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The typing approach to Church-Fitch's knowability paradox and its revenge form

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ABSTRACT: Williamson, Linsky, Paseau and others proposed a solution to Church-Fitch's knowability paradox that is based on typing knowledge; however, it received some criticism. Carrara and Fassio objected that the approach has no paradox-independent motivation, it is thus *ad hoc*. In the first part of the paper, I dismiss such criticism by carefully stating typing approach principles that are based on non-circular formation of propositions and intensional operators operating on them. In the second part of the paper, I demonstrate that the firm foundation of the approach prevents the variants of the paradox by Florio, Murzi and Jago that were developed as allegedly unresolvable by typing knowledge. The revenge form of Church-Fitch's knowability paradox, which had been proposed by Williamson, Hart, Carrara and Fassio, fares badly as well, since it is likewise based on violation of reasonable typing rules.

KEY WORDS: Church-Fitch's knowability paradoxrevenge paradox typing knowledgeramified theory of types Vicious Circle Principle

1 Introduction

The famous *Church-Fitch's knowability paradox (FP;* see Sec. 1.1 below) seems to prove that the epistemic optimism known as *verificationism,* which maintains that every truth is knowable, is wrong.[1] As a remedy to FP, several writers – most notably (Williamson 2000) – offered a *typing approach (TA).* It consists in *typing knowledge (TK),* i.e. dividing our intuitive notion of knowledge

[1] FP was discovered by Church who reviewed Fitch's paper in 1945; the review is now published as (Church 2009). FP was first published in (Fitch 1963, 138). For an overview of the topic, including various approaches to FP, see (Brogaard and Salerno 2008).

into type variants; knowledge operators are then distinguished by their range of applicability. In appropriate contexts, I will use the narrower term "TK" instead of "TA".

Two kinds of criticism target the TA to FP. (i) (Carrara and Fassio 2011) and others objected that the TK is not well-founded; the TA is thus an *ad hoc* approach. (ii) Another criticism attempted to show that the TA is unsatisfactory, for it is incapable of solving a. paradoxes similar to FP that were constructed by (Florio and Murzi 2009) and (Jago 2010), and b. the *Revenge Form of FP for TA* (as I will call it) that was proposed by (Williamson 2000), (Hart 2009), (Carrara and Fassio 2011).

In response to the (i)-type criticism, I will distinguish two kinds of typing, the *Tarskian* and the *Russellian TA*. While the former is based on *language/metalanguage distinction* (Tarski 1956), the latter is framed within the Russellian *ramified theory of types (RTT),* cf. (Russell 1908), (Whitehead and Russell 19 10–13). Church's reconstruction of Russell's RTT, the theory of *r*-types (Church 1976), is the most prominent RTT; it was applied to FP by (Linsky 2009) and (Giaretta 2009). Other theoreticians are not committed to RTT, which might be the reason of their reluctant or deprecatory attitudes to the TA. The details of the Russellian TA, which I will state in Sec. 2.1, have not been mentioned in the discussion yet; arguably, they provide the best justification of the approach.

The firm foundation of the Russellian TA yields a straightforward rejection of both a.-type and b.-type paradoxes, thus dismissing the (ii)-type criticism, because they harbour errors, as I will demonstrate in Secs. 3 and 4. The Revenge Form of FP for TA in particular violates the principle of non-circular formation of propositions and intensional operators (such as knowledge operator) operating on them, which is embodied in the *Vicious Circle Principle (VCP).*

1.1 Fitch's knowability paradox

FP is generated by a combination of apparently acceptable principles concerning knowledge and possibility. These are often expressed using a familiar notation of multimodal logic, when

"K*p*" represents 'It is known (by someone) that *p*'

and

"◇K*p*" represents 'It is possible to know (by someone) that *p*'.[2]

[2] I use double quotation marks for quotations of expressions; single quotation marks are used to indicate propositions and other 'extralinguistic' entities, or for shift in meaning.

The denotatum of "K" is a subset of all propositions that are currently true.

The thesis of verificationism is then formalised as

(Ver) $\forall p(p \supset \Diamond Kp)$ verificationism (knowability principle)

and the most important derivation rules capturing the semantic properties of "K" are[3]

(Fact)	$Kp \vdash p$	factivity of knowledge (K-axiom)
(Dist)	$K(p \land q) \vdash Kp \land Kq$	distributivity rule for K

FP also employs the following, quite unproblematic rules of modal logic:

(Nec)	if $\vdash p$, then $\vdash \Box p$	necessitation rule
(ER)	$\Diamond \neg p \dashv\vdash \neg\Box p$	exchange rule for modal operators

The source of FP's paradoxicality is the surprising fact that (Ver) is incompatible with the evident principle that we are not omniscient:

(NonOmn) $\exists p(p \land \neg Kp)$ non-omniscience

Applying classical logic and the above rules, the addition of (Ver) to (NonOmn) leads to the contradictory of (NonOmn), which is

(Omn) $\forall p(p \supset Kp)$ omniscience

Here is the full inference, i.e. FP:[4]

Fitch's knowability paradox (FP)

1. $\exists p(p \land \neg Kp)$	(NonOmn)
2. $p \land \neg Kp$	an instance of 1.
3. $\forall p(p \supset \Diamond Kp)$	(Ver)
4. $(p \land \neg Kp) \supset \Diamond K(p \land \neg Kp)$	substitution of 2. for p in 3.
5. $\Diamond K(p \land \neg Kp)$	from 4. and 2. by MP
6. $K(p \land \neg Kp)$	assumption *per absurdum*
7. $Kp \land K\neg Kp$	from 6. by (Dist)
8. $Kp \land \neg Kp$	from 7. by (Fact) on its right conjunct
9. $\neg K(p \land \neg Kp)$	*reductio*

[3] I assume that $\vdash$ and $\dashv\vdash$ are 'metalinguistic' variants of the material conditional $\supset$ and equivalence $\equiv$. Some rules with $\dashv\vdash$ may be considered as (formal) definitions.

[4] My exposition largely follows (Brogaard and Salerno 2008). Various deduction systems may amend the inference in different ways. For example, in step 4. some would consider substitution of 2. to $p \supset \Diamond Kp$, which would be derived from 3. by application of Universal Instantiation (i.e. $\forall$-elimination).

10. $\Box\neg K(p \wedge \neg Kp)$ from 9. by (Nec)
11. $\Diamond\neg K(p \wedge \neg Kp)$ from 10. by (ER)

Thus, 11. contradicts 5., which means that the addition of (Ver) to (NonOmn) leads to (Omn).

1.2 Two kinds of typing knowledge and their criticism

Some approaches attempt to solve FP by restricting the formulation of verificationism, while others revise its underlying logic. The TK is adjacent to both approaches; it consists in stratification of the knowledge operator into an infinite hierarchy of the operators K_1, K_2, …, K_n, while each has a restricted range of applicability (the thesis of verificationism is thus restricted); derivation rules are amended accordingly. The crucial proposition 8. is then shown to be non-contradictory, the *reductio* part of the inference is therefore prevented.

The TA to FP was first considered by Church in his famous review of Fitch's paper, since typing is involved in both the then standard devices for solving paradoxes, i.e. Russell's RTT and Tarski's hierarchy of metalanguages:[5]

> Of course the foregoing refutation of Fitch's definition of value is strongly suggestive of the paradox of the liar and other epistemological paradoxes. It may be, therefore that Fitch can meet this particular objection by incorporating into the system of his paper one of the standard devices for avoiding the epistemological paradoxes. (Church 2009, 17)

The first use of the TA as a solution to FP was proposed in (Williamson 2000, 280–28 1). It is evidently the Tarskian TA, since Williamson explicitly evoked the stratification of T-predicate, whereas he did not mention Russell, or any Russellian topic.

The Tarskian TA has in fact been adopted by many writers: (Paseau 2008), (Halbach 2008), (Paseau 2009),[6] (Florio and Murzi 2009), (Jago 2010), (Carrara and Fassio 2011). As mentioned above, an application of the Russellian TA to FP was published by (Linsky 2009) and (Giaretta 2009);

[5] This is somehow inaccurate. Russell himself typed propositions and even truths (Whitehead and Russell 1910–13, 44–45), but not intensional operators such as belief or knowledge because he did not model belief attitudes as attitudes towards propositions, as e.g. Church did. Church proposed a hierarchy of belief-predicates as a solution to the *paradox of Bouleus* in (Church 1973–74, 23–24).

[6] Halbach suggested a paradox similar to FP, yet Paseau noticed its closer similarity to the *Knower paradox* (Kaplan and Montague 1960); I will thus not discuss it in this paper. Further investigation was made in (Rosenblatt 2014).

they used a version of *r*-types by (Church 1976). However, justification of RTT is rudimentary in Linsky's paper; and Giaretta's paper is rather focused on the relationship of FP to the Russellian logical framework.

Most of the previously-mentioned writers, notably (Carrara and Fassio 2011), objected that the TA is an *ad hoc* approach, for it has no paradox-independent motivation. However, if one clearly distinguishes the Tarskian and the Russellian TA, and embraces the latter, such objection seems unfounded, since it is largely inapplicable, which I am going to show in both this and the following section.

In (Tarski 1956), the *hierarchy of T-predicates* is forced by the existence of the Liar paradox. Each *n*th-level predicate "T_n" (where $1 \leqslant n \in \mathbb{N}^+$) is only applicable to (the names of) sentences S_{n-1} of the level n–1, thereby preventing the building up of paradox-producing sentences S_n such as "$\neg T_n \ulcorner S_n \urcorner$". One may naturally object with (Kripke 1975) as to why our sole intuitive T-predicate is split into infinitely many partial ones, and why a sentence of a certain level *n* cannot contain a T-predicate belonging to the same or a higher level. Exactly the same questions can be raised about the Tarskian hierarchy of K-predicates. On the other hand, however, one should not disregard an important argument in favour of the Tarskian TA: the purpose of typing is to help deliver a formally correct explication of the intuitive notions. If the TA is successful in this respect, philosophical 'worries' should be sidelined.

In contrast to the Tarskian TA, the Russellian TA assumes a slightly different picture of our conceptual scheme. Being historically related to Russell's paradox and the Russell-Myhill paradox, the proper cause of typing are individuation and the forming of propositions and intensional operators, which is governed by the VCP. These are exposed in details in Sec. 2; they provide the best justification of the TA.

Since the crucial features of the Russellian TK have not been addressed by the aforementioned criticism, one may provisionally adopt the conclusion that the Russellian TK is untouched by it – being therefore a viable approach to FP. This assessment is also supported by the solutions to paradoxes similar to FP and its revenge form. Moreover, (Linsky 2009) documented that the TA is even capable of solving many other epistemic paradoxes. As regards various possible philosophical objections targeting the material adequacy of the TK, see the discussion in (Paseau 2008).

Finally, some philosophers' worry that type theories are obsolete noble ruins belonging to the history of logic is a weighty underestimation of the current development. Though Russell's type theory is being indeed studied by philosophers and historians of logic (cf. e.g. (Landini 1998), (Linsky 1999), or (Irvine 2016)) modification of the type theory by (Church 1940), which evolved from Russell's, Chwistek's and Ramsey's type theories, has constituted the leading logical framework in computer science for several decades (cf. e.g. (Andrews 2014) and (Coquand 2015) for introduction). Formalisation of language is dominated by Montagovians (cf. e.g. (Thomason 1974), or (Janssen 2017) for introduction), or their rivals (e.g. (Chatzikyriakidis and Luo 2017)) who also use modifications of Church's type theory. New RTTs have been proposed and advocated e.g. in (Tichý 1988) and (Kamareddine, Laan and Nederpelt 2010); ramification is also evoked in the Homotopy Type Theory (Univalent Foundations Program 2013), which has been much discussed recently.

2 The Russellian typing approach

2.1 Propositions, types, and the Vicious Circle Principle

It is important to realise that *propositions* are considered within the Russellian TA as meanings of sentences, i.e. abstract entities distinct from sentences (which are sequences of letters).

Propositions are *intensional entities,* they have intensional (i.e. not extensional) individuation: two such propositions can be equivalent (congruent) without being identical. A proposition in this sense is an entity with fine-grained structure,[7] not a possible-world proposition, which is a flat mapping from possible worlds to truth values. *Intensional operators,* such as the *knowledge operator,* operate on propositions, not on their values.[8] As argued by Church, and e.g. by (Tichý 1988), the adoption of RTT is then inevitable:

> If, following early Russell, we hold that the object of assertion or belief is a proposition and then impose on propositions the strong conditions of identity which it requires, while at the same time undertaking to formulate a logic that will suffice for classical mathematics, we therefore find no alternative except for the ramified type theory with axioms of reducibility. (Church 1984, 521)

Remark. Within possible-world semantics, the notion of intension has been redefined: intension is a mapping from possible worlds. What is traditionally called "intension" is then often called *hyperintension,* cf. e.g. (Cresswell 1975).

[7] For an overview of explications of the notion of structured proposition see e.g. (King 2016). The consequences of the fact that FP presupposes the notion of fine-grained proposition was investigated by (Giaretta 2009).

[8] Such analysis of belief sentences is one of the leading proposals, cf. e.g. (McKay and Nelson 2014).

As is well known, (classical) *type theories* classify entities into (pairwise disjoint) sets of objects called *types*;[9] in logic (as a set of derivation rules) based on type theory types provide variability-ranges for variables, while no variable takes more than one type as its variability-range. A certain individual, for example, is then said to *be of type* – i.e. *belonging to the type* – of individuals. I will call types that classify extensional entities *types for extensions.* Examples: the type of individuals, the type of characteristic functions of individuals. On the other hand, *types for intensions* classify intensional entities. Examples: the type of propositions, the type of monadic intensional operators.

Every RTT involves at least one type for intensions and this type is *ramified,* i.e. split into its *orders* 1, 2, …, *n*. The types are 1*st-order,* 2*nd-order, …,* n*th-order types for intensions.* Examples: the type of 1st-order propositions, the type of 2nd-order propositions, . . ., the type of *n*th-order propositions; the type of (say) 2nd-order propositions and the type of 2nd-order monadic intensional operators are examples of 2nd-order types. If it will be clear from the context that an object under consideration is a proposition, I will briefly say that it *is of order k* instead of "belongs to the type of *k*th-order propositions".

Not every RTT has to involve all the types mentioned above. Russell's RTT, for example, contains only the type of individuals and many ramified types for intensions: for propositions, for propositional functions of one variable, for propositional functions of two variables, … In the RTT by (Tichý 1988), which is implicitly assumed in this paper, there is only one ramified type for intensions but many types for extensions.

In RTTs similar to Church's, *cumulativity* is inbuilt to avoid a certain restrictiveness (Church 1976, 747). Here I formulate it for the case of propositions:

The Principle of Cumulativity of Propositions

> Every *k*th-order proposition (for $1 \leqslant k \leqslant n$) is also a $k + 1$st-order proposition.

The type of $k + 1$st-order propositions is thus a superset of the type of *k*th-order propositions, i.e. the types are not pairwise disjoint. The order of proposition, i.e. its belonging to a type of propositions of a certain order, can vary from context to context of consideration – being, of course, always restricted by the VCP. Nevertheless, there is always a unique lowest possible order of a

[9] The following description of type matters aims to cover the ideas of almost all classical type theories. However, in modern type theories used in computer science things can be different. Especially, types are routinely considered within computer science as letters whose interpretations are certain sets; in this paper, I speak directly about such sets as types.

proposition. With few clear exceptions, I will usually speak directly about the lowest possible order of a proposition.

Categorisation of entities into types is technically implemented in the typing rule, see Sec. 2.2 below. Admittedly, it needs a justification, which is provided mainly by the VCPs that are exposed in the rest of this section.

(Churchian) *simple theory of types,* e.g. (Church 1940), can be understood as implementing the principle I will call the *VCP for Extensions*:

The VCP for Extensions

> No function (as mapping) can be its own argument or value, or a part of these.

If this principle is applied, characteristic functions (i.e. functional correlates of sets) are stratified into hierarchy. Consequently, the famous *Russell's paradox* (Russell 1903) is prevented.[10]

Russell even realised that the simple theory of types cannot prevent the *Russell-Myhill paradox* he subsequently discovered (Russell 1903, Klement 2016). The paradox presupposes a proposition that quantifies over the totality of propositions of which it is a member – which is circular. Russell estimated that hierarchisation of propositions would be a suitable solution to the problem. His supreme logical theory, RTT, provides this (Russell 1908, Whitehead and Russell 1910–13), while it also hierarchises propositional functions.

Constructing his RTT, Russell deployed the *VCP for Intensions,* as I will call it. He formulated it in more variants; the formulation in (Russell 1908, 237), resembles.

The VCP for Intensions

> Whatever contains a variable (in the objectual sense) cannot be in the range of the variable, it is thus of (i.e. belongs to) a higher type.

To illustrate, a composed proposition containing p cannot be in the range of p. This has a consequence for quantification because a proposition which quantifies over propositions collected in the type of kth-order propositions has to be of a higher order, i.e. it must belong to a (higher-order) type over which it cannot quantify.

[10] Since functions as mappings were not present in (Russell 1903), the above-evoked solution to Russell's paradox with functions-as-mappings does not occur there (but e.g. in Church 1940).

Critics of RTT seems to overlook the message conveyed by the VCPs, though they are entailed by an even more fundamental and unimpeachable principle, namely that an entity cannot be fully specified (defined, …) by means of the entity itself (cf. e.g. (Whitehead and Russell 1910–13, 41)).

2.2 The Russellian typing rule

In conformity with what has been said in the previous section, the order of a proposition commensurates with the number of its (successively) embedded intensional operators. Thus a 'base' proposition such as 'Fido is a dog' is of order 1, whereas an *epistemic proposition* (as we may call it) such as 'Xenia knows that Fido is a dog' is of order 2; the proposition 'Yannis knows that Xenia knows that Fido is a dog' is of order 3.[11]

The existence of such intuition related to the notion of knowledge can be documented e.g. on the familiar *Socrates' paradox* (as one may call it). When uttering that he knows nothing, Socrates hardly stated a contradiction as its formalisation $\mathrm{K}\forall p\neg\mathrm{K}p$ suggests. According to the TK, Socrates rather expressed the 3rd-order proposition (the notation conforms to the rule exposed below).

$$\mathrm{K}^2\forall p^1\neg\mathrm{K}^1p^1$$

This captures our intuition that Socrates can be ignorant^1 of all 1st-order propositions while knowing^2 the 2nd-order proposition that he knows^1 no 1st-order proposition. Socrates has a kind of 'meta-knowledge'.

As indicated above, several typing rules implementing the idea have been formulated in literature (cf. esp. (Linsky 2009), (Giaretta 2009)). My following *Rule for Typing Propositions* attempts to cover various cumulative RTTs, while it abstracts from details peculiar to the particular theories.[12]

Firstly, some auxiliary terminology. Propositions built up from subordinate propositions by means of extensional connectives, e.g. $p^{\mathrm{k}} \wedge q^{k-1}$, will be called *extensionally compound propositions,* while propositions built up using

[11] (Carrara and Fassio 2011, 189–90) doubted the very discrimination between 'base' and epistemic propositions that is involved in the TA. According to their intuition, the proposition 'Xenia lies in the bed' can be seen as an epistemic proposition because it informs us that Xenia does not know what happens in the kitchen. However, this objection can be challenged by referring to the fact that the latter proposition is logically unrelated to the former, and that typing is based on the nature of the former proposition, not some other proposition entailed by it in the minds of some agents.

[12] The RTT capable of convenient treatment of all details of the notions such as knowledge, modality, belief attitudes, etc., is (Tichý 1988). In that framework, "K^k" expresses knowledge operator and denotes possible-world intension whose values are relations-in-extension between individuals and propositions.

intensional operators, e.g. $K^m p^k$, will be called *intensionally compound propositions.* (Step ii. needs supplementation which occurs below.)

The Rule for Typing Propositions

i. The lowest possible order of any proposition involving no intensional operator is 1. Let p^k be any proposition of order k, for $1 \leqslant k$.

ii. The lowest possible order of an intensionally compound proposition such as $K^m p^k$, for $1 \leqslant k \leqslant m$, is $m + 1$.

iii. The lowest possible order of an extensionally compound proposition is identical with the order of its subproposition that has the greatest order in it.

Obviously, the language $\mathcal{L}$ of multimodal logic is modified in such a way that "p" ("q", etc.) is excluded in favour of "p^k" ("q^k", etc.) and "Kp" is excluded in favour of "$K^m p^k$, for $k \leqslant m$; "k" is omitted if "p" is not a propositional letter. Though modal operators □ and ◇ can be treated as intensional operators operating on propositions, to simplify typing I will consider them as operating on possible-world propositions by means of which formulas such as "p^k" are interpretable (on the denotational level); as mere extensions, possible-world propositions belong to one unramified type.

Examples.

a. The proposition

$$p^1$$

is a 1st-order proposition.

b. As regards compound propositions, their (lowest possible) order is not written in the superscript, it is easily derivable using the Rule for Typing Proposition. For instance,

$$K^1 p^1$$

is of order 2 (cf. Step ii.). Thanks to the Cumulativity Principle, m in (the record of) $K^m p^k$ need not equal k, it can be a greater number. In the 3rd-order proposition $K^2 p^1$, for example, the operator K^2 is applied to the 1st-order proposition p^1 that serves here as a 2nd-order argument for K^2.

c. The extensionally compound proposition

$$p^1 \wedge \neg K^1 p^1$$

is of order 2 because its subproposition $K^1 p^1$, which has the greatest order in it, is of order 2 (cf. Step iii.). The epistemic proposition

$$K^2(p^1 \wedge \neg K^1 p^1)$$

is of order 3 (cf. Steps ii. and iii.).

d. Expressions

$$\text{“}K^1 p^2\text{”, “}K^1 K^2 p^1\text{”, “}K^2 K^2 p^1\text{”, “}K^1(p^2 \wedge q^1)\text{”}$$

do not express any propositions – because of VCP, no such propositions are specifiable. In other words, the formulas such as "$K^1 p^2$" are meaningless strings of letters that are not coherently interpretable (I will return to this issue following the next paragraph).

A comment on Step ii. Step ii. partly presupposes definition of the *order of the operator* K^m (which is usually offered within RTTs). Such definition says that if, in some context, the order $K^m p^k$ is $m + 1 + l$, for $m \geqslant k$ and $l \geqslant 0$, the order of K^m is $m + 1 + l$ as well. To explain: using λ-abstraction one can transform the proposition $K^m p^k$ into the operator

$$\lambda p^k K^m p^k$$

which is convertible, by η-rule of λ-calculus, to K^m. In $\lambda p^k K^m p^k$, p^k is a variable whose range is the set S of propositions on which the operator K^m is applicable. The VCP for Intensions tells us that the proposition $K^m p^k$ cannot be in S. S must consist of propositions of a lower order than the order of $K^m p^k$. Consequently, if the order of p^k is k, the order of $K^m p^k$, and thus also of K^m as such, must be strictly greater than k.

Some philosophers might perhaps object that the incapability of the formulas listed in the above point d. to represent anything only indicates the restrictiveness of the adopted formalism. However, the view would be a serious misunderstanding of what I have stated so far. It is undoubtedly true that the language $\mathcal{L}$ considered here is restrictive. This is because its purpose is to represent existing propositions. As is clear from Sec. 2.1, no proposition seemingly represented by (say) "$K^1 p^2$" exists. A possible way out is to assume a knowledge operator that does not operate on propositions, but e.g. on (the names of) sentences. Since the VCP for Intensions has nothing to do with sentences, "$K^1 \ulcorner p^2 \urcorner$" may be treated as wff. But this construal of knowledge is incompatible with the greatest assumption of this paper, namely that sentences express (structured, fine-grained) propositions.

3 The solution to FP by Russellian typing knowledge

Knowledge operators and propositions containing them are governed by the system of derivation rules that are the results of type-theoretic specification of their untyped predecessor.

$(\mathrm{Dist}^{2(1,2)})$	$\mathrm{K}^2(p^1 \wedge q^2) \vdash \mathrm{K}^2 p^1 \wedge \mathrm{K}^2 q^2$
$(\mathrm{Fact}^{2(2)})$	$\mathrm{K}^2 p^2 \vdash p^2$
(MP^2)	$p^2 p^2 \supset q^2 \vdash q^2$

The TA to FP reveals the consequences of such specification.

Keeping the order as low as possible, while appropriately also typing the theses of non-omniscience and verificationism, the key part of FP is amended as follows.

The typed version of the first part of FP

1'. $\exists p^1(p^1 \wedge \neg \mathrm{K}^1 p^1)$	$(\mathrm{NonOmn}^{1(1)})$
2'. $p^1 \wedge \neg \mathrm{K}^1 p^1$	an instance of 1.
3'. $\forall p^2(p^2 \supset \Diamond \mathrm{K}^2 p^2)$	$(\mathrm{Ver}^{2,2(2)})$
4'. $(p^1 \wedge \neg \mathrm{K}^1 p^1) \supset \Diamond \mathrm{K}^2\, (p^1 \wedge \neg \mathrm{K}^1 p^1)$	substituting 2'. for p^2 in 3'.
5'. $\Diamond \mathrm{K}^2(p^1 \wedge \neg \mathrm{K}^1 p^1)$	from 4'. and 2'. by(MP^2)
6'. $\mathrm{K}^2(p^1 \wedge \neg \mathrm{K}^1 p^1)$	assumption *per absurdum*
7'. $\mathrm{K}^2 p^1 \wedge \mathrm{K}^2 \neg \mathrm{K}^1 p^1$	from 6'. by $(\mathrm{Dist}^{2(1,2)})$
8'. $\mathrm{K}^2 p^1 \wedge \neg \mathrm{K}^1 p^1$	from 7'. by $(\mathrm{Fact}^{2(2)})$

The proposition 8'. is not a contradiction, and so the *reductio* is thus blocked.

As rightly noticed by (Williamson 2000, 281), the proposition 8'. is only non-contradictory provided the following rule is not valid:

$$\mathrm{K}^2 p^1 \vdash \mathrm{K}^1 p^1$$

The reasons for its invalidity have been discussed by several authors, cf. (Williamson 2000), (Paseau 2008), (Linsky 2009), my reason is given in Sec. 3.1 below.

Before proceeding further, I will deflect an important argument raised against the TA by (Carrara and Fassio 2011, 191). The argument attempts to show that the TA to FP is internally incoherent, since it misapplies its own typing principles. Its authors argued that the TA must admit (notation adapted)

$$\mathrm{K}^1 p^1 \wedge \mathrm{K}^2 \neg \mathrm{K}^1 p^1$$

instead of 7'. because $\mathrm{K}^2 p^1$ is (allegedly) not type-theoretically possible. Using (Fact), one would then derive the contradiction $\mathrm{K}^1 p^1 \wedge \neg \mathrm{K}^1 p^1$ and the *reductio* would not be blocked. However, the TA employing a Churchian RTT is not affected by this argument because such RTT implements the Principle of Cumulativity and so embraces propositions such as K^2p^1. One is thus not forced to adopt $K^1p^1 \wedge K^2\neg K^1p^1$ instead of 7'. In other words, the argument is only directed against a TA framed within a non-cumulative type-theoretic framework.[13]

3.1 The invalidity of the rule $K^2p^1 \vdash K^1p^1$ *and two inconclusive arguments against typing knowledge*

In this section, I am going to reject three arguments against the TA. Two of them (c. and b.) were constructed by Florio with Murzi and Jago, but the first one (a.) was proposed, as well as dismissed, by the authors who offered the TA to FP.

a. As mentioned in the preceding section, the typing solution to FP requires that the rule $K^2p^1 \vdash K^1p^1$ is considered invalid. However, mere principles of type-theoretic correctness provide no reason for its adoption or rejection. (Williamson 2000), (Linsky 2009) and mainly (Paseau 2008; 2009) attempted to explain the invalidity of the rule by appealling to *epistemic access* to p^1.

(Linsky 2009) also cautiously wrote that propositions are typed due to their content, while logical relations between them reflect procedures for determination of epistemological states. This idea can be generalised. I suggest that it is sufficient to maintain that p^1 is known if it is *justified* by a certain *reason.* The reason surely constitutes the attitude towards p^1 being knowledge, rather than mere belief or contemplation.

$$\text{Justified}^m p^k \dashv\vdash \exists q^m\ \text{ReasonFor}^m(q^m,p^k) \qquad \text{for } k \leqslant m$$

The reasons q^m serving for justification of p^k can be, say, inevitable steps in a derivation of p^k. Some q^m can be epistemic propositions or propositions about an epistemic route to p^k – these can be e.g. *m*th-order propositions certifying that p^k was acquired from a reliable source. But one can leave the exact nature of q^m undecided and so not relying exclusively on the 'epistemic explanation' of the invalidity of the rule in question.

It follows from these considerations that for knowing^2 p^1 one needs a certain q^2 which helps to justify^2 p^1. The respective reason^2 thus makes K^2p^1 irreducible to K^1p^1. The rule $K^2p^1 \vdash K^1p^1$ is therefore invalid.

b. This conclusion even leads to rejection of the argument raised against the TA by (Jago 2010). Jago based it on the principle (notation adapted)

[13] The consequences of acceptance of the Cumulativity Principle for the solution of paradoxes were studied by (Peressini 1997); the present case is not mentioned there.

$$(\mathrm{KT}^{2(2)}) \qquad \mathrm{K}^1 p^1 \equiv \mathrm{K}^2\mathrm{T}^1 p^1$$

where T^1 operates on typed propositions. Here I show only four most important steps of Jago's modification of FP:

1^j. $\mathrm{K}^2(\mathrm{T}^1 p^1 \wedge \neg\mathrm{K}^1 p^1)$	assumption
2^j. $\mathrm{K}^2\mathrm{T}^1 p^1 \wedge \mathrm{K}^2\neg\mathrm{K}^1 p^1$	from 1^j. by $(\mathrm{Dist}^{2(2,2)})$
3^j. $\mathrm{K}^1 p^1 \wedge \mathrm{K}^2\neg\mathrm{K}^1 p^1$	from 2^j. by $(\mathrm{KT}^{2(2)})$
4^j. $\mathrm{K}^1 p^1 \wedge \neg\mathrm{K}^1 p^1$	from 3^j. by $(\mathrm{Fact}^{2(2)})$

However, one should notice that $\mathrm{T}^1 p^1$ is equivalent to p^1 by the well-known T-axiom. Jago's $(\mathrm{KT}^{2(2)})$ is thus equivalent to

$$\mathrm{K}^1 p^1 \equiv \mathrm{K}^2 p^1$$

which can hardly be maintained by a proponent of the TK, cf. the considerations stated in a. (and at the beginning of Sec. 2.2).

c. The *Paradox of Idealization* by (Florio and Murzi 2009), also allegedly unresolvable by the TA, differs from FP even more. The paradox utilises the modified (Ver)

$$\exists p(p \wedge \forall x(\mathrm{K}_x p \supset \mathrm{I}x))$$

where "Kx" represents 'it is known to x that' and "I" represents 'has epistemic abilities which finitely exceed ours'. It is further assumed that the proposition (which is derived from other assumptions that need not to concerns us now)

$$\mathrm{K}_a(q \wedge \neg\exists x\mathrm{I}x)$$

where q is only knowable by idealised agents, and so a is an idealised agent:

$$\mathrm{I}a$$

By (Dist), (Fact) and Existential Generalization one easily derives the contradiction

$$\exists x\mathrm{I}x \wedge \neg\exists x\mathrm{I}x$$

Regardless of the character of q, however, the puzzle is evidently based on the assumption that an idealised agent 'knows' that no idealised agent exists. This is obviously self-refuting, so one need not to deploy the TK for rejection of the inference.

4 The solution to the Revenge Form of FP by the Russellian typing knowledge

As mentioned in the introductory section, the first version of the *Revenge Form of FP for the TA*,[14] which serves as an argument against the TA to FP, was suggested by Williamson:

> We seem able to grasp the idea that p is *totally unknown,* in a sense which entails that *p* is unknowni for each level *i*, but which does not entail that *p* is untrue. If so, we can simply adapt Fitch's argument by considering the proposition that *p* is totally unknown truth, since that proposition cannot be knowni for any level *i*. Naturally, such quantification over levels must be handled with great care (Williamson 2000, 281)

The inference considered by Williamson, and also mentioned in (Carrara and Fassio 2011, 188), was further elaborated in (Hart 2009, 322–323). Hart explicitly stated its conclusion that every truth is necessarily known at some type level t.[15]

Here I offer symbolisation of the whole intended inference.[16]

The Revenge Form of FP for the TA

1^r.	$p \wedge \forall t \neg \mathrm{K}^t p$	assumption (of order t + 1) *per absurdum*
2^r.	$\Diamond \mathrm{K}^{t+1}(p \wedge \forall t \neg \mathrm{K}^t p)$	from 1^r. by (Ver)
3^r.	$\Diamond(\mathrm{K}^{t+1} p \wedge \mathrm{K}^{t+1} \forall t \neg \mathrm{K}^{\mathrm{t}} p)$	from 2^r. by '(Dist)'
4^r.	$\Diamond(\mathrm{K}^{t+1} p \wedge \forall t \neg \mathrm{K}^t p)$	from 3^r. by '(Fact)'
5^r.	$\Diamond(\mathrm{K}^{t+1} p \wedge \neg \mathrm{K}^{t+1} p)$	from 4^r. by Universal Instantiation
6^r.	$\neg(p \wedge \forall t \neg \mathrm{K}^t p)$	*reductio* (since 5^r. is a contradiction)
7^r.	$\Box\neg(p \wedge \forall t \neg \mathrm{K}^t p)$	from 6^r. by (Nec)
8^r.	$\Box(p \supset \neg\forall t \neg \mathrm{K}^t p)$	from 7^r. by classical propositional logic
9^r.	$\Box(p \supset \exists t \mathrm{K}^t p)$	from 8^r. by De Morgan Law for Quantifiers
10^r.	$\Box\forall p(p \supset \exists t \mathrm{K}^t p)$	from 9^r. by Universal Generalization

[14] The notion of revenge paradox here employed conforms to (Beall 2008): the strengthened Liar based on "This sentence is not true.", for instance, is the revenge form of the Liar for the trivalent approach that relies on the weak falsity predicate "is false" and cannot safely handle its strengthened form "is not true".

[15] The criticism resembles the criticism of Russell's RTT by (Gödel 1944), (Fitch 1964) and (Priest 2006), which I cannot discuss here for reasons of brevity.

[16] The key formula "$\forall t \neg \mathrm{K}^t p$" was suggested by (Carrara and Fassio 2011). The inference does not entirely parallel FP but it is largely similar.

However, the inference is suspicious from the very beginning. Consider the formula "$\forall t \neg K^t p$" first: if the use of $\forall$ is not hollow, $\forall$ binds some variable within "$\neg K^t p$". Evidently, that variable is *t*. Yet having such an assumption, one departs from the official notation of type theories in which "K^k" is a primitive symbol, so the numeral "*k*" is its irreplaceable part – it is not a variable.

Because the inference presupposes that *t* is a variable bound by $\forall$, to challenge the (Russellian) TA one must admit that, since formulas (and propositions) of the TA have to be typed,

> "$K^t p$" stands for "$K'^m(p^k, t)$", for $1 \leqslant k \leqslant m$, where K'^m is a novel binary operator.

There are several possibilities as to how to interpret "*t*" and, consequently, how to understand "K'^m". On the most probable reading, *t* ranges over natural numbers $\mathbb{N}^+$ that represent orders. Let us adopt this as a working hypothesis. For further simplification of our considerations assume $m = k$. (As we will see below, my findings will not hinge on the two particular assumptions.)

Now let us focus on the crucial steps of the correctly rendered version of the Revenge Form of FP. Its second proposition is

$$2^{r}.\ \Diamond K'^{k+1}(p^k \wedge \forall t \neg K'^k(p^k, t), t + 1)$$

Applying appropriate rules of distributivity and factivity that govern these two novel operators K'^k and K'^{k+1} one infers

$$5^{r}.\ \Diamond(K'^{k+1}(p^k, t + 1) \wedge \forall t \neg K'^k(p^k, t))$$

Applying Universal Instantiation to the right conjunct of 5^{r}. one gets

$$6^{r}.\ \Diamond(K'^{k+1}(p^k, t + 1) \wedge \neg K'^k(p^k, t + 1))$$

However, 6^{r}. is not a contradiction. Consequently, the rest of the untyped Revenge Form of FP cannot be derived.

The Revenge Form of FP for the TA is thus based on a hidden equivocation. If the (seemingly correctly typed) formulas of the inference are disambiguated and properly typed, and thus matched with some existing propositions, the inference does not go through.

5 Conclusions

In this paper, I discriminated between two kinds of the TA to FP, the Tarskian and the Russellian (Sec. 1.2); they both provide a solution to FP. The distinction helps to offer a compelling justification of the TA because the Russellian TA is induced by non-circular formation of intensional entities, namely propositions and operators operating on them, most notably the knowledge operator; this results in the VCP for Intensions and the Typing Rule for Propositions (Sec. 2.1). Thus, the Russellian TA cannot be refused as an *ad hoc* solution, which was an important point of criticism e.g. in (Carrara and Fassio 2011). The TA solution to FP is in fact only a by-product of the approach.

The TA solution to FP provides a successful blocking of the respective paradoxical inference (Sec. 3), which has been known since (Williamson 2000). The blocking assumes invalidity of a certain rule widely discussed in literature, for which I offered another reason (Sec. 3.1); this also helped to solve the modified FP by (Jago 2010).

The Russellian TA reinforced in this paper enabled an easy rejection (Sec. 4) of the Revenge Form of FP for the TA by (Williamson 2000), (Hart 2009), (Carrara and Fassio 2011). The paradox covertly deploys an unusual binary operator of knowledge at type-level, while violating typing principles.[17]

References

Andrews, P. 2014. "Church's Type Theory", in E.N. Zalta (ed.), The Stanford Encyclopedia of Philosophy (Spring 2014 Edition), https://plato.stanford. edu/archives/spr2014/entries/type-theory-church/ [accessed February 23rd 2017].

Beall, J. C. (ed.) 2008. *Revenge of the Liar: New Essays on the Paradox* (Oxford University Press).

Brogaard, B., Salerno, J. 2013. "Fitch's Paradox of Knowability", in E.N. Zalta (ed.), *The Stanford Encyclopedia of Philosophy (Winter 2013 Edition),* https://plato.stanford.edu/archives/win2013/entries/fitch-paradox/ [accessed February 23rd 2017].

Carrara, M., Fassio, D. 2011. "Why Knowledge Should Not Be Typed: An Argument against the Type Solution to the Knowability Paradox", *Theoria* 77(2), pp. 180–193.

Chatzikyriakidis, S., Luo, Z. (eds.) 2017. *Modern Perspectives in Type-Theoretical Semantics* (Springer).

Church, A. 1940. "A Formulation of the Simple Theory of Types", *Journal of Symbolic Logic* 5(2), 56–68.

Church, A. 1973–74. "Russellian Simple Type Theory", *Proceedings and Addresses of the American Philosophical Association* 47(2), 1–33.

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Church, A. 1976. "A Comparison of Russell's Resolution of the Semantical Antinomies with that of Tarski", *Journal of Symbolic Logic* 41(4), 747–760.

Church, A. 1984. "Russell's Theory of Identity of Propositions", *Philosophia Naturalis* 41(2/4), 513–522.

Church, A. 2009. "Referee Reports on Fitch's "A Definition of Value" ", in J. Salerno (ed.), *New Essays on the Knowability Paradox* (Oxford University Press), 13–20.

Coquand, T. 2015. "Type Theory", in E.N. Zalta (ed.), *The Stanford Encyclopedia of Philosophy (Summer 2015 Edition)*, https://plato.stanford. edu/archives/sum2015/entries/type-theory/ [accessed February 23rd 2017].

Cresswell, M. J. 1975. 'Hyperintensional Logic", *Studia Logica* 34(1), 26–38.

Fitch, F. B. 1963. "A Logical Analysis of Some Value Concepts", *The Journal of Symbolic Logic* 28(2), 135–142.

Fitch, F. B. 1964. "Self-reference in Philosophy", *Mind* 55(217), 64–73.

Florio, S., Murzi, J. 2009. "The Paradox of Idealization", *Analysis* 69(3), 461–469.

Giaretta, P. 2009. "The Paradox of Knowability from a Russellian Perspective", *Prolegomena* 8(2), 14 1–158.

Gödel, K. 1944. "Russell's Mathematical Logic", in P.A. Schilpp *(ed.),'The Philosophy of Bert rand Russell* (Evanston, Chicago: Northwestern University), 125–153.

Halbach, V. 2008. "On a Side Effect of Solving Fitch's Paradox by Typing Knowledge", *Analysis* 68(2), 114–120.

Hart, W. D. 2009. "Invincible Ignorance", in Salerno, J. (ed.), *New Essays on the Knowability Paradox* (Oxford University Press), 321–323.

Irvine, A. D. 2016. "Principia Mathematica", in E.N. Zalta (ed.), *The Stanford Encyclopedia of Philosophy (Winter 2016 Edition)*, https://plato.stanford. edu/archives/win2016/entries/principia-mathematica/ [accessed February 23rd 2017].

Jago, M. 2010. "Closure on Knowability", *Analysis* 70(4), 648–659.

Janssen, T. M. V. 2017. "Montague Semantics", in E.N. Zalta (ed.), *The Stanford Encyclopedia of Philosophy (Spring 2017 Edition)*, https://plato.stanford. edu/archives/spr2017/entries/montague-semantics/ [accessed March 13th 2017].

Kamareddine, F., Laan, T., Nederpelt, R. (2010), *A Modern Perspective on Type Theory. From Its Origins until Today* (Springer).

Kaplan, D., Montague, R. 1960. "A Paradox Regained", *Notre Dame Journal of Formal Logic* 1(3), 79–90.

King, J. C. 2016. "Structured Propositions", in E.N. Zalta (ed.), *The Stanford Encyclopedia of Philosophy (Winter 2016 Edition),* https://plato.stanford. edu/archives/win2016/entries/propositions-structured/ [ac-cessed February 23rd 2017].

Klement, K. C. 2016. "Russell-Myhill Paradox", in *The Internet Encyclopedia of Philosophy, http://www. iep. utm. edu/par-rusm/* [accessed February 23rd 2017].

Kripke, S. A. 1975. "Outline of a Theory of Truth", *Journal of Philosophy* 72(19), 690–716.

Landini, G. 1998. *Russell's Hidden Substitutional Theory* (Oxford University Press).

Linsky, B. 1999. *Russell's Metaphysical Logic* (Stanford, CSLI Publications).

Linsky, B. 2009. "Logical Types in Some Arguments about Knowability and Belief", in J. Salerno (ed.), *New Essays on the Knowability Paradox* (Oxford University Press), 163–179.

McKay, T., Nelson, M. 2014. "Propositional Attitude Reports", in E.N. Zalta (ed.), *The Stanford Encyclopedia of Philosophy (Spring 2014 Edition)*, https://plato.stanford.edu/archives/spr2014/entries/prop-attitude-reports/ [accessed February 23rd 2017].

Paseau, A. 2008. "Fitch's Argument and Typing Knowledge", *Notre Dame Journal of Formal Logic* 49(2), 153–176.

Paseau, A. 2009. "How to Type: Reply to Halbach", *Analysis* 69(2), 280–286.

Peressini, A. F. 1997. "Cumulative versus Noncumulative Ramified Types", *Notre Dame Journal of Formal Logic* 38(3), 385–397.

Priest, G. 2006. *In Contradiction: A Study of the Transconsistent,* 2nd edition (Oxford University Press).

Rosenblatt, L. 2014. "The Knowability Argument and the Syntactic Type-Theoretic Approach", *Theoria: An International Journal for Theory, History and Foundations of Science* 29(2), 201–221.

Russell, B. 1903. *Principles of Mathematics* (Cambridge University Press).

Russell, B. 1908. "Mathematical Logic as Based on the Theory of Types", *American Journal of Mathematics* 30(3), 222–262.

Salerno, J. (ed.) 2009. *New Essays on the Knowability Paradox,* (Oxford University Press).

Tarski, A. 1956. "The Notion of Truth in Formalized Languages", in *Logic, Semantics and Metamathematics* (Oxford University Press), 152–278.

Thomason, R. H. (ed.) 1974. *Formal Philosophy. Selected Papers of Richard Montague* (New Haven: Yale University Press).

Tichý, P. 1988. *The Foundations of Frege's Logic* (Walter de Gruyter).

The Univalent Foundations Program 2013. *Homotopy Type Theory: Univalent Foundations of Mathematics* (Institute for Advanced Study), https://homotopytypetheory.org/book/ [accessed February 23rd 2017].

Williamson, T. 2000. *Knowledge and its Limits* (Oxford University Press).

Whitehead, A. N., Russell, B. 1910–13. *Principia Mathematica* (Cambridge University Press).