

Residual Filtering Methods for the Purpose of Computing the Coordinate Transformation Distortion Model

Marko ŠLJIVARIĆ, Milan REZO, Marko PAVASOVIĆ

Abstract: For the purpose of coordinate transformation between two different horizontal geodetic data, the inherited one, the Croatian State Coordinate System (HDKS) and the new (official) one, Croatian Terrestrial Reference System 1996 (HTRS96) utilizing seven parameters coordinate transformation (Bursa–Wolf model) and distortion model, three methods of residual filtering were introduced in this paper, using our own developed algorithms. Each algorithm is based on a different ratio: the ratio of resultant residual and surface function of point's environment, the ratio of difference and mutual distance of neighboring resultants in the area, and the ratio of difference and mutual distance of neighboring resultants in tangential plane. Thus, the concept of the theoretical residuals has been introduced, as well as the idea of semi-iterative procedure that is conducted through the automatization of filtering in spatial or just in plane sense and operates on the principle of a combination of classic Bursa–Wolf transformation and TIN tessellation under the terms of Delaunay's triangulation and Voronoi's polygons.

Keywords: distortion; polygon; residual; triangulation; TIN

1 INTRODUCTION

As a former member of Austrian–Hungarian Monarchy (in 1901), the Republic of Croatia has historically inherited the horizontal (2D) coordinate reference system called *Croatian State Coordinate System* (cro. *Hrvatski državni koordinatni sustav*– HDKS), *Hermannskogel* or *HR1901*. HDKS was established by means of astro–geodetic measurements of the Military Geodetic Institute (MGI) Vienna in 1st order triangulation network, adjusted in 7 separate blocks [4]. The fundamental point (the origin) of HDKS was situated at Hermannskogel hill, near Vienna. For the mathematical approximation of the Earth's shape, the *Bessel 1841* rotation ellipsoid was selected.

Following the modern European trends, as well as the development of satellite positioning systems (primarily GPS and GLONASS), poor documentation of inherited horizontal reference system and its inhomogeneity (1–2 m at state level, few dm at county level and ≈10 cm at city levels), were more than a good reason to introduce changes, that is, to redefine a national horizontal (as well as height and gravimetric) system [4].

On 4th August 2004, the Government of the Republic of Croatia passed the *Decree on Establishing New Official Geodetic Datums and Map Projections of the Republic of Croatia* [8]. According to the *Decree*, the Republic of Croatia has adopted a new horizontal reference system called *Croatian Terrestrial Reference System 1996* (cro. *Hrvatski terestrički referentni sustav 1996*– HTRS96), based on *European Terrestrial Reference System 1989* (ETRS89) with *Geodetic Reference System 1980* (GRS80) level–ellipsoid as a mathematical and physical approximation of the Earth's shape [4].

In the procedure of developing coordinate transformation model between these two coordinate reference systems of the Republic of Croatia, coordinate residual filtering represents an important part and will be discussed further in this paper.

2 PROBLEMS WITH RESIDUAL FILTERING OPTIMIZATION

Besides collinearity of individual points in two separate coordinate reference systems and its exclusion from further calculations, zero iteration of coordinate transformation parameters calculation indicates the residual amounts at some points of a few dozens and greater than 100 m. In general, those deviations are easily spotted, and outlier points are progressively excluded from further calculations.



Figure 1 Zero iteration residuals [1]

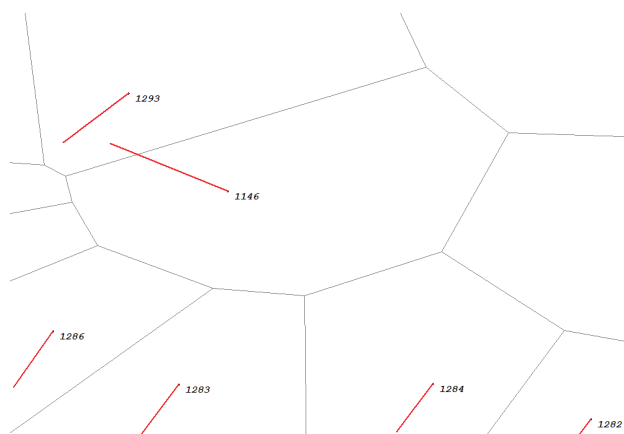


Figure 2 Zero iteration residuals [1]

Residuals that possibly meet the expected accuracy related to their amount (length) but locally do not match other nearby residuals in length and direction – e.g. point 270 (Fig. 1) or point 1146 (Fig. 2) represent a much greater problem.

It is also important to point out that the residual at some point with relatively small amount does not have to be acceptable for the whole model – e.g. point 1134 (Fig. 3).

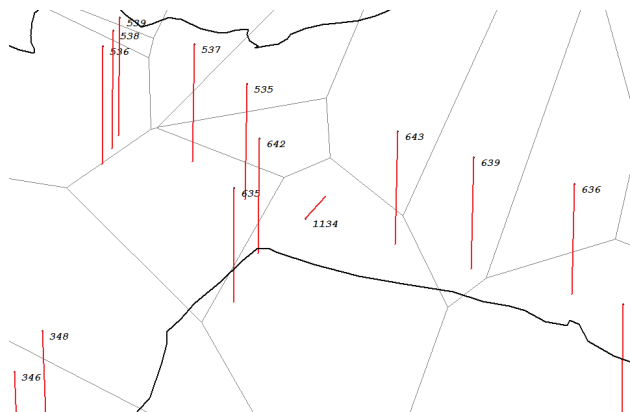


Figure 3 Zero iteration residuals [1]

Borders of Voronoi’s polygons drawn with grey lines in Fig. 1, 2 and 3, i.e. with blue lines in Fig. 4 define geometrical–topological neighborhood between the points.

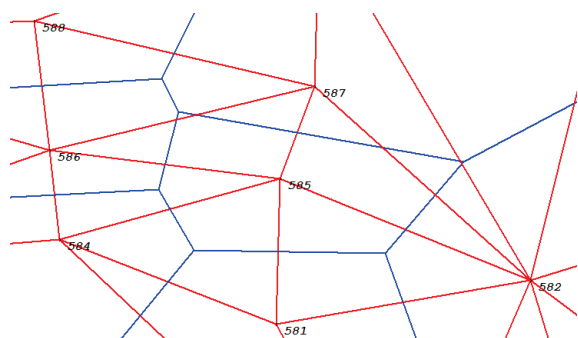


Figure 4 Relationship between Voronoi’s polygons (blue) and Delaunay’s triangles (red) [1]

The borders of Voronoi’s polygons define the area which is bounded by the part of a plane with specific property indicating that every point of that plane is the nearest one to the point that a polygon is formed around. Voronoi’s polygons or diagram [2] are a topological variant of Delaunay’s triangulation [3] (Fig. 4) that consists of points connected to a network of triangles – so called triangulated irregular network (TIN) provided that no point in a set is located inside a described circle of each triangle in the network. Unlike interpolation or approximation modeling of a statistical sample of any value in plane or space being a regionalized variable, the models based on TIN are more deterministically oriented and therefore more suitable for local analysis, i.e. the detection of vicinity.

3 TERM OF "THEORETICAL RESIDUAL"

The closure of Voronoi’s polygon for an individual point is made by connecting the centers of all described circles of triangles with the mentioned point being their vertex, hence, the point 585 in Fig. 4 has 5 neighbors (in counterclockwise direction): the points 581, 582, 587, 586

and 584. Taking into consideration the term "neighborhood" defined in that way in every point in a set, the theoretical value of residual can be computed as a weighted mean of adjacent real residuals. For the situation from Fig. 4, the theoretical residual is given by:

$$\vec{w}_{585} = \frac{P_{581}\vec{v}_{581} + P_{582}\vec{v}_{582} + P_{587}\vec{v}_{587} + P_{586}\vec{v}_{586} + P_{584}\vec{v}_{584}}{P_{581} + P_{582} + P_{587} + P_{586} + P_{584}} \quad (1)$$

where \vec{w}_i is vector of theoretical residual, \vec{v}_i vector of real residual, p_i weight of individual vector given by inverse squared distance between neighboring points:

$$p_i = \frac{1}{d_{i,j}^2} \quad (2)$$

The difference between real and theoretical vector of residuals will be:

$$\vec{r} = \vec{v} - \vec{w} \quad (3)$$

that is, by each spatial Cartesian vector components:

$$r_X = v_X - w_X \quad (4)$$

$$r_Y = v_Y - w_Y \quad (5)$$

$$r_Z = v_Z - w_Z \quad (6)$$

The module of resultant vector is the value that points to how good the real residual fits in its environment. In Fig. 5, the vectors of real residuals are colored red, the theoretical blue and the resultant green.

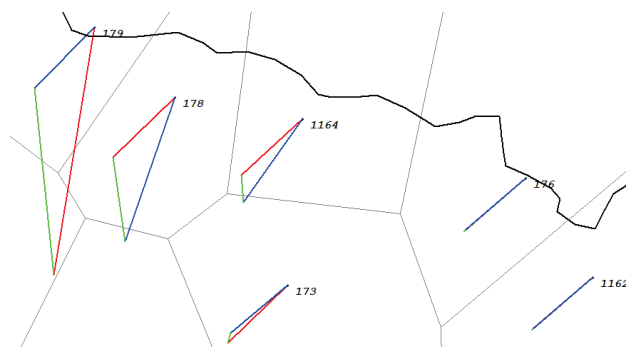


Figure 5 Vectors of real (red), theoretical (blue) and resultant (green) residuals [1]

Table 1 Numerical values for all three vectors of all three selected points [1]

Point 178	Point 179	Point 176
Real distortion:		
$v_X = +8,109 \text{ m}$	$v_X = +22,940 \text{ m}$	$v_X = +7,713 \text{ m}$
$v_Y = -9,316 \text{ m}$	$v_Y = -0,463 \text{ m}$	$v_Y = -9,384 \text{ m}$
$v_Z = -5,212 \text{ m}$	$v_Z = -20,018 \text{ m}$	$v_Z = -4,622 \text{ m}$
Theoretical distortion:		
$v_X = +14,672 \text{ m}$	$v_X = +8,077 \text{ m}$	$v_X = +7,393 \text{ m}$
$v_Y = -4,874 \text{ m}$	$v_Y = -8,886 \text{ m}$	$v_Y = -9,218 \text{ m}$
$v_Z = -11,868 \text{ m}$	$v_Z = -5,251 \text{ m}$	$v_Z = -4,635 \text{ m}$
Distortion resultant:		
$v_X = -6,563 \text{ m}$	$v_X = +14,862 \text{ m}$	$v_X = -0,320 \text{ m}$
$v_Y = -4,442 \text{ m}$	$v_Y = +8,423 \text{ m}$	$v_Y = -0,167 \text{ m}$
$v_Z = -6,656 \text{ m}$	$v_Z = -14,766 \text{ m}$	$v_Z = +0,013 \text{ m}$
$d_{YW} = 10,35 \text{ m}$	$d_{YW} = 22,58 \text{ m}$	$d_{YW} = 0,36 \text{ m}$
$d/P = 0,000 \text{ m}^{-1}$	$d/P = 0,000 \text{ m}^{-1}$	$d/P = 0,001 \text{ m}^{-1}$
$P_{VOR} = 682889 \text{ m}^2$	$P_{VOR} = 5271632 \text{ m}^2$	$P_{VOR} = 514 \text{ m}^2$

Let's pay attention to the points 178 and 179 (Fig. 5). While the point 179 has a bigger real (red) than a theoretical (blue) residual, the point 178 has a bigger theoretical than a real residual due to the influence of observed values from the point 179. At first glance, the point 176 (Fig. 5) has got hardly any resultant vector although its module is approx. 36 cm (Tab. 1) which is not immediately seen, because its real and theoretical residuals coincide in direction.

4 SIMPLIFIED APPROACH TO RESIDUAL FILTERING

The general idea is to develop the algorithm for automatic filtering of "unwanted" residuals iteratively in several general steps:

- computation of transformation parameters,
- triangulation of Delaunay's/Voronoi's TIN network,
- computation of theoretical and resultant residuals,
- exclusion of only "the worst" point from set,
- repetition of steps from a) to d) or completion of algorithm according to some "criteria".

Simplified approach to residual filtering – e.g. in each iteration, the point with max. real residual \bar{v}_{max} to be excluded is not appropriate because the area with identical points in two geodetic data is gradually getting smaller (Fig. 6).

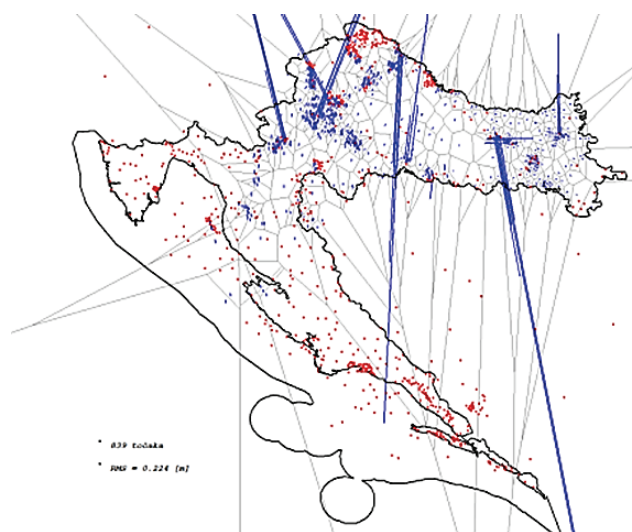


Figure 6 Gradual disappearance of points (red) by increasing the number of iterations [1]

A significant loss of points (marked red in Fig. 6) is noticeable in the Istra and Primorsko–Goranska county already in the two–hundredth step. In the three–hundredth step, the points are starting to significantly vanish in Dalmatia region, and in the four–hundredth, they totally vanish as well as in the Međimurje region. Finally, the points are vanishing from the whole Lika region, and the continuation of this procedure would probably result in reducing the Croatian territory (without Slavonija and Baranja region) to "leftovers".

The reason for such behavior of the designed algorithm is certainly the heterogeneous character of inherited geodetic basis at the territory of the Republic of Croatia,

i.e. the blocks of Austrian–Hungarian triangulation (Fig. 7) [4, 5].

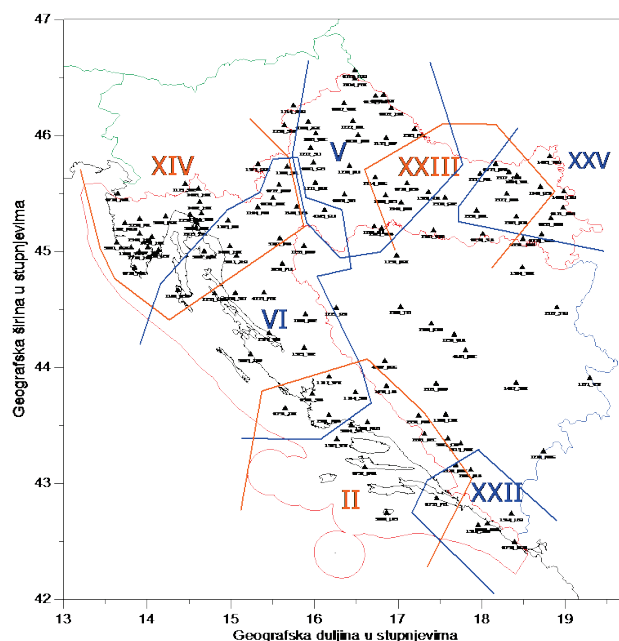


Figure 7 Adjustment blocks of Austrian–Hungarian triangulation [4]

In Fig. 8 it is possible to see the orientation of the mentioned geodetic basis according to the new (official) geodetic datum of the Republic of Croatia (HTRS96). Rotation groups are seen and are classified according to azimuth in the southern part (mostly 2nd quadrant for 2nd block) and directly towards south in the extreme southern part (22nd block). In most part of western Croatia, azimuths are oriented in 1st quadrant (14th block), in Slavonia towards southwest (25th block) and in northern part of the state towards west and northwest (5th block). The orientation for 5th block of Austrian–Hungarian triangulation in the area of Lika shows very similar character like the one in the area of Dalmatia (2nd block), while for 23rd block in western Slavonija, no significant characteristics can be applied.

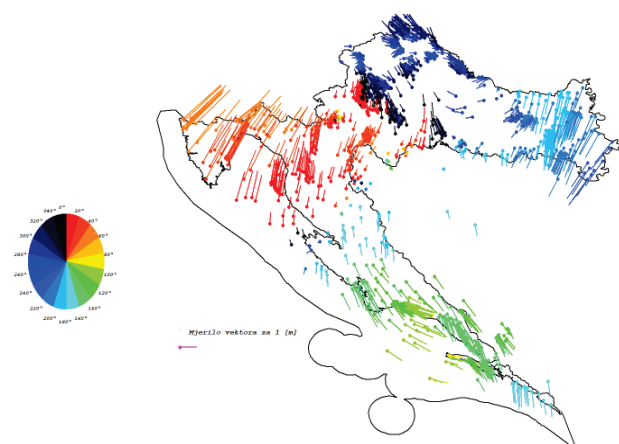


Figure 8 Classification of directions of residuals on Croatian state territory [1]

Furthermore, the examples of different optimization filtering residuals based primarily on the use of Delaunay triangulation are given.

5 RESIDUAL FILTERING METHOD USING THE RATIO OF RESULTANT RESIDUAL AND SURFACE FUNCTION OF POINT'S ENVIRONMENT

For structuring the TIN, so called "main triangle" needs to be selected, i.e. the area defined where the triangulation will take its place. By selecting the North Pole ($\varphi = 90^\circ$; $\lambda = 0^\circ$), the intersection of the equator and Greenwich meridian ($\varphi = 0^\circ$; $\lambda = 0^\circ$) and some arbitrary point in the central Asia ($\varphi = 45^\circ$; $\lambda = 90^\circ$), the whole territory of the Republic of Croatia is situated inside the "main triangle" (Fig. 9).

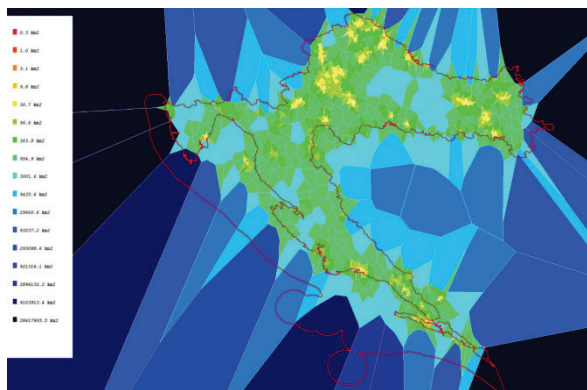


Figure 9 Classification of surface surroundings of identical points in two geodetic data (HDKS and HTRS96) of the Republic of Croatia [1]

On the other hand, by selecting such a big "main triangle", the area of boundary Voronoi's triangles would be significantly larger than the areas that are closer to the centroid of the whole set of datum points (Fig. 9).

Therefore, the above mentioned leads to the idea that criteria for the exclusion of the least suitable point require the ratio between the resultant residual of individual point \vec{r}_i and surface function of associated Voronoi's polygon to that point $f(Pov_i)$ to be analyzed to avoid unwanted disappearance of block of points.

Thus, the area function of Voronoi's polygons can be given by the expression in numerator and can be normalized by the sum of all area functions of all Voronoi's polygons in a given set of points:

$$Pov_i = \frac{\sum_{BS_i} N_j (E_{j+1} - E_{j-1})}{\sum Pov_{UK}} \quad (7)$$

Where BS_i is the number of neighbors of the i^{th} point, that is, the number of associated triangles, N_j is a north coordinate component of the center of described circle of the j^{th} triangle in DEG format, E_j is an east coordinate component of the center of described circle of j triangle in DEG format, and $\sum Pov_{UK}$ the total area of all Voronoi's cells under "main triangle".

Experimenting with the invariants of logarithmic, exponential and Gaussian functions led to the empirical conclusion that the best results are yielded by exponential functions with exponent in interval between 0 and 1, so the selected criterion for the exclusion of the least suitable point is the highest ratio between module of resultant

vector with selected exponential function of surface, that is:

$$\left\{ \frac{|\vec{r}_i|}{Pov_i^{\frac{1}{n}}} \right\} \xrightarrow{\text{tends}} \text{Max} \quad (8)$$

where $|\vec{r}_i|$ is module (length) of resultant vector in (m) and $\frac{1}{Pov_i^{\frac{1}{n}}}$ is the surface at selected exponent, that is the n^{th} root of surface in (km²).

Before determining the optimal exponent n , the criterion for iteration needs to be defined. According to statistical principle which is usually used in geodesy, the border error of 95,45% probability is only applied in the adjustment of high accuracy measurements [6]. In case of transformations between geodetic data, two sets of points with unequal internal accuracy are compared –results of GPS measurements with below centimeter accuracy and Austrian–Hungarian triangulation in seven separate blocks with accuracy of few decimeters. It certainly means that we need less strict criteria according to which the set of errors is freed from rough values, i.e., all errors are considered to be random with the probability of 99,73% [6].

In this particular case, the observed value is the module of resultant vector $|\vec{r}_i|$. The mean value is the arithmetic mean of all values:

$$\mu_R = \frac{\sum_{BT} |\vec{r}_i|}{BT} \quad (9)$$

where μ_R is the arithmetic mean of all resultant vectors modules and BT is the number of points that decreases with every step of iteration.

The criterion for the mentioned value of 99,73% is that every individual error (including also the maximal one) needs to be within the interval in every step of iteration:

$$m_i^{MAX} < \langle \mu_R \pm 3m_R \rangle, \quad (10)$$

where m_R is mean square error given by:

$$m_R = \pm \sqrt{\frac{\sum_{BT} (\mu_R - |\vec{r}_i|)^2}{BT - 1}} \quad (11)$$

and m_i^{MAX} is the maximal value of resultant vector module in current iteration step.

The empirical determination of optimal value for the exponent of Voronoi's surface function represents the biggest problem because only one point is excluded from further computation according to the ratio of resultant residual module with Voronoi's surface in each step of iteration, which results in a different number of excluded points after the statistical criteria of ending the iteration procedure have been met. Based on the set of 1638 identical points in HDKS and HTRS96, different selection

of mentioned exponent gives significant results to be seen in Tab. 2.

Table 2 Optimization of exponent of function in denominator of Eq. (8) [1]

Exponent n	1	8	320	325	1000
No. of left points BT	25	6	1467	1467	1467
No. of excluded points 1638 – BT	1613	1630	171	171	171
Percentage (%)	6452	27200	12	12	12
Accuracy RMS (cm)	36,73	15,48	42,47	42,47	42,47
Max. resultant (cm)	68,61	42,71	43,75	43,75	43,75

From the values shown in Tab. 2, it can be seen that the values of parameters are stabilizing for the selected exponent of $n > 320$ so it can be empirically deduced that for this approach, 171 points (12%) can be excluded from the original set of 1638 identical points in HDKS and HTRS96.

However, although this approach filters the largest number of unwanted residuals (Fig. 13), it does not solve local non-coincidence of residuals – e.g. point 395 (Fig. 10) or point 534 (Fig. 11).

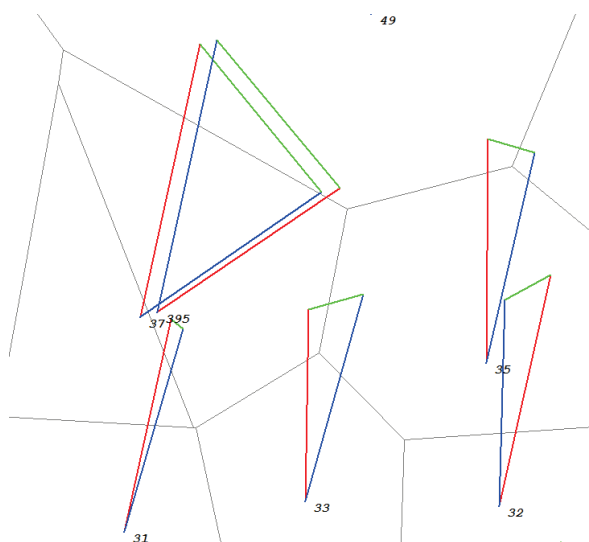


Figure 10 The rest of unfiltered residual at point 395 (Horvati, southwest from Zagreb) [1]

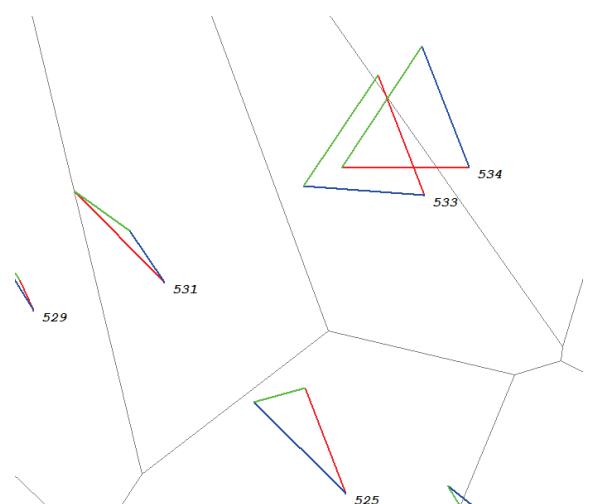


Figure 11 The rest of unfiltered residual at point 534 (Northwest from Sisak) [1]

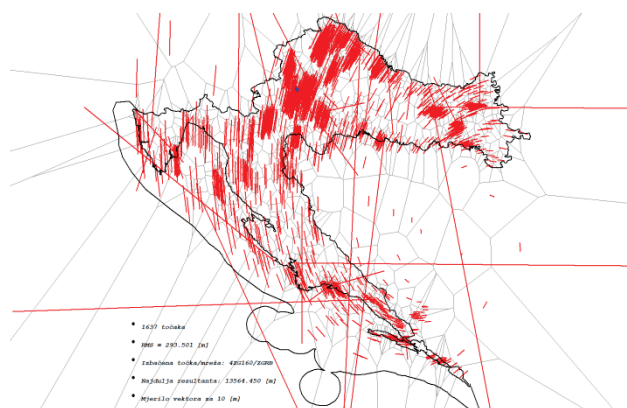


Figure 12 The beginning of residual filtering after the first iteration step [1]

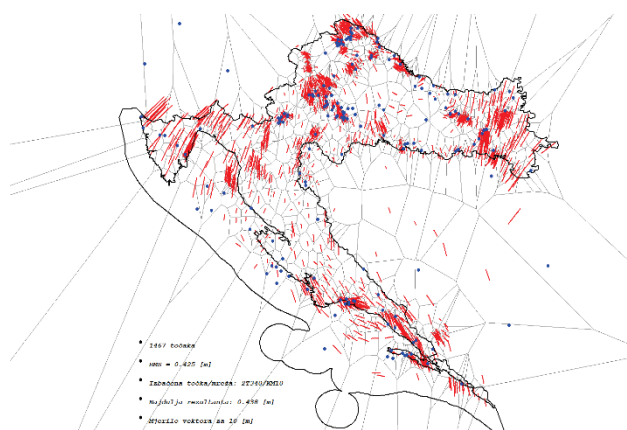


Figure 13 Filtered residuals after 171 excluded points (blue) [1]

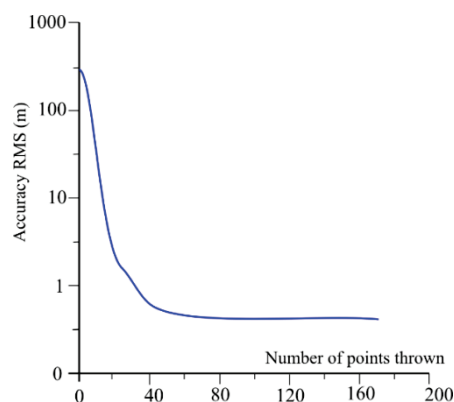


Figure 14 Increase of the accuracy computation of transformation parameters with the increase of the number of rejected (excluded) points for $n = 320$ [1]

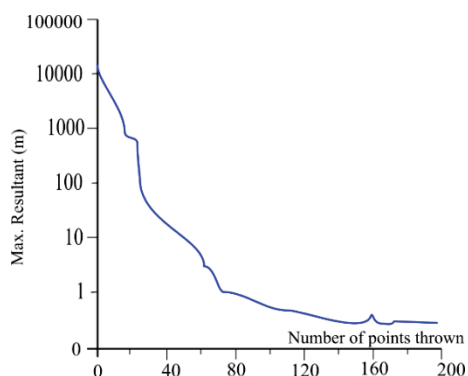


Figure 15 Value of maximum module of resultant residual for $n = 320$ [1]

Fig. 12 and Fig. 13 show the difference in scale of drawn real residuals (12:1) as well as approx. 15 rough

residuals values (e.g. 13,5 km) that are rejected during the iterative process and the computed standard deviation of adjustment for transformation parameters determination is decreasing and getting more accurate (Fig. 14). The value of maximum resultant module also decreases with the increase of iteration steps (Fig. 15).

6 RESIDUAL FILTERING METHOD USING RATIO OF DIFFERENCE AND MUTUAL DISTANCE OF NEIGHBORING RESULTANTS IN THE AREA

As shown in Fig. 10 and Fig. 11, the previous filtering method does not yield satisfactory results because some obviously unacceptable residuals were not rejected from the sample. On the other hand, those two mentioned cases give the idea for the next hypothesis: the pair of residuals will be locally accepted if two neighboring resultants (differences between real and theoretical residual vector at both neighboring points) are significantly different. So, how much can they differ from each other? Logically, a simple answer would be: if points are farther from each other, all the more, and if points are very close, as less as possible (e.g. points 37 and 395 in Fig. 10). The big resultant vector at the point 37 (which otherwise satisfactorily fits in a set of other neighboring residuals) is a direct consequence of proximity of the point 395 (whose real residual, colored red, locally does not fit by direction). The difference of both resultants can be written as:

$$\bar{\Delta}_{37,395} = \bar{r}_{37} - \bar{r}_{395} \quad (12)$$

where $\bar{r}_{37} = \bar{v}_{37} - \bar{w}_{37}$ is the resultant at the point 37 and $\bar{r}_{395} = \bar{v}_{395} - \bar{w}_{395}$ is the resultant at the point 395.

The distance between the mentioned points (37 and 395), i.e. the initial, exponentially again optimized function will be:

$$d_{OPT} = d_{EL}^{\frac{1}{n}} \quad (13)$$

where $d_{EL} = \sqrt{(\varphi_{37} - \varphi_{395})^2 + (\lambda_{37} - \lambda_{395})^2}$ is the distance defined in a simplified way (using ellipsoidal coordinates) between the points 37 and 395.

The criteria for the selection of individual point which would be excluded in every iteration step would be:

$$\frac{|\bar{\Delta}_{m,n}|}{d_{OPT}^{m,n}} \xrightarrow{tends} Max \quad (14)$$

It means that the observed value is the ratio of difference of resultant vectors and the function of distance between neighboring points for every pair of points (m, n), i.e. of all triangle sides of TIN that defines the neighborhood. Between two possible points for exclusion in each iteration step, the one that has larger resultant residual, i.e. that fits in the neighborhood due to other points would be excluded.

The criterion for ending the iteration process is again a traditional one, i.e. the maximum value of mentioned ratio fits into statistically reliable interval of 99,73%:

$$m_{m,n}^{MAX} < \langle \mu_D \pm 3m_D \rangle \quad (15)$$

where $m_{m,n}^{MAX}$ is the maximum value of the ratio of resultant difference and the function of distance between neighboring points, μ_D is the average value of mentioned ratio (arithmetic mean) and m_D is the mean square error of ratio computed by the following expression:

$$m_D = \sqrt{\frac{\sum_{1}^{BS} (\mu_D - m_{m,n})^2}{BS - 1}} \quad (16)$$

where $m_{m,n}$ is the value of ratio for every individual side of triangulated network and BS the total number of sides in triangulated network in each iteration step.

Already in the first attempt, the results for random selected exponent $n = 1$ are seen in Fig. 16. The excluded points are colored blue (909 of them; $1638 - 729 = 909$).

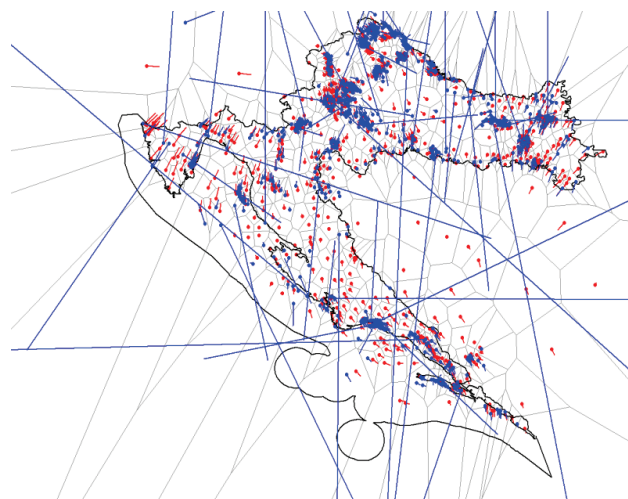


Figure 16 Method of ratio of neighboring resultants for $n = 1$ [1]

Table 3 Empirical optimization of exponent n ($n < 1$) for the method of ratio of difference between neighboring resultants and function of distance [1]

Exponent	No. of left points	No. of excluded points	Percentage	Accuracy T.P.C.
n^{-1}	BT	$1638 - BT$	(%)	RMS (cm)
1	729	909	55	45,74
2	288	1350	82	52,95
3	53	1585	97	53,02
4	30	1608	98	47,23
5	23	1615	99	47,17
6	11	1627	99	53,53
7	8	1630	100	59,39
10	6	1632	100	65,15

Excluded points are more or less equally distributed at the whole area and significantly more often at critical locations of higher density, i.e. of territory coverage. On the other hand, the value of 909 excluded points in the sample of 1638 points (55%) is again too rigid so the value of exponent n is further analyzed empirically. As seen in figures above, for some parts of Croatian territory (e.g.

Karlovac, Sisak, Okučani) the filtering is too rigid, i.e. too many points have been excluded from computation. By decreasing the value for exponent n , results are still unsatisfactory because the number of rejected points increases (Tab. 3).

Therefore, the search for optimal value of the exponent n continues in other direction – for higher values $1 < n < 10$ as shown in Tab. 4.

Table 4 Empirical optimization of exponent n ($n > 1$) for the method of ratio of difference between neighboring resultants and function of distance [1]

Exponent	No. of left points	No. of excluded points	Percentage	Accuracy T.P.C.	Max. resultant residual
n	BT	$1638 - BT$	(%)	RMS (cm)	(cm)
1,5	1047	591	36	44,00	183,86
2,0	1186	452	28	43,04	143,61
2,5	1233	405	25	42,49	65,97
3,0	1240	398	24	42,38	65,92
3,5	1260	378	23	42,31	65,97
4,0	1297	341	21	42,31	65,95
4,5	1313	325	20	42,18	65,99
5,0	1309	329	20	42,14	65,99
5,5	1307	331	20	42,12	44,02
6,0	1311	327	20	42,12	44,01
6,5	1314	324	20	42,06	44,02
7,0	1312	326	20	42,03	44,01
7,5	1310	328	20	41,99	44,02
8,0	1311	327	20	41,87	44,03
8,5	1309	329	20	41,73	44,05
9,0	1306	332	20	41,72	44,06
9,5	1307	331	20	41,71	44,06
10,0	1311	327	20	41,71	44,06

As seen in Tab. 4, the number of left points, accuracy of transformation parameters computation (T.P.C.) and the maximum module of resultant vector were not changed significantly by selecting the exponent greater than $n = 5,5$: $BT = 1310 \pm 4$, RMS in the amount of tenths of millimeters, and the maximum resultant module in the amount of hundredths of millimeters.

For $n = 5,5$, the mentioned critical parts of Croatian territory are weighted with non-parallel residuals, but the set of points is not so rigidly reduced.

The next presentation refers to the situations for local area with higher density of identical points in both data, Varaždin and Međimurje, shown in Fig. 17, Fig. 18 and Fig. 19.

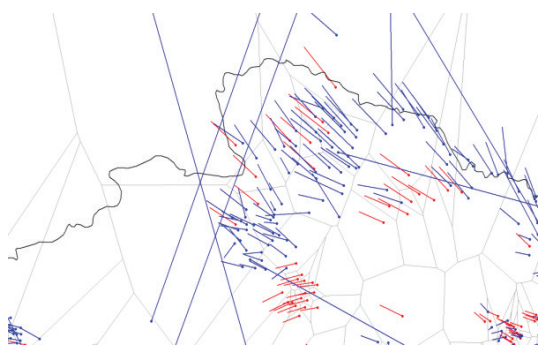


Figure 17 Exponent of rigid filtering $n = 1$ [1]

Fig. 17 shows rigid filtering with exponent $n = 1$, Fig. 18 with empirically obtained exponent of less rigid filtering $n = 5,5$ and Fig. 19 with the least rigid criteria $n = 10,0$. All three mentioned figures are rotated counter clockwise in a way that the direction of north is always to the reader's left.

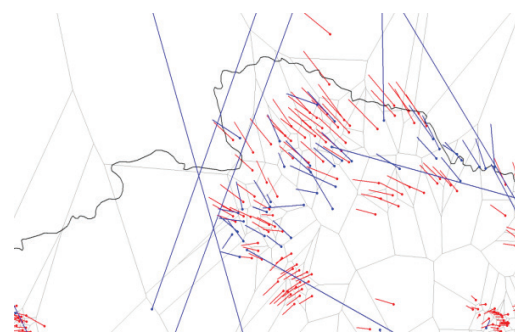


Figure 18 Exponent of less rigid filtering $n = 5,5$ [1]

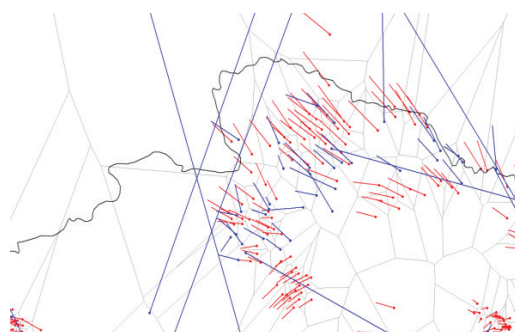


Figure 19 Exponent of the least rigid filtering $n = 10,0$ [1]

7 RESIDUAL FILTERING METHOD USING RATIO OF DIFFERENCE AND MUTUAL DISTANCE OF NEIGHBORING RESULTANTS IN TANGENTIAL PLANE

Unlike the computed spatial theoretical residual in previous chapters, components of residual vector in tangential plane at ellipsoid would be:

$$r_{\phi} = v_{\phi} - w_{\phi}, \quad (17)$$

$$r_{\lambda} = v_{\lambda} - w_{\lambda}, \quad (18)$$

where the real residual of individual point in meridian and parallel direction of the ellipsoid are expressed as functions of residuals in Cartesian coordinate system [7]:

$$v_{\phi} = -\sin \phi \cos \lambda \cdot v_X - \sin \phi \sin \lambda \cdot v_Y + \cos \lambda \cdot v_Z \quad (19)$$

$$v_{\lambda} = -\sin \lambda \cdot v_X + \cos \lambda \cdot v_Y \quad (20)$$

Similar, values for theoretical residual \bar{w} are obtained from v_{ϕ} and v_{λ} . Other equations, the criteria for point exclusion in each iteration step and the criteria for iteration end are the same as the ones described in previous method (chapter 6).

Further on, the procedure of empirical selection of optimum exponent n is shown in Tab. 5.

After the exponent value $n = 11$, the number of points varies 1377 ± 3 , root mean square (RMS) has a constant value of 42,3 cm, and maximum value of resultant vector is also the same within 1/10 mm so it could be said that $n = 11$ is the optimal lowest exponent which gives stable results. However, during a detail check of all areas, especially ones with higher sample of points, filtering with higher exponent e.g. $n = 20$ is a wanted result, less rigid than using lower exponent. Fig. 20, Fig. 21 and Fig. 22 show the results for selected exponents $n = 1$, $n = 11$ and $n = 21$ for the areas of the city of Varaždin and the region of Međimurje.

Table 5 Empirical optimization of exponent n for residual filtering method using ratio of difference of neighboring resultants and function of distance on ellipsoid surface [1]

Exponent	No. of left points	No. of excluded points	Percentage	Accuracy T.P.C.	Max. resultant residual
n	BT	$1638 - BT$	(%)	RMS (m)	(m)
1	772	866	53	0,4418	1,7439
2	1150	488	30	0,4341	0,4594
3	1298	340	21	0,4301	0,4598
4	1322	316	19	0,4276	0,4382
5	1314	324	20	0,4274	0,3442
6	1329	309	19	0,4266	0,3003
7	1357	281	17	0,4247	0,3005
8	1366	272	17	0,4240	0,3005
9	1366	272	17	0,4239	0,3005
10	1366	272	17	0,4239	0,3005
11	1374	264	16	0,4233	0,3005
12	1375	263	16	0,4235	0,3005
13	1378	260	16	0,4232	0,3005
14	1374	264	16	0,4229	0,3004
15	1376	262	16	0,4230	0,3004
16	1376	262	16	0,4230	0,3004
17	1379	259	16	0,4230	0,3004
18	1379	259	16	0,4230	0,3004
19	1380	258	16	0,4231	0,3004
20	1380	258	16	0,4231	0,3004
21	1380	258	16	0,4231	0,3004
22	1378	260	16	0,4233	0,3004
23	1377	261	16	0,4234	0,3004
24	1378	260	16	0,4233	0,3004
25	1378	260	16	0,4233	0,3004
26	1376	262	16	0,4227	0,3004
27	1376	262	16	0,4227	0,3004
28	1376	262	16	0,4227	0,3004
29	1376	262	16	0,4227	0,3004
30	1376	262	16	0,4227	0,3004

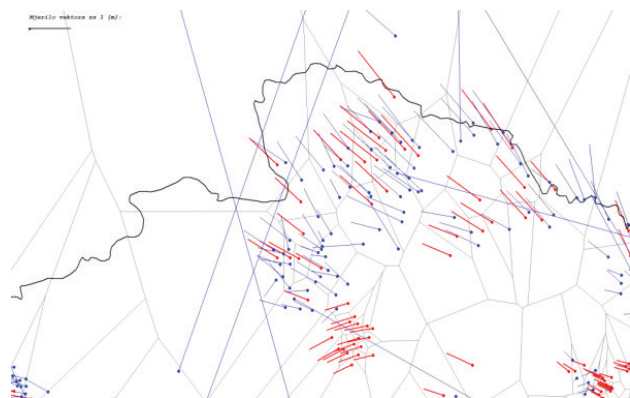


Figure 20 Filtering exponent $n = 1$ [1]

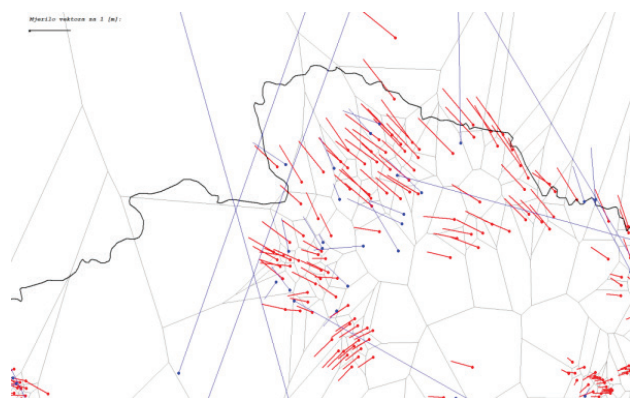


Figure 21 Filtering exponent $n = 11$ [1]

The results with exponent $n = 11$ and $n = 21$ show almost no differences between each other.

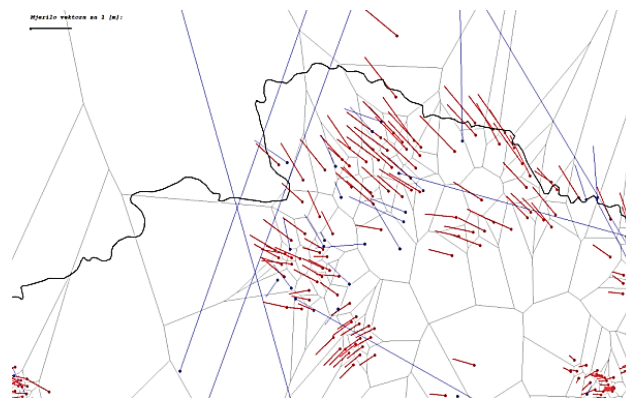


Figure 22 Filtering exponent $n = 21$ [1]

The adequacy of traditional criteria for the completion of iterative procedure ($\pm 3m$) compared to today's modern, more precise and more reliable criteria of $\pm 2m$ for this approach to filter residuals is logically more justified if we take a look at the results shown in Tab. 6.

Table 6 Comparison of criteria of border error for ending of the iterative procedure [1]

Exponent	Probability 99,7% Criteria $\pm 3m$		Probability 95,4% Criteria $\pm 2m$	
	No. of left points	Accuracy T.P.C.	No. of left points	Accuracy T.P.C.
n	BT	RMS (m)	BT	RMS (m)
1	772	0,4418	11	0,2938
5	1314	0,4274	28	0,1300
9	1366	0,4239	10	0,2174
12	1375	0,4235	29	0,1506
15	1376	0,4230	13	0,1315

It should of course be logical to assume that for another set of data, i.e. another sample of residuals between two geodetic data, a different optimal exponent n can be expected to be the best one, and that it always must be possible to include a certain point again manually in a sample allowing the distortion model to be computed.

8 CONCLUSION

Three originally developed methods of residual filtering can certainly be considered as a significant scientific contribution of this work.

The last method of ratio of differences and mutual distance of neighboring resultant of the tangential plane is shown as the most optimal and rational applied later in the transformation between HDKS and HTRS96.

Although rough and badly distorted residuals have been more than satisfactorily filtered out, complete automation is still not recommended because the operator must always have the opportunity to review and analyze the remaining residuals and logically intervene if they are needed. By applying the optimization method of filtering residuals, i.e. the automation of usually long-term iterative process of transformation was significantly speeded up by about 100 times.

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Contact information:

Marko ŠLJIVARIĆ, PhD, Assis. Prof.
Evropski univerzitet Brčko,
Ul. Bijeljinska cesta 72-74
76120 Brčko distrikt BiH, Bosnia and Herzegovina
Tel: +385 91 4444 998
E-mail: marko.sljivacic@gmail.com

Milan REZO, PhD, Assis. Prof.
University of Zagreb,
Faculty of Geotechnical Engineering,
Hallerovaaleja 7, 42000 Varaždin, Croatia
Tel: +385 42 408 950
E-mail: mrezo@gfv.hr

Marko PAVASOVIĆ, PhD, Assis. Prof.
University of Zagreb,
Faculty of Geodesy,
Fra Andrije Kačića Miošića 26
10000 Zagreb, Croatia
Tel: +385 1 4639 163
E-mail: mpavasovic@geof.hr