffness parameters as mode dependent quantities. In this way more accurate results for the uncoupled girder vibration are obtained.

In this paper the beam finite element method is elaborated for coupled horizontal and torsional vibration of ship hull with large hatch openings. For this purpose a beam finite element of eight degrees of freedom is developed. Application of the theory is illustrated in case of a large container ship.

2 Differential equations of beam vibrations

Referring to the flexural beam theory [11], the total beam deflection, w, consists of the bending deflection, w_b, and the shear deflection, w_s, i.e., Figure 1

$$w = w_b + w_s . \tag{1}$$

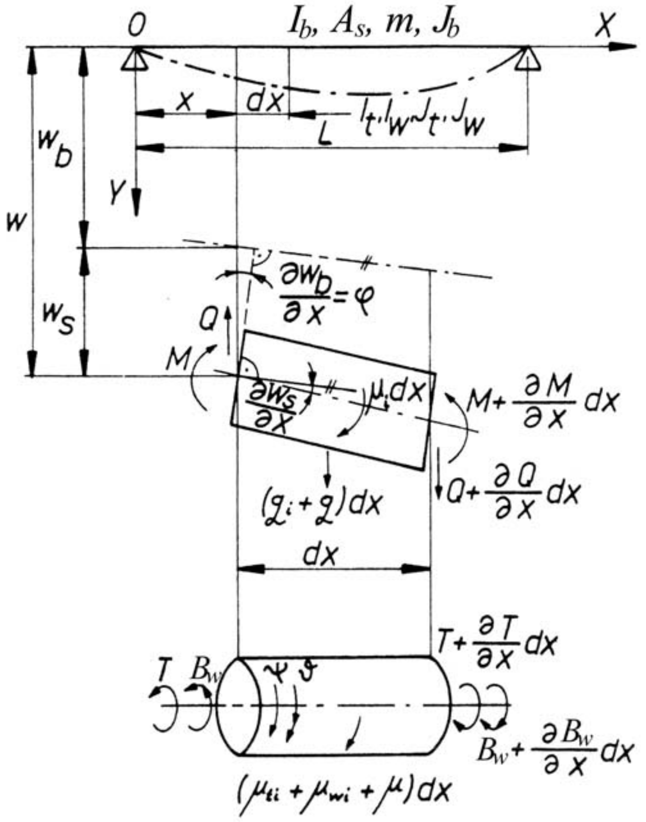


Figure 1 **Beam bending and torsion**
Slika 1 **Savijanje i uvijanje grede**

The shear deflection is a function of w_b

$$w_s = -\frac{EI_b}{GA_s}\frac{\partial^2 w_b}{\partial x^2}, \tag{2}$$

where E and G are the Young's and shear modulus respectively, while I_b and A_s are the bending modulus and shear area respectively. In (2) a term depending on mass rotation is ignored because of low influence and for the reason of simplicity. The angle of cross-section rotation also depends on w_b

$$\varphi = \frac{\partial w_b}{\partial x} . \tag{3}$$

The cross-sectional forces include the bending moment

$$M = -EI_b \frac{\partial^2 w_b}{\partial x^2} \tag{4}$$

and the shear force

$$Q = GA_s \frac{\partial w_s}{\partial x} . \tag{5}$$

The inertia load consists of the distributed load, q_i, and the bending moment, μ_i, specified as

$$q_i = -m\left(\frac{\partial^2 w}{\partial t^2} + c\frac{\partial^2 \psi}{\partial t^2}\right) \tag{6}$$

$$\mu_i = -J_b \frac{\partial^3 w_b}{\partial x \partial t^2} , \tag{7}$$

where m is the distributed ship and added mass, J_b is the moment of inertia of ship mass about the centre line, and c is the distance between the centre of gravity and the deformation centre, $c = z_G - z_D$, Figure 2. Deformation centre is a shear centre i.e. torsional centre.

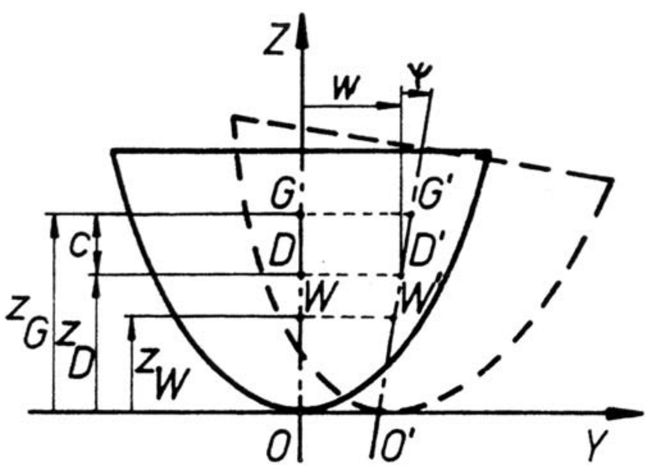


Figure 2 **Cross section of a thin-walled girder**
Slika 2 **Poprečni presjek tankostijenog nosača**

Concerning torsion, the displacements are the twist angle ψ and its variation

$$\vartheta = \frac{\partial \psi}{\partial x} \tag{8}$$

as a warping measure, Figure 1. The sectional forces include the total torque, T, consisting of the pure torsional torque, T_t, and the warping torque T_w [12] i.e.

$$T = T_t + T_w , \tag{9}$$

where

$$T_t = GI_t \frac{\partial \psi}{\partial x} \tag{10}$$

$$T_w = -EI_w \frac{\partial^3 \psi}{\partial x^3} \tag{11}$$

and the bimoment given by

$$B_w = EI_w \frac{\partial^2 \psi}{\partial x^2} \ . \tag{12}$$

The inertia load includes the distributed torque, μ_{ti}, and the bimoment, μ_{wi}, presented in the following form:

$$\mu_{ti} = -J_t \frac{\partial^2 \psi}{\partial t^2} - mc \frac{\partial^2 w}{\partial t^2} \tag{13}$$

$$\mu_{wi} = J_w \frac{\partial^4 \psi}{\partial x^2 \partial t^2} \ , \tag{14}$$

where J_t is polar moment of inertia of ship and added mass about the deformation centre and J_w is bimoment of inertia of ship mass about the warping centre, Figure 2.

Considering the equilibrium of a differential element, one may write

$$\begin{aligned} \frac{\partial M}{\partial x} &= Q + \mu_i \\ \frac{\partial Q}{\partial x} &= -q_i - q \\ \frac{\partial T}{\partial x} &= -\mu_{ti} - \mu_{wi} - \mu \ . \end{aligned} \tag{15}$$

Substituting corresponding sectional forces and distributed loads into (15) and condensing the first two equations (15), the following system of differential equations is obtained for a prismatic beam [13], [14]:

$$\begin{aligned} & EI_b \frac{\partial^4 w_b}{\partial x^4} - \left(\frac{EI_b}{GA_s} m + J_b \right) \frac{\partial^4 w_b}{\partial x^2 \partial t^2} + \\ & + m \left(\frac{\partial^2 w_b}{\partial t^2} + \frac{J_b}{GA_s} \frac{\partial^4 w_b}{\partial t^4} + c \frac{\partial^2 \psi}{\partial t^2} \right) = q \\ & EI_w \frac{\partial^4 \psi}{\partial x^4} - GI_t \frac{\partial^2 \psi}{\partial x^2} + J_t \frac{\partial^2 \psi}{\partial t^2} - J_w \frac{\partial^4 \psi}{\partial x^2 \partial t^2} + \\ & + mc \left(\frac{\partial^2 w_b}{\partial t^2} - \frac{EI_b}{GA_s} \frac{\partial^4 w_b}{\partial x^2 \partial t^2} \right) = \mu \ . \end{aligned} \tag{16}$$

Since

$$G = \frac{E}{2(1+\nu)}$$

the ratio E/G may be expressed by Poisson's ratio ν.

3 Beam finite element

The properties of a finite element for the coupled horizontal and torsional vibration analysis may be derived from the element total energy. The total energy consists of the deformation energy, the inertia energy, the work of the external lateral load, q, and the torque, μ, and the work of the boundary forces. Thus [14],

$$\begin{aligned} E_{tot} = & \frac{1}{2} \int_0^l \left[\begin{aligned} & EI_b \left(\frac{\partial^2 w_b}{\partial x^2} \right)^2 + GA_s \left(\frac{\partial w_s}{\partial x} \right)^2 + \\ & + EI_w \left(\frac{\partial^2 \psi}{\partial x^2} \right)^2 + GI_t \left(\frac{\partial \psi}{\partial x} \right)^2 \end{aligned} \right] \mathrm{d}x + \\ & + \frac{1}{2} \int_0^l \left[\begin{aligned} & m \left(\frac{\partial w}{\partial t} \right)^2 + J_b \left(\frac{\partial^2 w_b}{\partial x \partial t} \right)^2 + \\ & + 2mc \frac{\partial w}{\partial t} \frac{\partial \psi}{\partial t} + \\ & + J_w \left(\frac{\partial^2 \psi}{\partial x \partial t} \right)^2 + J_t \left(\frac{\partial \psi}{\partial t} \right)^2 \end{aligned} \right] \mathrm{d}x - \\ & - \int_0^l (qw + \mu\psi) \mathrm{d}x + (Qw - M\varphi + T\psi + B_w \vartheta)_0^l \ , \end{aligned} \tag{17}$$

where l is the element length.

Since the beam has four displacements, ω, φ, ψ, ϑ, a two-nodded finite element has eight degrees of freedom, i.e. four nodal shear-bending and torsion-warping displacements respectively, Figure 3,

$$\{U\} = \begin{Bmatrix} w(0) \\ \varphi(0) \\ w(l) \\ \varphi(l) \end{Bmatrix}, \quad \{V\} = \begin{Bmatrix} \psi(0) \\ \vartheta(0) \\ \psi(l) \\ \vartheta(l) \end{Bmatrix} \tag{18}$$

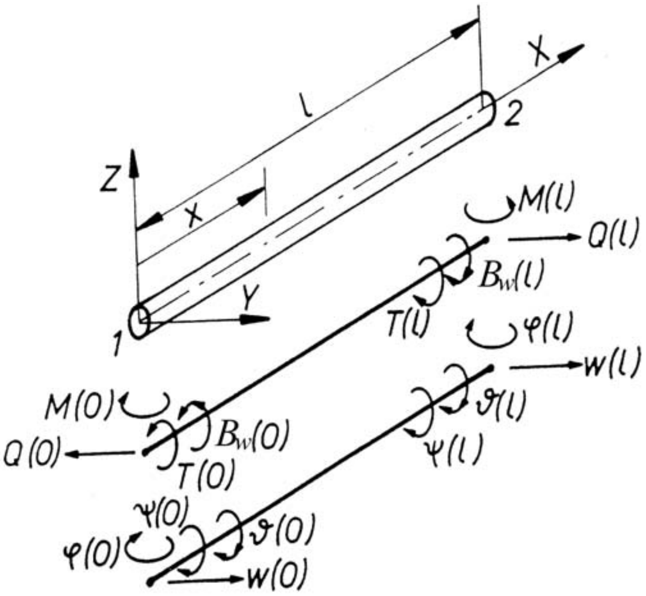


Figure 3 **Beam finite element**
Slika 3 **Gredni konačni element**

Therefore, the basic beam displacements, w_b and ψ, may be presented by the third-order polynomials

$$w_b = \langle a_k \rangle \{\xi^k\}, \quad \xi = \frac{x}{l}$$
$$\psi = \langle d_k \rangle \{\xi^k\}, \quad k = 0,1,2,3. \tag{19}$$

Furthermore, satisfying alternately the unit value for one of the nodal displacement $\{U\}$ and zero values for the remaining displacements, and doing the same for $\{V\}$, it follows that:

$$\begin{aligned} w_b &= \langle w_{bi} \rangle \{U\}, \quad i = 1,2,3,4 \\ w_s &= \langle w_{si} \rangle \{U\} \\ w &= \langle w_i \rangle \{U\}, \\ \psi &= \langle \psi_i \rangle \{V\}, \quad \langle \dots \rangle = \{ \dots \}^T, \end{aligned} \tag{20}$$

where w_{bi}, w_{si}, w_i and ψ_i are the shape functions given below

$$\begin{aligned} w_{bi} &= \langle a_{ik} \rangle \{\xi^k\} \\ w_{si} &= \langle b_{ik} \rangle \{\xi^k\} \\ w_i &= \langle c_{ik} \rangle \{\xi^k\} \\ \psi_i &= \langle d_{ik} \rangle \{\xi^k\} \end{aligned} \tag{21}$$

$$[a_{ik}] = \frac{1}{1+12\beta}\begin{bmatrix} 1+6\beta & 0 & -3 & 2 \\ -4\beta(1+3\beta)l & (1+12\beta)l & -2(1+3\beta)l & l \\ 6\beta & 0 & 3 & -2 \\ -2\beta(1-6\beta)l & 0 & -(1-6\beta)l & l \end{bmatrix} \tag{22}$$

$$[b_{ik}] = \frac{1}{1+12\beta}\begin{bmatrix} 6\beta & -12\beta & 0 & 0 \\ 4\beta(1+3\beta)l & -6\beta l & 0 & 0 \\ -6\beta & 12\beta & 0 & 0 \\ 2\beta(1-6\beta)l & -6\beta l & 0 & 0 \end{bmatrix} \tag{23}$$

$$[c_{ik}] = [a_{ik}] + [b_{ik}], \quad \beta = \frac{EI_b}{GA_s l^2} \tag{24}$$

$$[d_{ik}] = \begin{bmatrix} 1 & 0 & -3 & 2 \\ 0 & l & -2l & l \\ 0 & 0 & 3 & -2 \\ 0 & 0 & -l & l \end{bmatrix} \tag{25}$$

Substituting Eqs. (20) into (17) one obtains

$$E_{tot} = \frac{1}{2}\begin{Bmatrix} U \\ V \end{Bmatrix}^T \begin{bmatrix} k_{bs} & 0 \\ 0 & k_{wt} \end{bmatrix} \begin{Bmatrix} U \\ V \end{Bmatrix} + \frac{1}{2}\begin{Bmatrix} \dot{U} \\ \dot{V} \end{Bmatrix}^T \begin{bmatrix} m_{sb} & m_{st} \\ m_{ts} & m_{tw} \end{bmatrix} \bullet$$
$$\bullet \begin{Bmatrix} \dot{U} \\ \dot{V} \end{Bmatrix} - \begin{Bmatrix} q \\ \mu \end{Bmatrix}^T \begin{Bmatrix} U \\ V \end{Bmatrix} - \begin{Bmatrix} P \\ R \end{Bmatrix}^T \begin{Bmatrix} U \\ V \end{Bmatrix}, \tag{26}$$

where, assuming constant values of the element properties,

$$[k]_{bs} = \left[EI_b \int_0^l \frac{\mathrm{d}^2 w_{bi}}{\mathrm{d}x^2} \frac{\mathrm{d}^2 w_{bj}}{\mathrm{d}x^2} \mathrm{d}x + GA_s \int_0^l \frac{\mathrm{d}w_{si}}{\mathrm{d}x} \frac{\mathrm{d}w_{sj}}{\mathrm{d}x} \mathrm{d}x \right]$$

– bending-shear stiffness matrix,

$$[k]_{wt} = \left[EI_w \int_0^l \frac{\mathrm{d}^2 \psi_i}{\mathrm{d}x^2} \frac{\mathrm{d}^2 \psi_j}{\mathrm{d}x^2} \mathrm{d}x + GI_t \int_0^l \frac{\mathrm{d}\psi_i}{\mathrm{d}x} \frac{\mathrm{d}\psi_j}{\mathrm{d}x} \mathrm{d}x \right]$$

– warping-torsion stiffness matrix,

$$[m]_{sb} = \left[m \int_0^l w_i w_j \mathrm{d}x + J_b \int_0^l \frac{\mathrm{d}w_{bi}}{\mathrm{d}x} \frac{\mathrm{d}w_{bj}}{\mathrm{d}x} \mathrm{d}x \right]$$

– shear-bending mass matrix,

$$[m]_{tw} = \left[J_t \int_0^l \psi_i \psi_j \mathrm{d}x + J_w \int_0^l \frac{\mathrm{d}\psi_i}{\mathrm{d}x} \frac{\mathrm{d}\psi_j}{\mathrm{d}x} \mathrm{d}x \right]$$

– torsion-warping mass matrix,

$$[m]_{st} = \left[mc \int_0^l w_i \psi_j \mathrm{d}x \right], \quad [m]_{ts} = [m]_{st}^T$$

– shear-torsion mass matrix,

$$\begin{aligned} \{q\} &= \left\{ \int_0^l q\, w_j \,\mathrm{d}x \right\} \text{ – shear load vector,} \\ \{\mu\} &= \left\{ \int_0^l \mu\, \psi_j \,\mathrm{d}x \right\} \text{ – torsion load vector,} \\ i, j &= 1,2,3,4\,. \end{aligned} \tag{27}$$

The vectors $\{P\}$ and $\{R\}$ represent the shear-bending and torsion-warping nodal forces, respectively,

$$\{P\} = \begin{Bmatrix} -Q(0) \\ M(0) \\ Q(l) \\ -M(l) \end{Bmatrix}, \quad \{R\} = \begin{Bmatrix} -T(0) \\ -B_w(0) \\ T(l) \\ B_w(l) \end{Bmatrix}. \tag{28}$$

The above matrices are specified in Appendix A, as well as the load vectors for loads linearly distributed along the element, i.e.

$$q = q_0 + q_1\xi, \quad \mu = \mu_0 + \mu_1\xi\,. \tag{29}$$

Also, shape functions of sectional forces are listed in Appendix B.

The total element energy has to be at its minimum. Satisfying the relevant conditions

$$\frac{\partial E_{tot}}{\partial \{U\}} = \{0\}, \quad \frac{\partial E_{tot}}{\partial \{V\}} = \{0\}, \tag{30}$$

and employing Lagrange equations of motion, the finite element equation yields

$$\{f\} = [k]\{\delta\} + [m]\{\ddot{\delta}\} - \{f\}_{q\mu} \tag{31}$$

where

$$\begin{aligned} &\{f\} = \begin{Bmatrix} P \\ R \end{Bmatrix}, \quad \{f\}_{q\mu} = \begin{Bmatrix} q \\ \mu \end{Bmatrix}, \quad \{\delta\} = \begin{Bmatrix} U \\ V \end{Bmatrix} \\ &[k] = \begin{bmatrix} k_{bs} & 0 \\ 0 & k_{wt} \end{bmatrix}, \quad [m] = \begin{bmatrix} m_{sb} & m_{st} \\ m_{ts} & m_{tw} \end{bmatrix}. \end{aligned} \tag{32}$$

It is obvious that coupling between the bending and torsion occurs through the mass matrix only, by the coupling matrices $[m]_{st}$ and $[m]_{ts}$.

4 Finite element transformation

In the finite element equation (31) first the element properties related to bending and then those related to torsion appear. To make a standard finite element assembling possible, it is necessary to transform Eq. (31) in such a way that first all properties related to the first node and then those belonging to the second one are specified. Thus, the nodal force and displacement vectors read

$$\{\tilde{f}\} = \begin{Bmatrix} -Q(0) \\ M(0) \\ -T(0) \\ -B_w(0) \\ Q(l) \\ -M(l) \\ T(l) \\ B_w(l) \end{Bmatrix}, \quad \{\tilde{\delta}\} = \begin{Bmatrix} w(0) \\ \varphi(0) \\ \psi(0) \\ \vartheta(0) \\ w(l) \\ \varphi(l) \\ \psi(l) \\ \vartheta(l) \end{Bmatrix}. \tag{33}$$

The same transformation has to be done for the load vector $\{f\}_{q\mu}$ resulting in $\{\tilde{f}\}_{q\mu}$. The above vector transformation implies the row and column exchange in the stiffness and mass matrix according to the following set form:

	1	2	5	6	3	4	7	8
1	11	12	15	16	13	14	17	18
2	21	22	25	26	23	24	27	28
5	51	52	55	56	53	54	57	58
6	61	62	65	66	63	64	67	68
3	31	32	35	36	33	34	37	38
4	41	42	45	46	43	44	47	48
7	71	72	75	76	73	74	77	78
8	81	82	85	86	83	84	87	88

(34)

The element deflection refers to the deformation centre as the origin of a local coordinate system. Since the vertical position of the deformation centre varies along the ship's hull, it is necessary to prescribe the element deflection for a common line, in order to be able to assemble the elements. Thus, choosing the x-axis (base line) of the global coordinate system as the referent line, the following relation between the former and the latter nodal deflections exists:

$$\begin{aligned} w(0) &= \overline{w}(0) + z_D \psi(0) \\ w(l) &= \overline{w}(l) + z_D \psi(l), \end{aligned} \tag{35}$$

where z_D is the coordinate of the deformation centre, Figure 2. The other displacements are the same in both coordinate systems. Twist angle ψ does not have influence on the cross-section rotation angle φ. The local displacement vector may be expressed as

$$\{\tilde{\delta}\}=[\tilde{T}]\{\bar{\tilde{\delta}}\} \tag{36}$$

where $[\tilde{T}]$ is the transformation matrix

$$[\tilde{T}]=\begin{bmatrix}[T] & [0]\\ [0] & [T]\end{bmatrix},\quad [T]=\begin{bmatrix}1 & 0 & z_D & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\end{bmatrix} \tag{37}$$

Since the total element energy is not changed by the above transformations, a new element equation may be derived taking (36) into account. Thus, one obtains in the global coordinate system

$$\{\bar{\tilde{f}}\}=[\bar{\tilde{k}}]\{\bar{\tilde{\delta}}\}+[\bar{\tilde{m}}]\{\ddot{\bar{\tilde{\delta}}}\}-\{\bar{\tilde{f}}\}_{q\mu}\,, \tag{38}$$

where

$$\begin{aligned}\{\bar{\tilde{f}}\}&=[\tilde{T}]^T\{\tilde{f}\}\\ [\bar{\tilde{k}}]&=[\tilde{T}]^T[\tilde{k}][\tilde{T}]\\ [\bar{\tilde{m}}]&=[\tilde{T}]^T[\tilde{m}][\tilde{T}]\\ \{\bar{\tilde{f}}\}_{q\mu}&=[\tilde{T}]^T\{\tilde{f}\}_{q\mu}\,.\end{aligned} \tag{39}$$

The first of the above expressions transforms the nodal torques into the form

$$\begin{aligned}-\bar{T}(0)&=-T(0)-z_DQ(0)\\ \bar{T}(l)&=T(l)+z_DQ(l)\,.\end{aligned} \tag{40}$$

5 Vibration analysis

A ship's hull is modelled by a set of beam finite elements. Their assemblage in the global coordinate system, performed in the standard way, results in the matrix equation of motion, which may be extended by the damping forces

$$[K]\{\Delta\}+[C]\{\dot{\Delta}\}+[M]\{\ddot{\Delta}\}=\{F(t)\}, \tag{41}$$

where $[K]$, $[C]$ and $[M]$ are the stiffness, damping and mass matrices, respectively; $\{\Delta\}$, $\{\dot{\Delta}\}$ and $\{\ddot{\Delta}\}$ are the displacement, velocity and acceleration vectors, respectively; and $\{F(t)\}$ is the load vector.

In case of natural vibration $\{F(t)\} = \{0\}$ and the influence of damping is rather low for ship structures so that the damping forces may be ignored. Assuming

$$\{\Delta\}=\{\phi\}e^{i\omega t} \tag{42}$$

where $\{\phi\}$ and ω are the mode vector and natural frequency respectively, Eq. (41) leads to the eigenvalue problem

$$\left([K]-\omega^2[M]\right)\{\phi\}=\{0\} \tag{43}$$

which may be solved employing different numerical methods [15]. The basic one is the determinant search method in which ω is found from the condition

$$\left|[K]-\omega^2[M]\right|=0 \tag{44}$$

by an iteration procedure. Afterwards, $\{\phi\}$ follows from (43) assuming unit value for one element in $\{\phi\}$.

The forced vibration analysis may be performed by direct integration of Eq. (41) or as well as by the modal superposition method. In the latter case the displacement vector is presented in the form

$$\{\Delta\}=[\phi]\{X\}, \tag{45}$$

where $[\phi] = [\{\phi\}]$ is the undamped mode matrix and $\{X\}$ is the generalised displacement vector. Substituting (45) into (41), the modal equation yields

$$[k]\{X\}+[c]\{\dot{X}\}+[m]\{\ddot{X}\}=\{f(t)\} \tag{46}$$

where

$$\begin{aligned}&[k]=[\phi]^T[K][\phi]\text{ - modal stiffness matrix}\\ &[c]=[\phi]^T[C][\phi]\text{ - modal damping matrix}\\ &[m]=[\phi]^T[M][\phi]\text{ - modal mass matrix}\\ &\{f(t)\}=[\phi]^T\{F(t)\}\text{ - modal load vector.}\end{aligned} \tag{47}$$

The matrices $[k]$ and $[m]$ are diagonal, while $[c]$ becomes diagonal only in a special case, for instance if $[C] = \alpha[M] + \beta[K]$, where α and β are coefficients [14].

Solving (46) for undamped natural vibration $[k] = [\omega^2 m]$ is obtained, and by its backward substitution into (46) the final form of the modal equation yields

$$[\omega^2]\{X\}+2[\omega][\zeta]\{\dot{X}\}+\{\ddot{X}\}=\{\varphi(t)\} \tag{48}$$

where

$$\begin{aligned}&[\omega]=\left[\sqrt{\frac{k_{ii}}{m_{ii}}}\right]\text{ - natural frequency matrix}\\ &[\zeta]=\left[\frac{c_{ij}}{2\sqrt{(k_{ii}m_{ii})}}\right]\text{ - relative damping matrix}\\ &\{\varphi(t)\}=\left\{\frac{f_i(t)}{m_{ii}}\right\}\text{ - relative load vector.}\end{aligned} \tag{49}$$

If $\{\zeta\}$ is diagonal, the matrix Eq. (48) is split into a set of uncoupled modal equations.

The ship vibration is caused by the engine and propeller excitation forces, which are of the periodical nature and therefore may be divided into harmonics. Thus, the ship's hull response is obtained solving either (41) or (46). In both cases the system of differential equations is transformed into a system of algebraic equations.

If hull vibration is induced by waves, the time integration of (41) or (46) can be performed. A few numerical methods are available for this purpose, as for instance the Houbolt, the Newmark and the Wilson θ method [15], as well as the harmonic acceleration method [16], [17].

6 Natural vibrations of container ship

The application of developed theory is illustrated in case of a 7800 TEU VLCS (Very Large Container Ship), Figure 4. The main vessel particulars are the following:

Length over all	L_{oa} = 334 m
Length between perpendiculars	L_{pp} = 319 m
Breadth	B = 42.8 m
Depth	H = 24.6 m
Draught	T = 14.5 m
Displacement, full load	Δ_f = 135336 t
Displacement, ballast	Δ_b = 68387 t
Engine power	P = 69620 kW
Ship speed	v = 25.4 kn

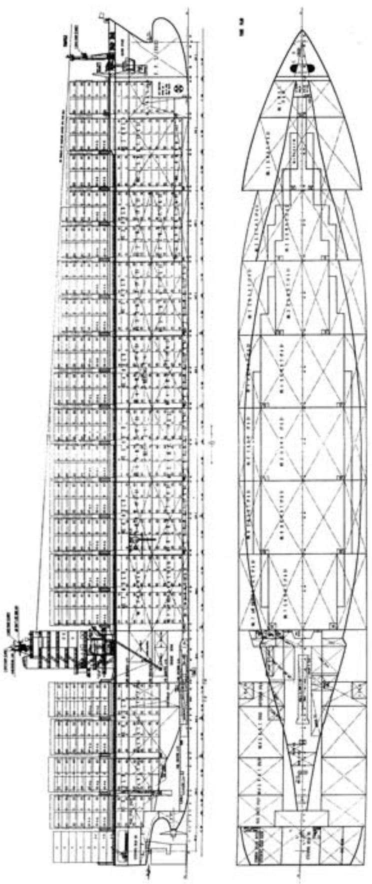


Figure 4 **7800 TEU container ship**
Slika 4 **Kontejnerski brod nosivosti 7800 TEU**

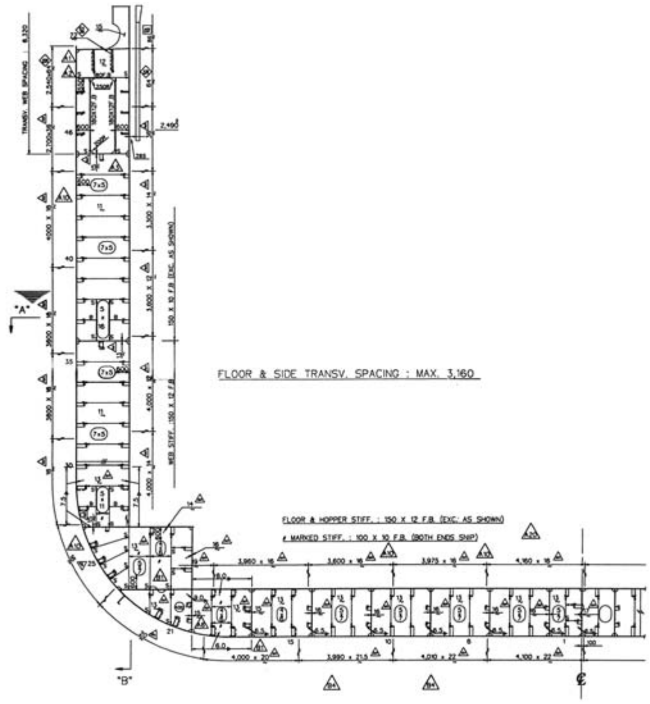


Figure 5 **Midship section**
Slika 5 **Glavno rebro**

The midship section, which shows a double skin structure with the web frames and longitudinals, is presented in Figure 5. Rows and tiers of containers at the midship section are indicated in Figure 6. Vertical positions of the neutral line, deformation (shear, torsional) centre, and the centre of gravity are also marked in the figure. A large distance between the gravity centre and the deformation centre causes high coupling of horizontal and torsional vibrations.

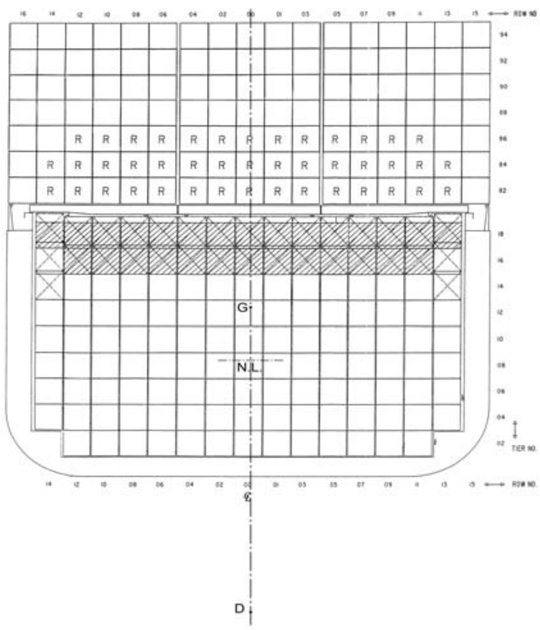
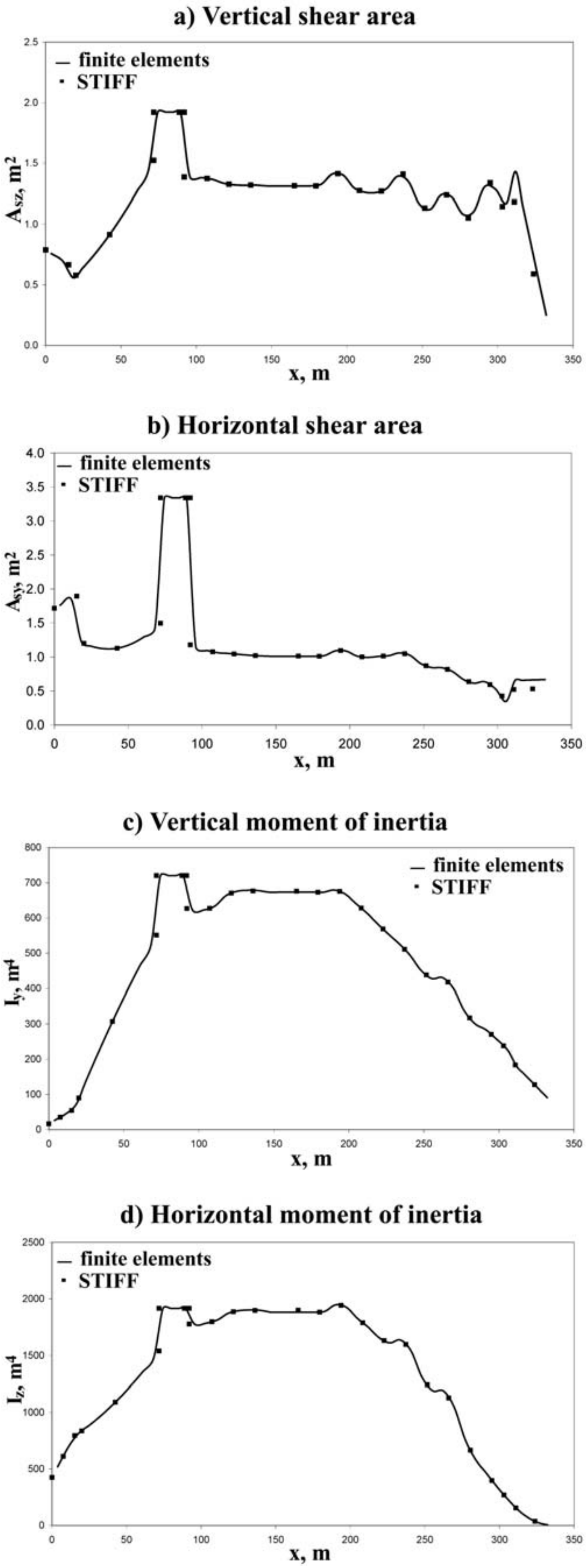


Figure 6 **Container distribution at midship section**
Slika 6 **Raspored kontejnera na glavnom rebru**



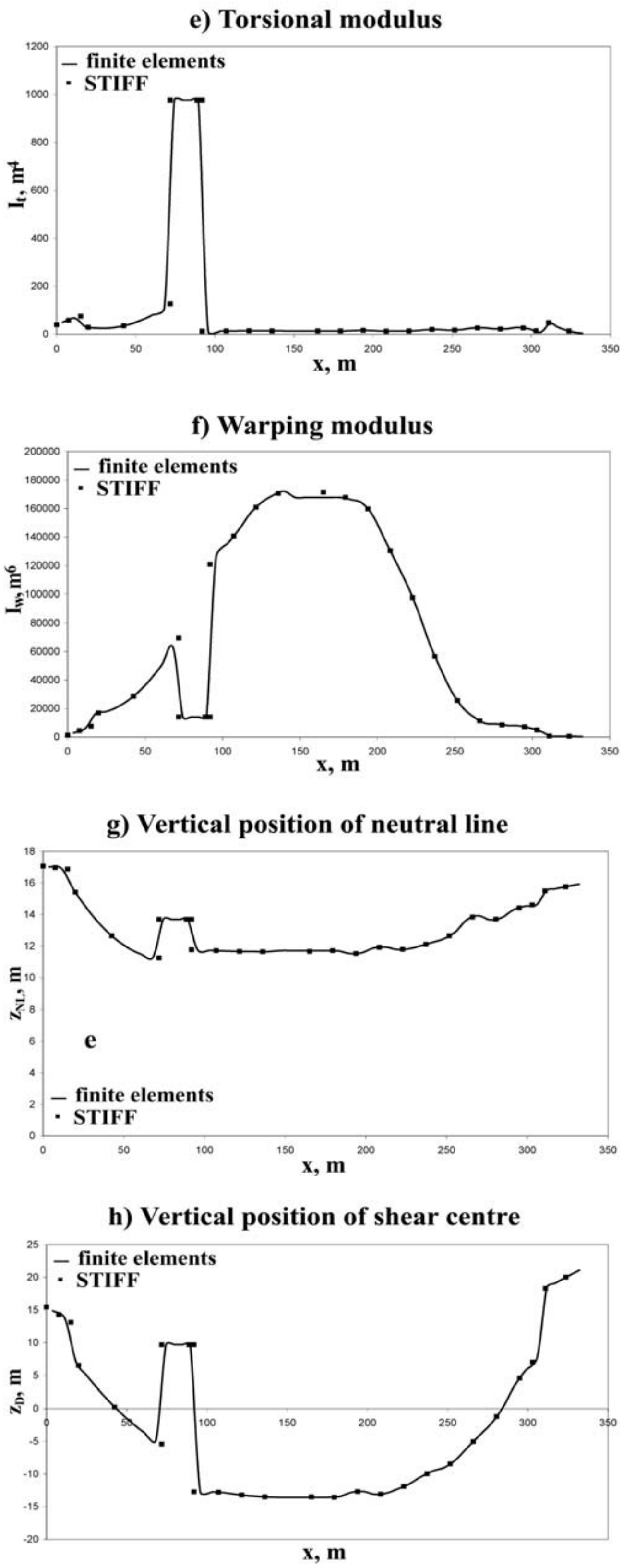
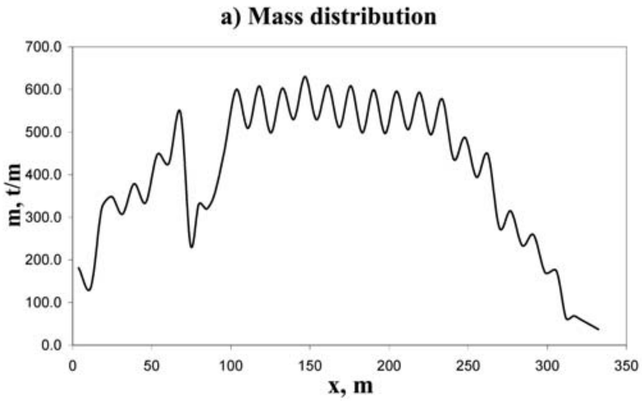
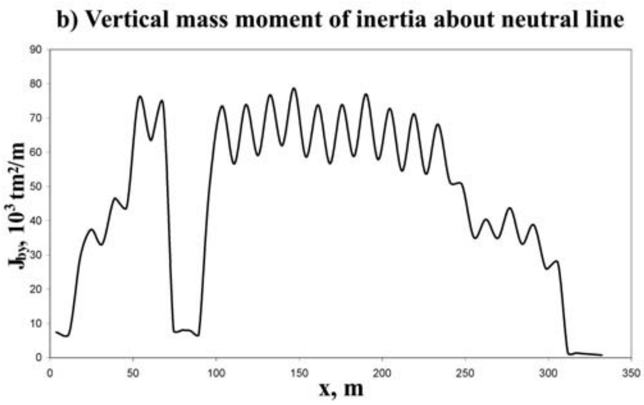
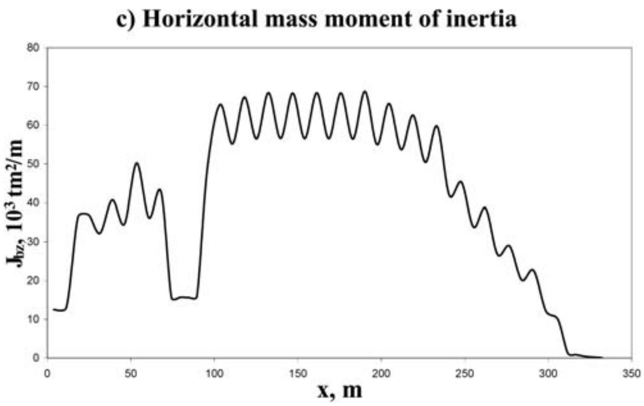


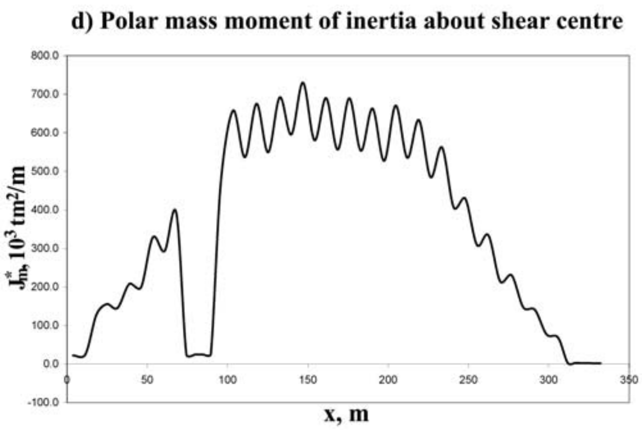
Figure 7 **Longitudinal distribution of ship cross-section geometrical properties**

Slika 7 **Uzdužna raspodjela geometrijskih značajki poprečnog presjeka brodskog trupa**









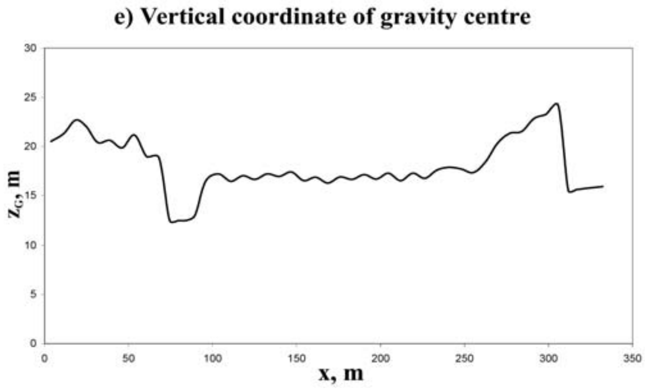
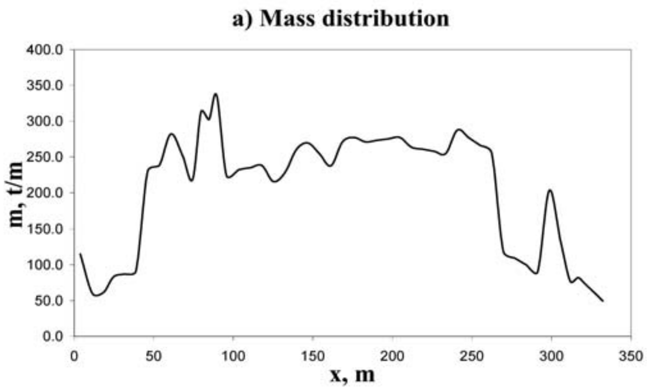
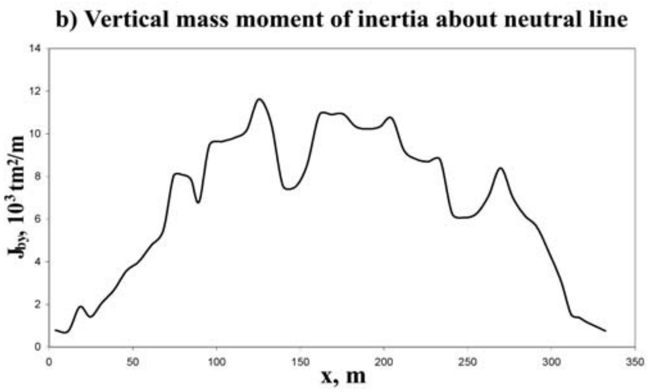
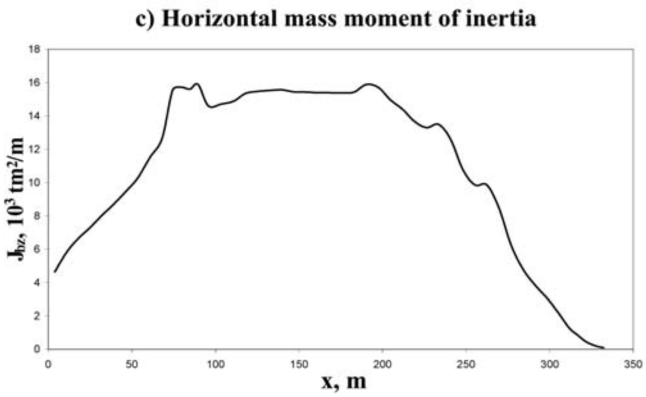


Figure 8 **Longitudinal distribution of ship mass properties, full load**

Slika 8 **Uzdužna raspodjela masenih značajki potpuno nakrcanog broda**







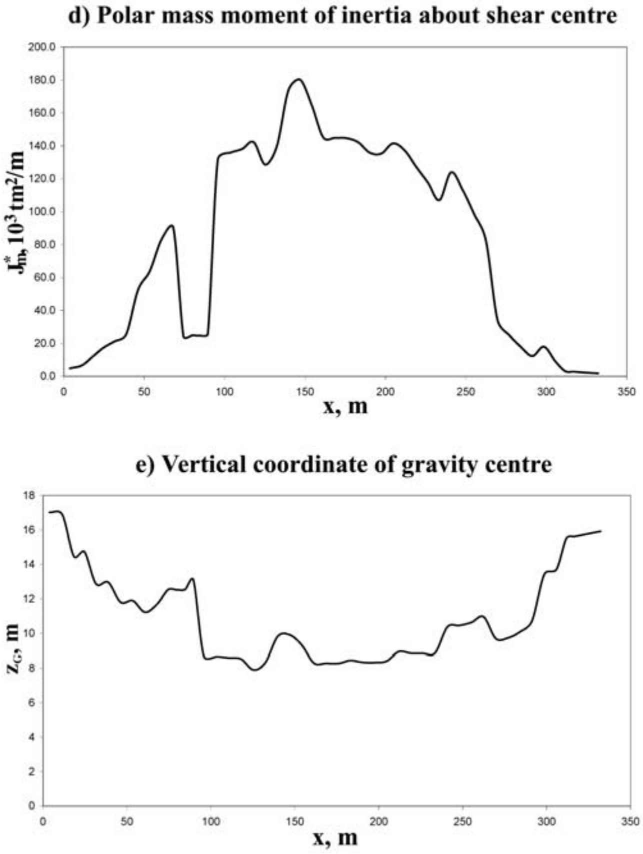


Figure 9 **Longitudinal distribution of ship mass properties, ballast condition**
Slika 9 **Uzdužna raspodjela masenih značajki broda u balastu**

The ship hull stiffness properties are calculated by program STIFF [18], based on the theory of thin-walled girders [10], [19]. Their distributions along the ship are shown in Figure 7. It is evident that geometrical properties rapidly change values in the engine and superstructure area due to closed ship cross-section. This is especially pronounced in the case of torsional modulus, which takes quite small values for open cross-section and rather high for closed one, Figure 7e.

Longitudinal distributions of ship mass properties for full load and ballast conditions are shown in Figures 8 and 9 respectively. The harmonic variation of diagrams in Figure 8 is a result of container arrangement.

Dry natural vibrations are calculated by the modified and improved program DYANA within [4]. The ship hull is divided into 50 beam finite elements. Finite elements of closed cross-section (6 d.o.f.) are used in the ship bow, ship aft and in the engine room area. Dry natural frequencies for vertical vibrations, and coupled horizontal and torsional vibrations for full load and ballast conditions are listed in Table 1. Their values for ballast condition are higher due to the lower mass. The lowest frequency value, which plays the main role in wave excitation, is detected for coupled vibrations. As a result of rather low torsional stiffness it belongs to primarily torsional mode.

The first two dry natural modes, i.e. total deflection w, of vertical vibrations for full load are shown in Figure 10. They are of ordinary shape. The corresponding diagrams of vertical bending moment M_y and shear force Q_z are presented in Figures 11 and 12 respectively.

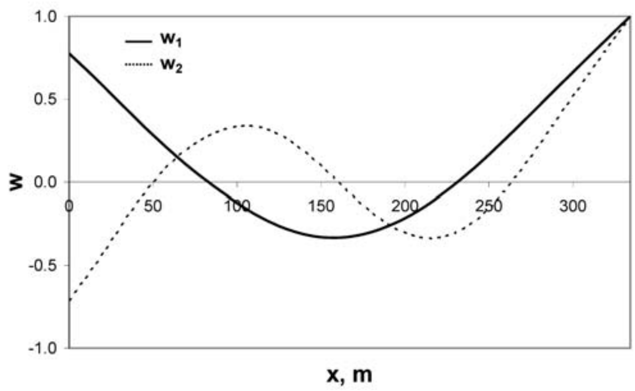


Figure 10 **Natural modes of vertical vibrations**
Slika 10 **Prirodni oblici vertikalnih vibracija**

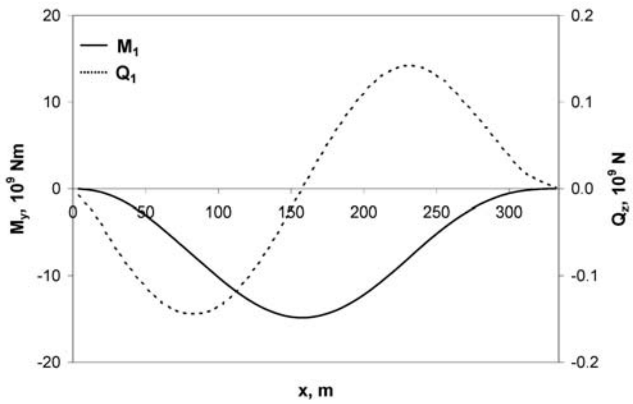


Figure 11 **Vertical bending moment and shear force of the first natural mode**
Slika 11 **Vertikalni moment savijanja i poprečna sila prvog oblika vibriranja**

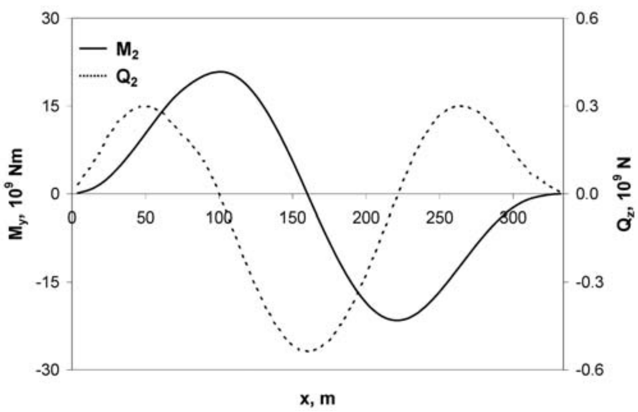


Figure 12 **Vertical bending moment and shear force of the second natural mode**
Slika 12 **Vertikalni moment savijanja i poprečna sila drugog oblika vibriranja**

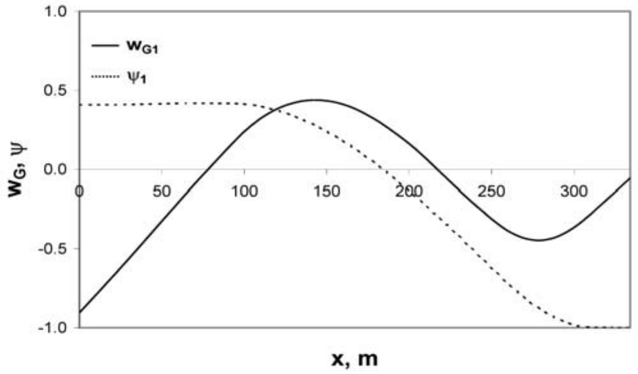


Figure 13 **The first natural modes of horizontal and torsional vibrations**
Slika 13 **Prvi prirodni oblici horizontalnih i torzijskih vibracija**

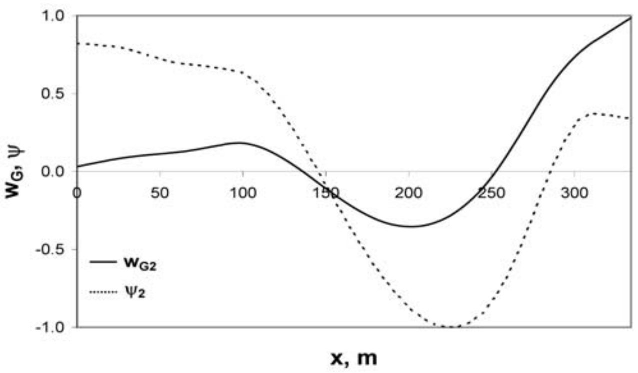


Figure 14 **The second natural modes of horizontal and torsional vibrations**
Slika 14 **Drugi prirodni oblici horizontalnih i torzijskih vibracija**

The first two natural modes of coupled horizontal and torsional vibrations for full load are shown in Figures 13 and 14 respectively. Symbol w_G denotes horizontal hull deflection at the level of gravity centre, while ψ denotes torsional angle. The corresponding sectional forces, i.e. horizontal bending moment M_z, shear force Q_y, torque T and warping bimoment B_w, are shown in Figures 15, 16, 17 and 18 respectively. Torque T is determined with respect to the shear centre of corresponding finite element. T and B_w show some discontinuity in the area of the engine room due to different type of the hull cross-sections.

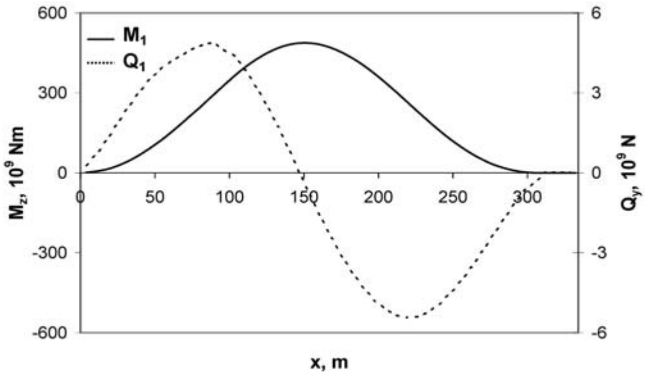


Figure 15 **Horizontal bending moment and shear force of the first natural mode**
Slika 15 **Horizontalni moment savijanja i poprečna sila prvog oblika vibriranja**

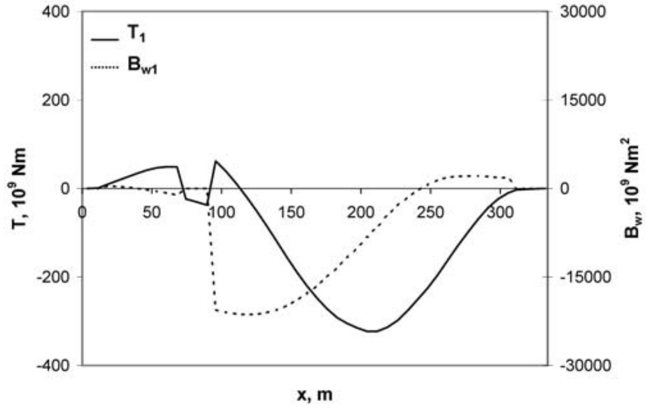


Figure 16 **Torque and warping bimoment of the first natural mode**
Slika 16 **Moment uvijanja i bimoment vitoperenja prvog prirodnog oblika**

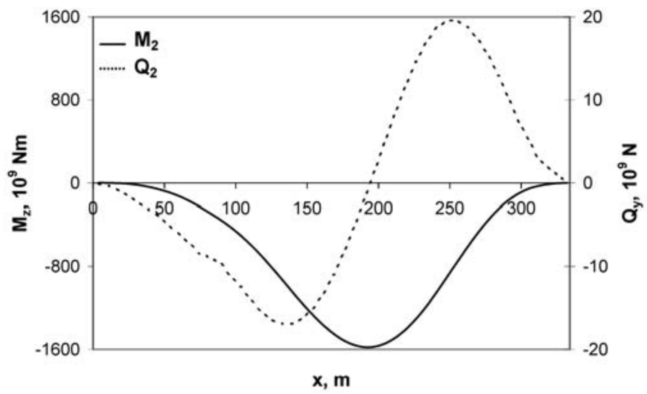


Figure 17 **Horizontal bending moment and shear force of the second natural mode**
Slika 17 **Horizontalni moment savijanja i poprečna sila drugog oblika vibriranja**

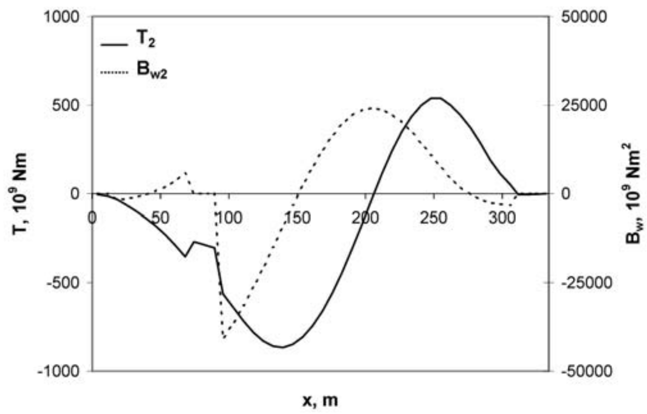


Figure 18 **Torque and warping bimoment of the second natural mode**
Slika 18 **Moment uvijanja i bimoment vitoperenja drugog prirodnog oblika**

Once the dry natural modes of ship hull are determined, it is possible to transfer the beam node displacements to the ship wetted surface for the hydrodynamic calculation. The transformation (spreading) functions for vertical and coupled horizontal and torsional vibration yield respectively [3]

$$\mathbf{h}_v = -\frac{\mathrm{d}\,w_v}{\mathrm{d}\,x}(Z - z_N)\,\mathbf{i} + w_v\mathbf{k} \qquad (50)$$

$$\mathbf{h}_{ht} = \left(-\frac{\mathrm{d}\,w_h}{\mathrm{d}\,x}Y + \frac{\mathrm{d}\,\psi}{\mathrm{d}\,x}\overline{u}\right)\mathbf{i} + \left[w_h + \psi(Z - z_S)\right]\mathbf{j} - \psi\,Y\,\mathbf{k} \qquad (51)$$

where w is hull deflection, ψ is twist angle, $\overline{u} = \overline{u}(x,Y,Z)$ is the cross-section warping function reduced to the wetted surface, z_N and z_S coordinate of neutral line and shear centre respectively, and Y and Z are coordinates of the point on the ship surface. The first two dry natural modes of the ship wetted surface in case of vertical and coupled horizontal and torsional vibrations are shown in Figures 19, 20, 21 and 22.

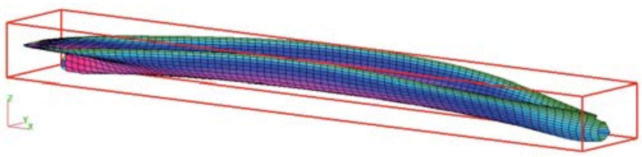


Figure 19 **The first natural mode of vertical vibrations, ω_1 = 4 rad/s**
Slika 19 **Prvi prirodni oblik vertikalnih vibracija, ω_1 = 4 rad/s**

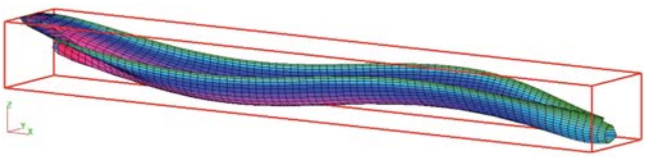


Figure 20 **The second natural mode of vertical vibrations, ω_2 = 8.41 rad/s**
Slika 20 **Drugi prirodni oblik vertikalnih vibracija, ω_2 = 8,41 rad/s**

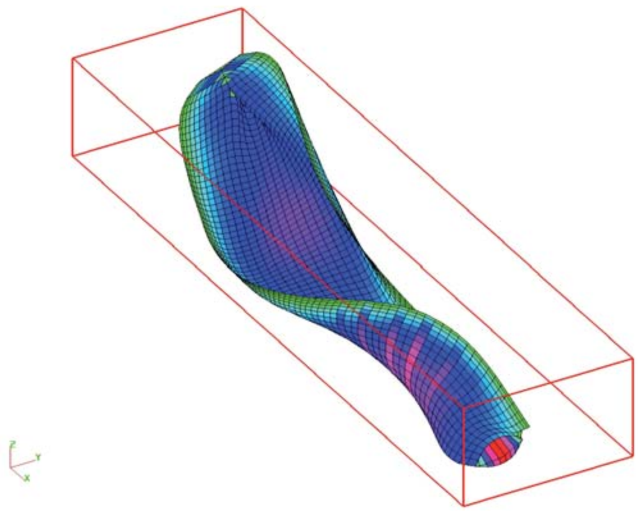


Figure 21 **The first natural mode of coupled horizontal and torsional vibrations, ω_1 = 2.18 rad/s**
Slika 21 **Prvi prirodni oblik spregnutih horizontalnih i torzijskih vibracija, ω_1 = 2,18 rad/s**

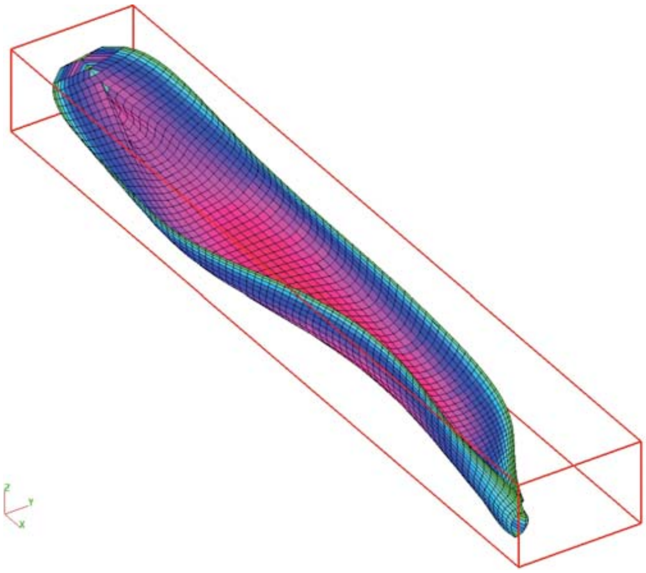


Figure 22 **The second natural mode of coupled horizontal and torsional vibrations, ω_2 = 4.23 rad/s**
Slika 22 **Drugi prirodni oblik spregnutih horizontalnih i torzijskih vibracija, ω_2 = 4,23 rad/s**

Table 1 **Dry natural frequencies, ω_i [rad/s]**
Tablica 1 **Prirodne frekvencije za trup u zraku, ω_i [rad/s]**

Mode no.	Full load		Ballast	
	Vertical	Horizontal + torsional	Vertical	Horizontal + torsional
1	4.00	2.18	5.59	3.98
2	8.41	4.23	11.60	6.79
3	13.22	7.08	17.85	11.55
4	18.07	9.23	25.02	14.90
5	23.04	13.19	32.52	20.59
6	28.09	15.37	39.21	23.03
7	32.77	18.22	46.29	27.98
8	37.22	22.65	53.25	32.64
9	41.73	23.75	60.40	36.33
10	42.27	28.38	66.96	40.38

7 Conclusion

Ultra large container ships are quite flexible and present classification rules cannot be used for reliable structure design. Therefore, hydroelastic analysis has to be performed. The methodology of hydroelastic analysis is elaborated in [3]. One of the basic steps is dry natural vibration analysis of ship hull. Vertical vibration calculation is performed in a standard way, while the coupled horizontal and torsional vibrations are rather complex as it is evident from this article.

The theory of coupled horizontal and torsional vibrations of ships with large deck openings is checked by analytical vibration solution for a prismatic pontoon [20], [21]. Furthermore, the implementation of the structural model in hydroelastic analysis is checked for the case of a flexible barge [3], [4], for which test results are available [22]. The influence of transverse bulkheads of container ships on hull stiffness is investigated in [23], where an increase of torsional stiffness of 127% is determined, while the influence on bending stiffness is almost negligible. Repercussion on torsional natural frequencies is ca 6%.

The reliability of the 1D FEM model and the software code is confirmed by 3D FEM analysis of dry vibrations for ship lightweight with displacement of 33 692 t. Natural frequencies of the basic modes are compared in Table 2 [5].

Table 2 **Natural frequencies of dry vibrations, ω_i [rad/s]** (*T* – torsion, *HB* – horizontal bending)
Tablica 2 **Prirodne frekvencije za trup u zraku, ω_i [rad/s]** (*T* – uvijanje, *HB* – horizontalno savijanje)

Mode No.	1 (*T*)	2 (*T+HB*)
1D FEM model	5.39	9.23
3D FEM model	5.41	9.42

The performed vibration analysis of the 7800 TEU container ship shows that the developed theory, utilising the 1D FEM model and the theory of thin-walled girders for determination of hull stiffness parameters, is an efficient tool for application in ship hydroelastic analyses.

Acknowledgment

The authors would like to express their gratitude to Ms. Estelle Stumpf, research engineer at Bureau Veritas, Marine Division - Research Department, for performing the 3D FEM vibration analysis of container ship and king permission to publish the valuable results.

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