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ISSN 0469-6255
(1-13)

DIFFERENT WAYS OF TRANS-OCEANIC PASSAGES

RAZLIČITI NAČINI PREKOOCEANSKIH PUTOVANJA

UDK 656.61.052

Review
Pregledni članak

1. Introduction
Uvod

The determination of sailing distance in navigation represents very important problem which can be solved with the knowledge of the shape and dimension of the Earth. Mariners crossing the ocean used the wind, resulting in today's navigational courses being considerably different from navigational courses of sailing vessels. Namely, only after use of independent drive it was possible to use theoretical directions defined minimal distance as well.

In this paper it is shown a different ways of trans-oceanic passages and some of them are very convenient for crossing the ocean.

2. Analyses
Analize

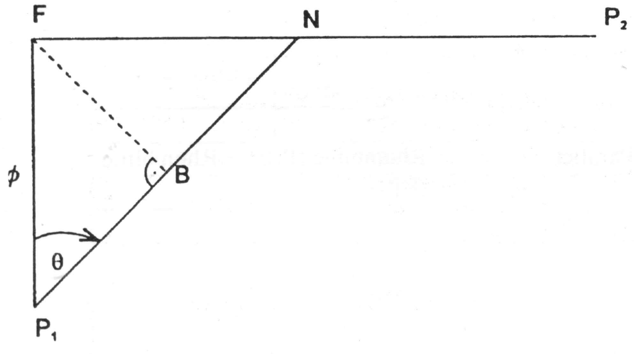

Figure 1. Illustration of Trans-oceanic passage by Rhumbline sailing

Form tirangle P_1BF (Fig. 1)

$$FB = M \cdot cos\phi_2$$

M is difference in meridional parts between position 1 and 2.

$$P_1B = \sqrt{(P_1F)-(FB)^2} = \sqrt{\delta^2\phi - (M \cdot cos\phi_2)^2}$$

(Journal of Navigation, May 1990)

Equation for the course is

$$sin\,\theta = \frac{F\ B}{P_1 F} = \frac{M \cdot cos\phi_2}{\delta\phi}$$

Departure = $\delta\phi \cdot \cos\phi_2$

For the Earth as a sphere $\cos\phi_{(40°)} = 0.70710678$

For the Bessel's elipsoid $\cos\phi_{(40°)} = 0.76839091$

$$cos\phi = \frac{6377397.155 \cdot cos 40}{\sqrt{1-0.08169683121^2 \cdot sin^2 40}} \div (1852 \cdot 3437.746771)$$

cos ϕ for an allipsoid can be shown in a simple table for each minute form 0° to 90° (Tijardović, I.: "Prekooceanske plovidbe - priručnik namijenjen zapovjednicima, Split, 1997).

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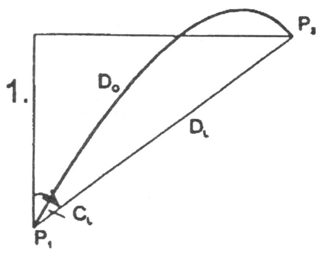
Determination of the position B

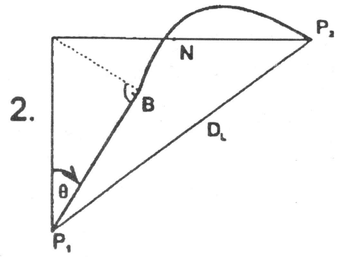
$\delta\phi_B = P_1B \cdot \cos\Theta$
$\delta\phi_B = \tan\theta \cdot M'$

$\phi_B = \phi_1 + \delta\phi_B$
$M' = (\pm M_B) - (\pm M_1)$

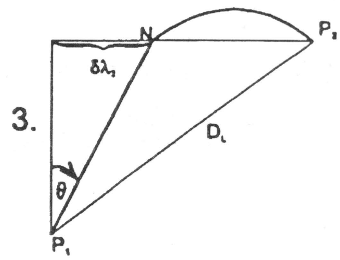
$\lambda_B = \lambda_1 + \delta\lambda_B$



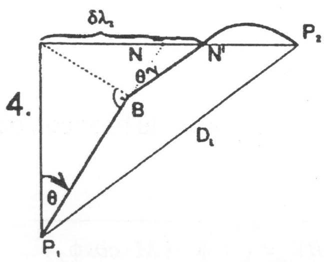
Great circle sailing



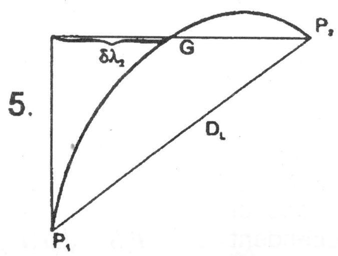
Rhumbline (P_1B) + Great circle (BP_2)



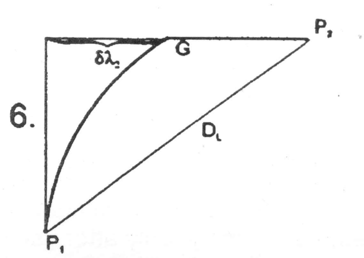
Rhumbline (P_1N) + Great circle (NP_2)



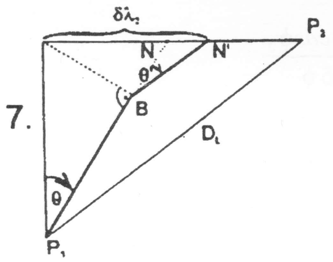
Rhumbline (P_1B) +
Rhumbline (BN') +
Great circle ($N'P_2$)



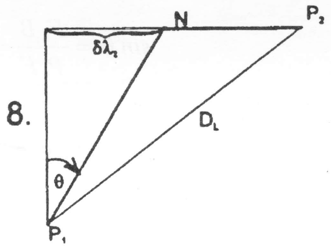
Great circle of composite sailing (P_1G) + Great circle (GP_2)



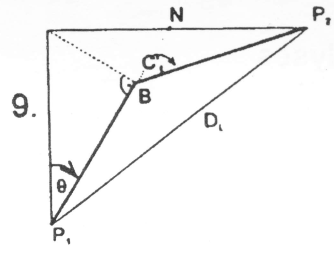
Great circle of composite sailing (P_1G) + Parallel sailing (GP_2)



Rhumbline (P_1B) +
Rhumbline (BN') +
Parallel sailing ($N'P_2$)



Rhumbline (P_1N) + Parallel sailing (NP_2)



Rhumbline (P_1B) + Rhumbline (BP_2)

Fig. 2. Different ways of Trans-oceanic Passages

D_L - Rhumbline distance
D_o - Theoretical Great circle distance
θ - Rhumbline course from P_1 to B or N
θ' - Rhumbline course from B to N'

C_L - Rhumbline course from P_1 to P_2
C_L' - Rhumbline course from B to P_2
$\delta\lambda_2$ - Difference of longitude between position of departure (P_1) and turnpoint (N,N',G)

Explanation of different ways of Trans-oceanic Passages

1. Great circle sailing - classic orthodrome
2. Rhumbline sailing with the course θ to the position B and then Great circle sailing
3. Rhumbline sailing with the course θ to the position N on the parallel of destination and then Great circle sailing
4. Rhumbline sailing with the course θ to the position B and then again Rhumbline sailing with the course θ' to the position N' on the parallel of destination and then Great circle sailing
5. Great circle of composite sailing with the parallel of destination as a limiting parallel and then from the position G to the position of destination (P_2) Great circle sailing
6. Great circle of composite sailing with the parallel of destination as a limiting parallel and then from position G to P_2 Parallel sailing
7. Rhumbline sailing with the course θ to the position B and then again Rhumbline sailing with the course θ' to the position N' on the parallel of destination and then Parallel sailing to the position of destination
8. Rhumbline sailing with the course θ to the position N on the parallel of destination and then Parallel sailing to the position of destination (Naše more, br. 1-2, Dubrovnik, 1991.)
9. Rhumbline sailing with the course θ to the position B and then classic Rhumbline sailing to the position of destination

$$\sin\theta = \frac{M \cdot \cos\phi_2}{\delta\phi} \qquad M = (\pm M_2) - (\pm M_1) \qquad \delta\phi = (\pm\phi_2) - (\pm\phi_1)$$

$$\sin\theta' = \frac{M' \cdot \cos\phi_2}{\delta\phi'} \qquad M' = (\pm M_2) - (\pm M_B) \qquad \delta\phi' = (\pm\phi_2) - (\pm\phi_B)$$

$$\tan C_L = \frac{\delta\lambda}{M} \qquad M = (\pm M_2) - (\pm M_1) \qquad \delta\lambda = (\pm\lambda_2) - (\pm\lambda_1)$$

$$\tan C_L' = \frac{\delta\lambda'}{M'} \qquad M' = (\pm M_2) - (\pm M_B) \qquad \delta\lambda' = (\pm\lambda_2) - (\pm\lambda_B)$$

Table 1. Comparison of results based on Theoretical example

($\phi_1 = 25°$, $\phi_2 = 40°$, $\delta\lambda = 120°$, $\delta\lambda_x = 10°$)

Way of sailing	Saving in relation to the Rhumbline	ϕ_V	$\delta\lambda_2$
1	459	52°55'	-
2	429	51°33'	-
3	372	47°44'	39°26'
4	357	47°08'	42°19'
5	315	44°40'	56°14'
6	250	ϕ_2	56°14'
7	234	ϕ_2	42°19'
8	233	ϕ_2	39°26'
9	38	-	-

ϕ_V - Latitude of Vertex
$\delta\lambda_x$ - Difference of longitude between waypoints

THEORETICAL EXAMPLE

Latitude of Position of Departure (P_1 or A_1) $\rightarrow \phi_1 = 25°$
Latitude of Position of Destination (P_2 or A_2) $\rightarrow \phi_2 = 40°$
Difference of longitude $\rightarrow \delta\lambda = 120°$
Difference of longitude between waypoints $\rightarrow \delta\lambda_x = 10°$

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1. Way of sailing

Rhumbline

$\delta\phi = (\pm\phi_2) - (\pm\phi_1) = 40° - 25° = 15° = 900'$
$M = (\pm M_2) - (\pm M_1) = 2607.9' - 1540.3' = 1067.6'$
$\delta\lambda = (\pm\lambda_2) - (\pm\lambda_1) = 120° = 7200'$

$$\tan C_L = \frac{\delta\lambda}{M} = \frac{7200'}{1067.6'} \rightarrow C_L = 081.5657634° \approx 081.6°$$

$D_L = \delta\phi \cdot \sec C_L = 900 \cdot \sec 081.5657634° = 6136.0$ Miles

Great Circle

$\cos D_o = \sin\phi_1 \cdot \sin\phi_2 + \cos\phi_1 \cdot \cos\phi_2 \cdot \cos\delta\lambda$
$= \sin 25° \cdot \sin 40° + \cos 25° \cdot \cos 40° \cdot \cos 120° = 94.32893115 \rightarrow 5659.7$ Miles

Theoretical Saving in relation to the Rhumbline = $D_L - D_o = 6136.0 - 5659.7 = 476.3$ Miles

$$\cos C_o = \frac{\sin\phi_2 - \cos D_o \cdot \sin\phi_1}{\sin D_o \cdot \cos\phi_1} \rightarrow C_o = \text{Great Circle Initial Course}$$

$$= \frac{\sin 40° - \cos 94.32893115° \cdot \sin 25°}{\sin 94.32893115° \cdot \cos 25°} \rightarrow C_o = 41.70626455°$$

$\cos\phi_v = \cos\phi_1 \cdot \sin C_o \rightarrow \phi_v$ = Latitude of Vertex
$= \cos 25° \cdot \sin 41.70626455° = 52.91656043° = \underline{52° 55'}$

$\text{ctn}\,\delta\lambda_v = \sin\phi_1 \cdot \tan C_o \rightarrow \delta\lambda_v$ = Difference of longitude between P_1 (Position of Departure) and P_v (Position of Vertex)
$= \sin 25° \cdot \tan 41.70626455° \rightarrow \delta\lambda_v = 069.36252044° = 069° 21.8'$

$\lambda_v = \lambda_1 + \delta\lambda_v \rightarrow \lambda_v$ = Longitude of Vertex
$= 0° + 69° 21.8' = 069° 21.8'$

Latitude (ϕ_x) and Longitude (λ_x) of Waypoints

$\tan\phi_x = \cos\delta\lambda_x \cdot \tan\phi_v$ $\qquad$ $\lambda_x = \lambda_v \pm \delta\lambda_x$

$= \cos 10° \cdot \tan 52° 55' \rightarrow \phi_A = 52° 29.6'$ $\qquad$ $\lambda_A = \lambda_v \pm 10° =$ 079° 21.8' / 059° 21.8'

$= \cos 20° \cdot \tan 52° 55' \rightarrow \phi_B = 51° 11.3'$ $\qquad$ $\lambda_B = \lambda_v \pm 20° =$ 089° 21.8' / 049° 21.8'

$= \cos 30° \cdot \tan 52° 55' \rightarrow \phi_C = 48° 53.2'$ $\qquad$ $\lambda_C = \lambda_v \pm 30° =$ 099° 21.8' / 039° 21.8'

$= \cos 40° \cdot \tan 52° 55' \rightarrow \phi_D = 45° 23.1'$ $\quad \lambda_D = \lambda_v \pm 40° = \begin{matrix} 109° 21.8' \\ 029° 21.8' \end{matrix}$

$= \cos 50° \cdot \tan 52° 55' \rightarrow \phi_E = 40° 22.7'$ $\quad \lambda_E = \lambda_v \pm 50° = \begin{matrix} 119° 21.8' \\ 019° 21.8' \end{matrix}$

$= \cos 60° \cdot \tan 52° 55' \rightarrow \phi_F = 33° 29.1'$ $\quad \lambda_F = \lambda_v - 60° = 009° 21.8'$

Rhumbline Course (C_L) and Distance (D_L) between Waypoints

P_1	$\phi_1 = 25°$	$\lambda_1 = 0°$		
			$C_L = 044°$	$D_L =$ 708 Miles
	$\phi_2 = 33° 29.1'$	$\lambda_2 = 009° 21.8'$		
			$C_L = 049.3°$	$D_L =$ 634.4 Miles
	$\phi_3 = 40° 22.7'$	$\lambda_3 = 019° 21.8'$		
			$C_L = 055.7°$	$D_L =$ 533.5 Miles
	$\phi_4 = 45° 23.1'$	$\lambda_4 = 029° 21.8'$		
			$C_L = 062.8°$	$D_L =$ 460 Miles
	$\phi_5 = 48° 53.2'$	$\lambda_5 = 039° 21.8'$		
			$C_L = 070.3°$	$D_L =$ 410.3 Miles
	$\phi_6 = 51° 11.3'$	$\lambda_6 = 049° 21.8'$		
			$C_L = 078.1°$	$D_L =$ 379.8 Miles
	$\phi_7 = 52° 29.6'$	$\lambda_7 = 059° 21.8'$		
			$C_L = 086°$	$D_L =$ 365.3 Miles
P_V	$\phi_8 = 52° 55'$	$\lambda_8 = 069° 21.8'$		
			$C_L = 094°$	$D_L =$ 365.3 Miles
	$\phi_9 = 52° 29.6'$	$\lambda_9 = 079° 21.8'$		
			$C_L = 101.9°$	$D_L =$ 379.8 Miles
	$\phi_{10} = 51° 11.3'$	$\lambda_{10} = 089° 21.8'$		
			$C_L = 109.7°$	$D_L =$ 410.3 Miles
	$\phi_{11} = 48° 53.2'$	$\lambda_{11} = 099° 21.8'$		
			$C_L = 117.2°$	$D_L =$ 460 Miles
	$\phi_{12} = 45° 23.1'$	$\lambda_{12} = 109° 21.8'$		
			$C_L = 130.7°$	$D_L =$ 533.5 Miles
	$\phi_{13} = 40° 22.7'$	$\lambda_{13} = 119° 21.8'$		
			$C_L = 127.8°$	$D =$ 37.1 Miles
P_2	$\phi_{14} = 40°$	$\lambda_{14} = 120°$		$\Sigma D_L = 5677.3$ Miles

ΣD_L = Practical Great Circle Distance

(Practical) Saving in relation to the Rhumbline Distance $= D_L - \Sigma D_L = 6136 - 5677.3 = 458.7 \approx$ 459 Miles

2. Way of sailing

$$\sin \theta = \frac{M \cdot \cos \phi_2}{\delta\phi} = \frac{1067.6' \cdot 0.76839091}{900} \rightarrow \theta = 065.71101503° \approx 065.7°$$

$$P_1B = \sqrt{\delta^2\phi - (M \cdot \cos\phi_2)^2} = \sqrt{900^2 - (1067.6 \cdot 0.76839091)^2} = 370.2052216' \approx 370.2 \text{ Miles}$$

$\delta\phi_B = P_1B \cdot \cos\theta = 370.2052216 \cdot \cos 065.71101503° = 152.2798956' \approx 152.3' = 2° 32.3'$

$\phi_B = \phi_1 + \delta\phi_B = 25° + 2° 32.3' = 27° 32.3'$

$M' = (\pm M_B) - (\pm M_1) = 1709.2' - 1540.3' = 168.9'$

$\delta\lambda_B = \tan\theta \cdot M' = \tan 065.71101503° \cdot 168.9' \approx 374.3' = 6° 14.3'$

$\lambda_B = \lambda_1 + \delta\lambda_B = 0° + 006° 14.3' = 006° 14.3'$

<u>B (ϕ_B = 27° 32.3', λ_B = 006° 14.3')</u>

<u>Great Circle Sailing between B and P_2</u>

<u>B (ϕ = 27° 32.3', λ = 006° 14.3'), P_2 (ϕ = 40°, λ = 120°)</u>

$\delta\lambda = 120° - 006° 14.3' = 113° 45.7'$

$\cos D_o = \sin\phi_1 \cdot \sin\phi_2 + \cos\phi_1 \cdot \cos\phi_2 \cdot \cos\delta\lambda$

$= \sin 27° 32.3' \cdot \sin 40° + \cos 27° 32.3' \cdot \cos 40° \cdot \cos 113° 45.7' \rightarrow D_o = 88.65375603° = 5319.2 \text{ Miles}$

$$\cos C_o = \frac{\sin\phi_2 - \cos D_o \cdot \sin\phi_1}{\sin D_o \cdot \cos\phi_1} = \frac{\sin 40° - \cos 88.65375603° \cdot \sin 27° 32.3'}{\sin 88.65375603° \cdot \cos 27° 32.3'}$$

$$C_o = 044.53139759°$$

$\cos\phi_v = \cos\phi_1 \cdot \sin C_o = \cos 27° 32.3' \cdot \sin 044.53139759° \rightarrow \phi_v = 51° 32.9' \approx \underline{51° 33'}$

$\text{ctn}\,\delta\lambda_v = \sin\phi_1 \cdot \tan C_o = \sin 27° 32.3' \cdot \tan 044.53139759° \rightarrow \delta\lambda_v = 065.32.5'$

$\lambda_v = \lambda_1 + \delta\lambda_v = 006° 14.3' + 065° 32.5' = 071° 46.8'$

$\tan\phi_x = \cos\delta\lambda_x \cdot \tan\phi_v$	$\lambda_x = \lambda_v \pm \delta\lambda_x$
$= \cos 10° \cdot \tan 51° 32.9' \rightarrow 51° 07.2'$	071° 46.8' ± 10° = 081° 46.8' / 061° 46.8'
$= \cos 20° \cdot \tan 51° 32.9' \rightarrow 49° 48.1'$	071° 46.8' ± 20° = 091° 46.8' / 051° 46.8'
$= \cos 30° \cdot \tan 51° 32.9' \rightarrow 47° 28.9'$	071° 46.8' ± 30° = 101° 46.8' / 041° 46.8'
$= \cos 40° \cdot \tan 51° 32.9' \rightarrow 43° 58.3'$	071° 46.8' ± 40° = 111° 46.8' / 031° 46.8'
$= \cos 50° \cdot \tan 51° 32.9' \rightarrow 38° 59.4'$	071° 46.8' - 50° = 021° 46.8'
$= \cos 60° \cdot \tan 51° 32.9' \rightarrow 32° 11.9'$	071° 46.8' - 60° = 011° 46.8'

P_B	$\phi_1 = 27° 32.3'$	$\lambda_1 = 006° 14.3'$		
			$C_L = 046°$	$D_L =$ 402.6 Miles
	$\phi_2 = 32° 11.9'$	$\lambda_2 = 011° 46.8'$		
			$C_L = 050.2°$	$D_L =$ 636.9 Miles
	$\phi_3 = 38° 59.4'$	$\lambda_3 = 021° 46.8'$		
			$C_L = 056.5°$	$D_L =$ 540.9 Miles
	$\phi_4 = 43° 58.3'$	$\lambda_4 = 031° 46.8'$		
			$C_L = 063.4°$	$D_L =$ 469.9 Miles
	$\phi_5 = 47° 28.9'$	$\lambda_5 = 041° 46.8'$		
			$C_L = 070.7°$	$D_L =$ 421.2 Miles
	$\phi_6 = 49° 48.1'$	$\lambda_6 = 051° 46.8'$		
			$C_L = 078.3°$	$D_L =$ 391.1 Miles
	$\phi_7 = 51° 07.2'$	$\lambda_7 = 061° 46.8'$		
			$C_L = 086.1°$	$D_L =$ 376.7 Miles
P_V	$\phi_8 = 51° 32.9'$	$\lambda_8 = 071° 46.8'$		
			$C_L = 093.9°$	$D_L =$ 376.7 Miles
	$\phi_9 = 51° 07.2'$	$\lambda_9 = 081° 46.8'$		
			$C_L = 101.7°$	$D_L =$ 391.1 Miles
	$\phi_{10} = 49° 48.1'$	$\lambda_{10} = 091° 46.8'$		
			$C_L = 109.3°$	$D_L =$ 421.2 Miles
	$\phi_{11} = 47° 28.9'$	$\lambda_{11} = 101° 46.8'$		
			$C_L = 116.6°$	$D_L =$ 469.9 Miles
	$\phi_{12} = 43° 58.3'$	$\lambda_{12} = 111° 46.8'$		
			$C_L = 122.9°$	$D_L =$ 438.2 Miles
P_2	$\phi_{13} = 40°$	$\lambda_{13} = 120°$		$\Sigma D_L = 5336.4$ Miles

$P_1B =$ 370.2 Miles (Rhumbline $\rightarrow \theta = 065.7°$)
$+ BP_2 =$ 5336.4 Miles (Great Circle Sailing)
5706.6 Miles

Saving in relation to the Rhumbline
6136 - 5706.6 = 429.4 ≈ 429 Miles

3. Way of sailing

$P_1N = \delta\phi \cdot \sec\theta = 900' \cdot \sec 065.71101503° = 2188$ Miles
$\delta\lambda_2 = M \cdot \tan\theta = 1067.6 \cdot \tan 065.71101503° = 2365.7' = 39° 25.7' \approx 39° 26'$
$\lambda_N = \lambda_1 + \delta\lambda_2 = 0° + 39° 25.7' = 039° 25.7'$

Great Circle Sailing between N and P_2

N ($\phi = 40°$, $\lambda = 039° 25.7'$), P_2 ($\phi = 40°$, $\lambda = 120°$)

$\delta\lambda = 120° - 039° 25.7' = 080° 34.3'$

$$\cos D_o = \sin\phi_1 \cdot \sin\phi_2 + \cos\phi_1 \cdot \cos\phi_2 \cdot \cos\delta\lambda$$
$$= \sin 40° \cdot \sin 40° + \cos 40° \cdot \cos 40° \cdot \cos 80° 34.3' \rightarrow D_o = 59.38239937° = 3562.9 \text{ Miles}$$

$$\cos C_o = \frac{\sin\phi_2 - \cos D_o \cdot \sin\phi_1}{\sin D_o \cdot \cos\phi_1} = \frac{\sin 40° - \cos 59.38239937° \cdot \sin 40°}{\sin 59.38239937° \cdot \cos 40°}$$

$$C_o = 061.4162269°$$

$\cos\phi_v = \cos\phi_1 \cdot \sin C_o = \cos 40° \cdot \sin 061.4162269° \rightarrow \phi_v = 47° 43.6' \approx 47° 44'$
$\text{ctn}\,\delta\lambda_v = \sin\phi_1 \cdot \tan C_o = \sin 40° \cdot \tan 061.4162269° \rightarrow \delta\lambda_v = 040° 17.1'$

$\lambda_v = \lambda_1 + \delta\lambda_v = 39° 25.7' + 040° 17.1' = 079° 42.8'$

$\tan\phi_x = \cos\delta\lambda_x \cdot \tan\phi_v$	$\lambda_x = \lambda_v \pm \delta\lambda_x$
$= \cos 10° \cdot \tan 47° 43.6' \rightarrow 47° 17.4'$	$079° 42.8' \pm 10° =$ 089° 42.8' / 069° 42.8'
$= \cos 20° \cdot \tan 47° 43.6' \rightarrow 45° 56.9'$	$079° 42.8' \pm 20° =$ 099° 42.8' / 059° 42.8'
$= \cos 30° \cdot \tan 47° 43.6' \rightarrow 43° 36.6'$	$079° 42.8' \pm 30° =$ 109° 42.8' / 049° 42.8'
$= \cos 40° \cdot \tan 47° 43.6' \rightarrow 40° 07.2'$	$079° 42.8' \pm 40° =$ 119° 42.8' / 039° 42.8'

N	$\phi_1 = 40°$	$\lambda_1 = 039° 25.7'$		
			$C_L = 061.3°$	$D_L =$ 15 Miles
	$\phi_2 = 40° 07.2'$	$\lambda_2 = 039° 42.8'$		
			$C_L = 065°$	$D_L =$ 494.8 Miles
	$\phi_3 = 43° 36.6'$	$\lambda_3 = 049° 42.8'$		
			$C_L = 071.8°$	$D_L =$ 449.7 Miles
	$\phi_4 = 45° 56.9'$	$\lambda_4 = 059° 42.8'$		
			$C_L = 079°$	$D_L =$ 421.1 Miles
	$\phi_5 = 47° 17.4'$	$\lambda_5 = 069° 42.8'$		
			$C_L = 086.3°$	$D_L =$ 407.4 Miles
P_V	$\phi_6 = 47° 43.6'$	$\lambda_6 = 079° 42.8'$		
			$C_L = 093.7°$	$D_L =$ 407.4 Miles
	$\phi_7 = 47° 17.4'$	$\lambda_7 = 089° 42.8'$		
			$C_L = 101°$	$D_L =$ 421.1 Miles
	$\phi_8 = 45° 56.9'$	$\lambda_8 = 099° 42.8'$		
			$C_L = 108.2°$	$D_L =$ 449.7 Miles
	$\phi_9 = 43° 36.6'$	$\lambda_9 = 109° 42.8'$		
			$C_L = 115°$	$D_L =$ 494.8 Miles
	$\phi_{10} = 40° 07.2'$	$\lambda_{10} = 119° 42.8'$		
			$C_L = 118.6°$	$D_L =$ 15.1 Miles
P_2	$\phi_{11} = 40°$	$\lambda_{11} = 120°$		$\Sigma D_L = 3576.1$ Miles

P_1N = 2188 Miles (Rhumbline $\rightarrow \theta = 065.7°$)
+ NP_2 = 3576.1 Miles (Great Circle Sailing)
5764.1 Miles

Saving in relation to the Rhumbline
6136 - 5764.1 = 371.9 ≈ 372 Miles

4. Way of sailing

B ($\phi = 27° 32.3'$, $\lambda = 006° 14.3'$), P_2 ($\phi = 40°$, $\lambda = 120°$)
$\delta\phi = 40° - 27° 32.3' = 12° 27.7' = 747.7'$
$M = 2607.9' - 1709.2' = 898.7'$
$\delta\lambda = 120° - 006° 14.3' = 113° 45.7'$

$$\sin\theta' = \frac{M \cdot \cos\phi_2}{\delta\phi} = \frac{898.7' \cdot 0.76839091}{747.7'} \rightarrow \theta = 067.45362603° \approx 067.5°$$

$BN' = \delta\phi \cdot \sec\theta' = 747.7' \cdot \sec 067.45362603° = 1950$ Miles

$\delta\lambda_2 = \lambda_B + M \cdot \tan\theta = 006°\ 14.3' + (898.7' \cdot \tan 067.45362603°) = 006°\ 14.3' + 2164.7' = 006°\ 14.3' + 036°\ 04.7' =$
$= \underline{042°\ 19'}$

$\lambda_N' = \lambda_1 + \delta\lambda_2 = 0° + 042°\ 19' = 042°\ 19'$

Great Circle Sailing between N' and P_2

N' ($\phi = 40°$, $\lambda = 042°\ 19'$), P_2 ($\phi = 40°$, $\lambda = 120°$)

$\delta\lambda = 120° - 042°\ 19' = 077°\ 41'$

$\cos D_o = \sin\phi_1 \cdot \sin\phi_2 + \cos\phi_1 \cdot \cos\phi_2 \cdot \cos\delta\lambda$
$= \sin 40° \cdot \sin 40° + \cos 40° \cdot \cos 40° \cdot \cos 077°\ 41' \rightarrow D_o = 57.42833795° = 3445.7$ Miles

$$\cos C_o = \frac{\sin\phi_2 - \cos D_o \cdot \sin\phi_1}{\sin D_o \cdot \cos\phi_1} = \frac{\sin 40° - \cos 57.42833795° \cdot \sin 40°}{\sin 57.42833795° \cdot \cos 40°}$$

$$C_o = 062.63461257°$$

$\cos\phi_v = \cos\phi_1 \cdot \sin C_o = \cos 40° \cdot \sin 062.63461257° \rightarrow \phi_v = 47°\ 07.9' \approx \underline{\underline{47°\ 08'}}$
$\text{ctn}\,\delta\lambda_v = \sin\phi_1 \cdot \tan C_o = \sin 40° \cdot \tan 062.63461257° \rightarrow \delta\lambda_v = 038°\ 50.5'$
$\lambda_v = \lambda_1 + \delta\lambda_v = 042°\ 19' + 038°\ 50.5' = 081°\ 09.5'$

$\tan\phi_x = \cos\delta\lambda_x \cdot \tan\phi_v$	$\lambda_x = \lambda_v \pm \delta\lambda_x$
$= \cos 10° \cdot \tan 47°\ 07.9' \rightarrow 46°\ 41.6'$	$081°\ 09.5' \pm 10° =$ 091° 09.5' / 071° 09.5'
$= \cos 20° \cdot \tan 47°\ 07.9' \rightarrow 45°\ 21.1'$	$081°\ 09.5' \pm 20° =$ 101° 09.5' / 061° 09.5'
$= \cos 30° \cdot \tan 47°\ 07.9' \rightarrow 43°\ 00.9'$	$081°\ 09.5' \pm 30° =$ 111° 09.5' / 051° 09.5'

N'	$\phi_1 = 40°$	$\lambda_1 = 042°\ 19$		
			$C_L = 065.6°$	$D_L =$ 437.8 Miles
	$\phi_2 = 43°\ 00.9'$	$\lambda_2 = 051°\ 09.5'$		
			$C_L = 072°$	$D_L =$ 453.9 Miles
	$\phi_3 = 45°\ 21.1'$	$\lambda_3 = 061°\ 09.5'$		
			$C_L = 079.1°$	$D_L =$ 425.6 Miles
	$\phi_4 = 46°\ 41.6'$	$\lambda_4 = 071°\ 09.5'$		
			$C_L = 086.3°$	$D_L =$ 412 Miles
P_V	$\phi_5 = 47°\ 07.9'$	$\lambda_5 = 081°\ 09.5'$		
			$C_L = 093.7°$	$D_L =$ 412 Miles
	$\phi_6 = 46°\ 41.6'$	$\lambda_6 = 091°\ 09.5'$		
			$C_L = 100.9°$	$D_L =$ 425.6 Miles

	$\phi_7 = 45° 21.1'$	$\lambda_7 = 101° 09.5'$		
			$C_L = 108°$	$D_L = 453.9$ Miles
	$\phi_8 = 43° 00.9'$	$\lambda_8 = 111° 09.5'$		
			$C_L = 114.4°$	$D_L = 437.8$ Miles
P_2	$\phi_9 = 40°$	$\lambda_9 = 120°$		$\Sigma D_L = 3458.6$ Miles

$P_1 B = 370.2$ Miles (Rhumbline $\rightarrow \theta = 065.7°$)
+ $BN' = 1950.0$ Miles (Rhumbline $\rightarrow \theta' = 067.5°$)
~~+ $N'P_2 = 3458.6$ Miles (Great Circle Sailing)~~
5778.8 Miles

Saving in relation to the Rhumbline
$6136 - 5778.8 = 357.2 \approx$ 357 Miles

5. Way of sailing

Limiting parallel = ϕ_G

$\cos \delta\lambda_2 = \tan \phi_1 \cdot \text{ctn}\, \phi_G = \tan 25° \cdot \text{ctn}\, 40° \rightarrow \delta\lambda_2 = 056° 14.4' \approx 056° 14'$

Great Circle Sailing between P_1 and G

P_1 ($\phi = 25°$, $\lambda = 0°$), G ($\phi = 40°$, $\lambda = 056° 14.4'$)

$\phi_V = \phi_G \qquad \lambda_V = \lambda_G = 056° 14'$

$\tan \phi_x = \cos \delta\lambda_x \cdot \tan \phi_v \qquad \lambda_x = \lambda_v - \delta\lambda_x$

$= \cos 10° \cdot \tan 40° \rightarrow 39° 34.1'$	$056° 14.4' - 10° = 046° 14.4'$
$= \cos 20° \cdot \tan 40°' \rightarrow 38° 15.3'$	$056° 14.4' - 20° = 036° 14.4'$
$= \cos 30° \cdot \tan 40°' \rightarrow 36° 00.3'$	$056° 14.4' - 30° = 026° 14.4'$
$= \cos 40° \cdot \tan 40° \rightarrow 32° 43.9'$	$056° 14.4' - 40° = 016° 14.4'$
$= \cos 50° \cdot \tan 40° \rightarrow 28° 20.4'$	$056° 14.4' - 50° = 006° 14.4'$

P_1	$\phi_1 = 25°$	$\lambda_1 = 0°$		
			$C_L = 059.2°$	$D_L = 391.5$ Miles
	$\phi_2 = 28° 20.4'$	$\lambda_2 = 006° 14.4'$		
			$C_L = 063.1°$	$D_L = 582.2$ Miles
	$\phi_3 = 32° 43.9'$	$\lambda_3 = 016° 14.4'$		
			$C_L = 068.5°$	$D_L = 534.8$ Miles
	$\phi_4 = 36° 00.3'$	$\lambda_4 = 026° 14.4'$		
			$C_L = 074.3°$	$D_L = 498.9$ Miles
	$\phi_5 = 38° 15.3'$	$\lambda_5 = 036° 14.4'$		
			$C_L = 080.5°$	$D_L = 475.3$ Miles
	$\phi_6 = 39° 34.1'$	$\lambda_6 = 046° 14.4'$		
			$C_L = 086.8°$	$D_L = 463.6$ Miles
P_2	$\phi_7 = 40°$	$\lambda_7 = 056° 14.4'$		$\Sigma D_L = 2946.3$ Miles

Great Circle Sailing between G and P_2

$G (\phi = 40°, \lambda = 056° 14.4')$, $P_2 (\phi = 40°, \lambda = 120°)$

$\delta\lambda = 120° - 056° 14.4' = 063° 45.6'$

$$\cos D_o = \sin\phi_1 \cdot \sin\phi_2 + \cos\phi_1 \cdot \cos\phi_2 \cdot \cos\delta\lambda$$
$$= \sin 40° \cdot \sin 40° + \cos 40° \cdot \cos 40° \cdot \cos 63° 45.6' \rightarrow D_o = 47.72964726° = 2863.8 \text{ Miles}$$

$$\cos C_o = \frac{\sin\phi_2 - \cos D_o \cdot \sin\phi_1}{\sin D_o \cdot \cos\phi_1} = \frac{\sin 40° - \cos 47.72964726° \cdot \sin 40°}{\sin 47.72964726° \cdot \cos 40°}$$

$$C_o = 068.20902255°$$

$\cos\phi_v = \cos\phi_1 \cdot \sin C_o = \cos 40° \cdot \sin 068.20902255° \rightarrow \phi_v = 44° 39.5' \approx \underline{44° 40'}$

$\text{ctn}\,\delta\lambda_v = \sin\phi_1 \cdot \tan C_o = \sin 40° \cdot \tan 068.20902255° \rightarrow \delta\lambda_v = 031° 52.8'$

$\lambda_v = \lambda_G + \delta\lambda_v = 056° 14.4' + 031° 52.8' = 088° 07.2'$

$\tan\phi_x = \cos\delta\lambda_x \cdot \tan\phi_v$	$\lambda_x = \lambda_v \pm \delta\lambda_x$	
$= \cos 10° \cdot \tan 44° 39.5' \rightarrow 44° 13.2'$	$088° 07.2' \pm 10° =$	098° 07.2' / 078° 07.2'
$= \cos 20° \cdot \tan 44° 39.5' \rightarrow 42° 52.7'$	$088° 07.2' \pm 20° =$	108° 07.2' / 068° 07.2'
$= \cos 30° \cdot \tan 44° 39.5' \rightarrow 40° 33.3'$	$088° 07.2' \pm 30° =$	118° 07.2' / 058° 07.2'

	ϕ	λ	C_L	D_L
G	$\phi_1 = 40°$	$\lambda_1 = 056° 14.4'$		
			$C_L = 068.9°$	$D_L =$ 92.6 Miles
	$\phi_2 = 40° 33.3'$	$\lambda_2 = 058° 07.2'$		
			$C_L = 072.8°$	$D_L =$ 470.6 Miles
	$\phi_3 = 42° 52.7'$	$\lambda_3 = 068° 07.2'$		
			$C_L = 079.5°$	$D_L =$ 443.7 Miles
	$\phi_4 = 44° 13.2'$	$\lambda_4 = 078° 07.2'$		
			$C_L = 086.5°$	$D_L =$ 430 7 Miles
P_V	$\phi_5 = 44° 39.5'$	$\lambda_5 = 088° 07.2'$		
			$C_L = 093.5°$	$D_L =$ 430.7 Miles
	$\phi_6 = 44° 13.2'$	$\lambda_6 = 098° 07.2'$		
			$C_L = 100.5°$	$D_L =$ 443.7 Miles
	$\phi_7 = 42° 52.7'$	$\lambda_7 = 108° 07.2'$		
			$C_L = 107.2°$	$D_L =$ 470.6 Miles
	$\phi_8 = 40° 33.3'$	$\lambda_8 = 118° 07.2'$		
			$C_L = 111.1°$	$D_L =$ 92.6 Miles
P_2	$\phi_9 = 40°$	$\lambda_{11} = 120°$		$\Sigma D_L = 2875.2$ Miles

P_1G = 2946.3 Miles (Great Circle Sailing)
+ GP_2 = 2875.2 Miles (Great Circle Sailing)
5821.5 Miles

Saving in relation to the Rhumbline
6136 - 5821.5 = 314.5 ≈ 315 Miles

6. Way of sailing

Parallel Sailing between G and P_2

G (ϕ = 40°, λ = 056° 14.4'), P_2 (ϕ = 40°, λ = 120°)

Departure = (120° - 056° 14.4')' · cos ϕ = 3825.6' · 0.76839091 = 2939.6 Miles

P_1G = 2946.3 Miles (Great Circle Sailing)
\+ GP_2 = 2939.6 Miles (Parallel Sailing → C_L = 090°)
5885.9 Miles

Saving in relation to the Rhumbline
6136 - 5885.9 = 250.1 ≈ 250 Miles

7. Way of sailing

Parallel Sailing between N' and P_2

N' (ϕ = 40°, λ = 042° 19'), P_2 (ϕ = 40°, λ = 120°)

Departure = (120° - 042° 19')' · cos ϕ = 4661' · 0.76839091 = 3581.5 Miles

P_1 B = 370.2 Miles (Rhumbline → θ = 065.7°)
\+ BN' = 1950.0 Miles (Rhumbline → θ' = 067.5°)
\+ N'P_2 = 3581.5 Miles (Parallel Sailing)
5901.7 Miles

Saving in relation to the Rhumbline
6136 - 5901.7 = 234.3 ≈ 234 Miles

8. Way of sailing

Parallel Sailing between N and P_2

N (ϕ = 40°, λ = 039° 25.7'), P_2 (ϕ = 40°, λ = 120°)

Departure = (120° - 039° 25.7')' · cos ϕ = 4834.3' · 0.76839091 = 3714.6 Miles

P_1N = 2188 Miles (Rhumbline → θ = 065.7°)
\+ NP_2 = 3714.6 Miles (Parallel Sailing → C_L = 090°)
5902.6 Miles

Saving in relation to the Rhumbline
6136 - 5902.6 = 233.4 ≈ 233 Miles

9. Way of sailing

Rhumbline between B and P_2

B (ϕ = 27° 32.3', λ = 006° 14.3'), P_2 (ϕ = 40°, λ = 120°)

$\delta\phi$ = 747.7'
M = 898.7'
$\delta\lambda$ = 113° 45.7' = 6825.7'

$$\tan C_L' = \frac{\delta\lambda}{M} = \frac{6825.7}{898.7} \rightarrow C_L' = 082.49858311 \approx 082.5°$$

$$D_L = \delta\phi \cdot \sec C_L' = 747.7 \cdot \sec 082.49858311 = 5727.3 \text{ Miles}$$

P_1B = 370.2 Miles (Rhumbline → θ = 065.7°)
+ BP_2 = 5727.3 Miles (Rhumbline → C_L = 090°)
6097.5 Miles

Saving in relation to the Rhumbline
6136 - 6097.5 = 38.5 ≈ 38 Miles

Table (1) shows savings in navigational distances between different ways of sailing and Rumbline sailing.

Saving by the Great circle sailing (1) with the latitude of vertex (52° 55') in relation to the Rhumbline sailing amounts 459 N. miles.

Saving when applying way (2) amounts 429 N. miles what means that way (1) is shorter for 30 N. miles but it leads vessel more than 80 N. miles further north.

Saving when applying way (3) amounts 372 N. miles but the vertex of the Great circle sailing is 47° 44' only.

Saving when applying way (4) amounts 357 N. miles and latitude of vertex is lower again (47° 08').

Saving when applying way (5) amounts 315 N. miles with the latitude of vertex of 44° 40' only.

Saving when applying way (6) amounts 250 N. miles with the limiting parallel of destination.

Saving when applying way (7) amounts 234 N. miles.

Saving when applying way (8 - Journal of Navigation, May, 1990) amounts 233 N. miles.

Saving when applying way (9) amounts 38 N. miles.

3. Conclusion
Zaključak

When planning a voyage the master makes a decision of which way to sail, taking into consideration weather conditions, type and quality of the ship, type of cargo, navigational distance, length of voyage... A "good master always tries to avoid high latitudes which can only be defined in the theory and not in practical navigation.

Calculating all distances and navigational courses, taking into account the aforementioned conditions, the master makes decision what way of navigation he will use to cross the ocean.

This paper shows several ways of navigation which don't guide the vessel in high latitudes. It is believed that further research of aforesaid ways of sailing would enrich this field.

Rukopis primljen: 13.3.1998.

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