Original scientific paper
Accepted 28.11.2007.

IBRAHIM GÜNALTILI

On the Projectively Extended Linear Spaces

On the Projectively Extended Linear Spaces

ABSTRACT

In this article, we show that a linear space whose parameters are those of the complement of a subset in a finite projective plane π of order n such that no line is removed and a sufficient number of lines lost only one point, is projectively extended linear space.

Key words: linear space, projective plane, affine plane, semiextension, parallel class, projectively extended linear space.

MSC 2000: 51E20, 51A45

O projektivno proširenim linearnim prostorima

SAŽETAK

U ovom članku se pokazuje da je projektivno prošireni linearni prostor onaj linearni prostor čiji parametri su parametri komplementa podskupa konačne projektivne ravnine π reda n tako da niti jedan pravac nije odstranjen, a dovoljan broj pravaca gubi samo jednu točku.

Ključne riječi: linearni prostor, projektivna ravnina, afina ravnina, poluproširenje, paralelna klasa, projektivno prošireni linearni prostor

1 Introduction

The complementation problem with respect to a projective plane is the following:
Remove a certain subset of points and lines from the projective plane. Determine the parameters of the resulting space. Now assume that you are starting with a space having these parameters. Does this some howforce this subset to reappear, thus giving an embedding in the original projective plane? A number of people have considered complementation problems $([1],[2],[3],\dots,[10])$. In 1970, Dickey solved the problem for the case where the configuration removed was a unital [6].
Let us first recall some definitions and results. For more details, see [5].

Definition 1.1 *A **finite linear space** is a pair $(\mathcal{P},\mathcal{L})$, where $\mathcal{P}$ is a finite set of points and $\mathcal{L}$ is a family of proper subsets of $\mathcal{P}$, which are called lines, such that*

(L1) Any two distinct poins lie on exactly one line,

(L2) Any line contains at least two points.

Definition 1.2 *A finite linear space $\mathcal{S}=(\mathcal{P},\mathcal{L})$ is called a non-trivial $(n+1)$-**regular linear space**, $n\geq 1$, if*

(i) Every point is on $n+1$ lines

(ii) No line contains all points of $\mathcal{S}$.

Definition 1.3 *Let $\mathcal{S}=(\mathcal{P},\mathcal{L})$ be a finite linear space. If there exists at least one parallel class in $\mathcal{S}$, this class is called **ideal point** of $\mathcal{S}$. We construct a new structure $\mathcal{S}^*=(\mathcal{P}^*,\mathcal{L}^*)$ which consists of the points of $\mathcal{S}$ along with the ideal points and the lines of $\mathcal{S}$ which are extended by those parallel classes to which belong. This structure $\mathcal{S}^*$ is called **semiextension** of $\mathcal{S}$. $\mathcal{S}$ is called **projectively extended linear space** if $\mathcal{S}^*$ is a projective plane.*

The cardinality of $\mathcal{P}$ (resp. $\mathcal{L}$) will be denoted by v (resp. b). The *degree* of a point p is the number $b(p)$ of lines on which it lies. The integer n, where $n+1=\max\{b(p):p\in\mathcal{P}\}$, is called the order of the space. The *size* or *degree* $v(l)$ (also denoted by $|l|$) of a line l is the number of points it contains. A k-line is a line of size k.
The difference between $n+1$ and the number of points on l is called a *deficiency* of l denoted $d(l)$ for any line l. Two lines l and l' are *parallel* (respectively *disjoint*) if $l=l'$ or $l\cap l'=\phi$ (respectively if $l\neq l'$ and $l\cap l'=\phi$).
A *parallel class* in the linear space $(\mathcal{P},\mathcal{L})$ is a subset of $\mathcal{L}$ with the property that each point of $\mathcal{P}$ is on a unique element of this subset.
A *finite projective plane* of order n, $n\geq 2$, is a non-trivial $(n+1)$-regular linear space in which all lines have the same size $n+1$.
A *finite affine plane* of order n, $n\geq 2$, is a non-trivial $(n+1)$-regular linear space in which all lines have the same size n.

In this paper, for any two disjoint lines l and l' which have size less than n in a finite linear space we will use $m(l,l')$ and $m_n(l,l')$ to denote respectively the total number of lines and $n-$lines meeting l or l' without l and l' are themselves included.

$\mu = \min\{n+1-v(l) \mid l \in \mathcal{L}, v(l) \lneqq n\}$ and

$\lambda = \max\{n+1-v(l) \mid l \in \mathcal{L}, v(l) \lneqq n\}$.

$M_n = \max\{m_n(l,l') \mid l', l \in \mathcal{L}, v(l) \lneqq n, v(l') \lneqq n\}$.

The positive integers μ and λ will denote the minimum and maximum of *deficiencies* of lines of $\mathcal{S}$ which have size less than n, respectively.

Lemma 1.1 [5, *Lemma 3.3*] *Suppose that $\mathcal{S} = (\mathcal{P}, \mathcal{L})$ is a non-trivial (n+1)-regular linear space with n^2+n+1 lines and $n^2+n+1-s$ points.*

(i) If b_n is the number of lines of size n, $b_n \geq s(n+2-s)$. In particular, $n^2+1 \leq v \leq n^2+n-1, b_n \geq 2n$.

(ii) If there is no line of size n, $\sum_{l\in\mathcal{L}} d(l)(d(l)-2) = s(s-2-n)$.

2 Main Results

Lemma 2.1 *If $\mathcal{S} = (\mathcal{P}, \mathcal{L})$ is a non-trivial (n+1)-regular linear space with n^2+n+1 lines which contains at least one n-line, $n \geq 2$, the semiextension of $\mathcal{S}$ is linear space.*

Proof. Let $\mathcal{S}^* = (\mathcal{P}^*, \mathcal{L})$ be a semiextension of $\mathcal{S}$. Fix an $n-$line l. Then the number of lines missing l is n. Hence each $n-$line induces a parallel class of $n+1$ lines. Since $\mathcal{S}$ is a $(n+1)$-regular linear space which contains sufficient number of n-lines, $\mathcal{S}$ contains at least one parallel class of lines.

It is clear that two old points (points of $\mathcal{P}$) or an old and a new point are on a unique line of $\mathcal{L}$.

Let x and y be new points. We must show that they determine a unique line of $\mathcal{L}$. Let l_x and l_y be $n-$lines which determine the parallel classes corresponding to x and y. If l_x and l_y do not meet, then $x = y$ which is a contradiction. So l_x and l_y meet. Each point of l_y is on a unique line of the parallel class determined by l_x. This leaves precisely one line of the parallel class parallel to both l_x and l_y. Therefore $\mathcal{S}^*$ is a linear space.

Lemma 2.2 *Let $\mathcal{S}$ be a non-trivial (n+1)-regular linear space with n^2+n+1 lines in which all lines in semiextension of $\mathcal{S}$ meet. Then semiextension of $\mathcal{S}$ is a projective plane of order n, $n \geq 2$.*

Proof. Let $\mathcal{S}^*$ be semiextension of $\mathcal{S}$. Then each line in $\mathcal{S}^*$ contains $n+1$ points since all lines in $\mathcal{S}^*$ meet. Therefore $\mathcal{S}^*$ is a linear space with n^2+n+1 lines and n^2+n+1 points. Hence $\mathcal{S}^*$ is a projective plane of order n.

Lemma 2.3 *Let $\mathcal{S}$ be a non-trivial (n+1)-regular linear space with $b = n^2+n+1$ lines and $v(l) = n+1-d(l)$ for every line l of $\mathcal{S}$. Then*

(i) The number of lines parallel to a line l is $d(l).n$

(ii) If l and l' intersect, the number of lines parallel to two lines l and l' is $d(l).d(l')$, and it is $(n-1)+(d(l)-1)(d(l')-1)$, if l and l' are parallel.

(iii) If M is the set of lines parallel to a given line l, $(v-v(l)).d(l) = \sum_{k\in M} v(k)$

Proof. It is trivial.

Lemma 2.4 *Let $\mathcal{S} = (\mathcal{P}, \mathcal{L})$ be an $(n+1)$-regular linear space with $b = n^2+n+1$ lines and v points having the property each point on an $n+1-d(l)$-line, $d(l) \geq 0$, is on at most $b-v-d(l)$ lines of size n.*

(i) If $b-v \leq 2\mu-1$, $M_n \leq 2(n+1-\mu)(b-v-\mu)$.

(ii) If $b-v \geq 2\lambda-1$, $M_n \leq (n+1-\mu)(b-v-1)$.

Proof. Suppose that l is an $n+1-d(l)$-line and l' is a $n+1-d(l')$-line, $d(l) \leq d(l')$. By all the assumptions of lemma, there are at most $(n+1-d(l))(b-v-d(l))$ lines of size n and $(n+1-d(l'))(b-v-d(l'))$ lines of size n meeting l and l', respectively. Let the number of n-lines which are (meeting l and missing l'), (meeting l' and missing l) or (meeting l and l') be x,y or z, respectively. Then $m_n(l,l') = x+y+z$. On the otherhand $y+z \leq (n+1-d(l'))(b-v-d(l'))$, since $d(l) \leq d(l')$.

If $b-v \leq 2\mu-1$, $b-v-d(l) \leq d(l')-1$ and $b-v-d(l') \leq d(l)-1$ since $d(l) \leq d(l')$ and $2\mu-1 \leq d(l)+d(l')-1$. Hence

$$\begin{aligned} x &\leq (n+1-d(l))(b-v-d(l)) \\ &\leq (n+1-\mu)(b-v-\mu) \\ y+z &\leq (n+1-d(l'))(b-v-d(l')) \\ &\leq (n+1-\mu)(b-v-\mu). \end{aligned}$$

Thus

$M_n \leq 2(n+1-\mu)(b-v-\mu)$.

If $b-v \geq 2\lambda-1$, $b-v-d(l) \geq d(l')-1$, $b-v-d(l) \geq d(l')-1$, since $d(l) \leq d(l')$ and $2\lambda-1 \geq d(l)+d(l')-1$.

Therefore,

$$(n+1-d(l))(b-v-d(l)) \geq (n+1-d(l))(d(l')-1)$$
and
$$(n+1-d(l'))(b-v-d(l') \geq (n+1-d(l'))(d(l)-1).$$

Hence

$$\begin{aligned} x &\leq (n+1-d(l))(d(l')-1) \\ y+z &\leq (n+1-d(l'))(b-v-d(l')) \\ &\leq (n+1-d(l))(b-v-d(l')). \end{aligned}$$

Therefore, $M_n \leq (n+1-\mu)(b-v-1)$, since

$m_n(l,l^{'}) \leq (n+1-d(l))(b-v-1)$ and $\mu \leq d(l) \leq d(l') \leq \lambda$.

Theorem 2.1 *Let $\mathcal{S}$ be a non-trivial $(n+1)$-regular linear space with n^2+n+1 lines and v points. Let the total number of n-lines in $\mathcal{S}$ be b_n. If $b_n > M_n$, $\mathcal{S}$ is projectively extended space of order n.*

Proof. Let $\mathcal{S}^*$ be semiextension of $\mathcal{S}$. By Lemma 2.1, $\mathcal{S}^*$ is a linear space. It follows from our method of construction that each point of $\mathcal{S}^*$ is on $n+1$ lines. Finally we prove that any two lines of $\mathcal{S}^*$ always meet. Let l and l' be lines of $\mathcal{S}^*$ which don't meet in $\mathcal{S}$. To prove that they meet in $\mathcal{S}^*$, it sufficies to find an n-line parallel to both.

If either l or l' is an n-line, we are done (and clearly neither can be an $n+1$-line)

Suppose that l is an $n+1-d(l)$-line and l' is an $n+1-d(l')$-line, $2 \leq d(l) \leq d(l')$. Since $b_n > M_n$, and there is at most M_n lines of size n which intersect at least one of two parallel lines l and $l^{'}$, there is at least another n-line. Thus l and l' meet in $\mathcal{S}^*$. Therefore, by Lemma 2.2, $\mathcal{S}^*$ is a projective plane of order n.

Theorem 2.2 *Let $\mathcal{S} = (\mathcal{P}, \mathcal{L})$ be an $(n+1)$-regular linear space with $b = n^2+n+1$ lines and v points having the property each point on an $n+1-d(l)$-line, $d(l) \geq 0$, is on at most $b-v-d(l)$ lines of size n. If $b_n > 2(n+1-\mu)(b-v-\mu)$, $\mathcal{S}$ is projectively extended space of order n.*

Proof. By Lemma 2.1, semiextension of $\mathcal{S}$ is linear space. Fix an n-line l. Then the number of lines missing l is n. Hence each n-line induces a parallel class of $n+1$ lines. Let $\mathcal{S}^*$ be a semiextension of $\mathcal{S}$. We must show that any two lines in $\mathcal{S}^*$ intersect. Let l and l' be lines of $\mathcal{S}^*$ which don't meet in $\mathcal{S}$. To prove that they meet in $\mathcal{S}^*$, it suffices to find an n-lines parallel to both.

If either l or l' is an n-line, we are done. (and clearly neither can be an $n+1$-line).

Suppose that l is a $n+1-d(l)$-line and l' is a $n+1-d(l')$-line, $2 \leq d(l) \leq d(l')$. By all the assumptions of theorem, there are at most $(n+1-d(l))(b-v-d(l))$ n-lines and $(n+1-d(l'))(b-v-d(l'))$ n-lines meeting l and l', respectively. Let the number of n-lines which are (meeting l and missing l'), (meeting l' and missing l) or (meeting l and l') be x, y or z, respectively. Then $y+z \leq (n+1-d(l'))(b-v-d(l^{'}))$, since $2 \leq d(l) \leq d(l^{'})$. Hence

$$\begin{aligned} x &\leq (n+1-d(l))(b-v-d(l)) \\ &\leq (n+1-\mu)(b-v-\mu) \\ y+z &\leq (n+1-d(l'))(b-v-d(l')) \\ &\leq (n+1-\mu)(b-v-\mu). \end{aligned}$$

Therefore, the number of n-lines which intersect at least one of lines l and l', the number of n-lines is at most $2(n+1-\mu)(b-v-\mu)$ and there is at least one n-line missing l and l' . Thus l and l' meet in $\mathcal{S}^*$. So by the Lemma 2.2, $\mathcal{S}^*$ is a projective plane of order n.

Corollary 2.1 *Let $\mathcal{S} = (\mathcal{P}, \mathcal{L})$ be an $(n+1)$-regular linear space with $b = n^2+n+1$ lines and v points having the property each point on an $n+1-d(l)$- line, $d(l) \geq 0$, is on at most $b-v-d(l)$ lines of size n.*
If $b_n \geq 1$ and $n \geq (b-v-\mu)(b-v-1)$, $\mathcal{S}$ is projectively extended space of order n

Proof. Let $b_n \geq 1$ and $n \geq (b-v-\mu)(b-v-1)$. By Lemma 1.1, $b_n > (b-v)(n+2-(b-v))$.
Since $n \geq (b-v-\mu)(b-v-1)$,

$$(b-v)(n+2-(b-v)) > 2(n+1-\mu)(b-v-\mu).$$

Therefore, by the Theorem 2.2, $\mathcal{S}$ is projectively extended space of order n.

Theorem 2.3 *Let $\mathcal{S} = (\mathcal{P}, \mathcal{L})$ be an $(n+1)$-regular linear space with $b = n^2+n+1$ lines and at most $n^2+n+2-2\lambda$ points having the property each point on an $n+1-d(l)$-line, $d(l) \geq 0$, is on at most $b-v-d(l)$ lines of size n.*
If $b_n > (n+1-\mu)(b-v-1)$ then $\mathcal{S}$ is projectively extended space of order n.

Proof. By Lemma 2.1, the semiextension of $\mathcal{S}$ is linear space. Fix an n-line l. Then the number of lines missing l is n. Hence each n-line induces a parallel class of $n+1$ lines. Let $\mathcal{S}^*$ be a semiextension of $\mathcal{S}$. We must show that all lines in $\mathcal{S}^*$ intersect. Let l and l' be lines which do not meet in $\mathcal{S}$. To prove that they meet in $\mathcal{S}^*$, it suffices to find an n-line parallel to both in $\mathcal{S}$.

If either l or l' is an n-line, we are done (and clearly neither can be an $n+1$-line).

Suppose that l is an $n+1-d(l)$-line and l' is an $n+1-d(l')$-line, $2 \leq d(l) \leq d(l')$. By all the assumptions of this theorem, there are at most $(n+1-d(l))(b-v-d(l))$ n-lines and $(n+1-d(l'))(b-v-d(l'))$ n-lines meeting l and l', respectively. Hence,

$$n+1-d(l') \leq n+1-d(l) \text{ and } b-v-d(l') \leq b-v-d(l).$$

$$(n+1-d(l))(b-v-d(l)) \geq (n+1-d(l'))(b-v-d(l'))$$

$b-v \geq 2\lambda-1$, since $v \leq n^2+n+2-2\lambda$ and $b = n^2+n+1$. Therefore, the number of n-lines meeting l or l' is at most $(n+1-\mu)(b-v-1)$, by Lemma 2.4(ii). Thus all lines in $\mathcal{S}^*$ intersect, since $\mathcal{S}$ contains at least $(n+1-\mu)(b-v-1)+1$ lines of size n. Hence, $\mathcal{S}^*$ is a projective plane of order n, by the Lemma 2.2

References

[1] I. GÜNALTILI, Ş. OLGUN, On the embedding some linear spaces in finite projective planes, J.geom. 68 (2000), pp. 96-99.

[2] I. GÜNALTILI, Pseudo-complements in finite projective planes, Ars Combinatoria, to appear

[3] I. GÜNALTILI, P. ANAPA, Ş. OLGUN, On the embedding of complements of some hyperbolic planes. Ars Combin. 80 (2006), pp. 205-214

[4] J. TOTTEN, Embedding the complement of two lines in a finite projective plane, J.Austral.Math.Soc. 22 (Series A) (1976), pp 27-34.

[5] L. M. BATTEN, A. BEUTELSPACHER, *Combinatorics of points and lines*, Cambridge University Press, 1993

[6] L. J. DICKEY, Embedding the complement of a unital in a projective plane, Atti del convegno di Geometria Combinatoria e sue Applicazioni, Perugia, 1971, pp. 199-203

[7] P. DE WITTE, The exceptional case in a Theorem of Bose and Shrikhande, J. Austral. Math. Soc. Ser. A 24 (1977), pp. 64-78

[8] R. H. BRUCK, Existence problems for classes of finite projective planes, Lectures delivered to the Canadian Math. Congress, Sask., Aug. 1963

[9] R.C. BOSE, S.S. SHRIKHANDE, Embedding the complement of a oval in a projective plane of even order, Discrete Math. 6 (1973), pp. 305-312

[10] R. KAYA, E. ÖZCAN, On the construction of B-L planes from projective planes, Rendiconti del Seminario Matematico Di Bresciot (1984), pp. 427-434

Ibrahim Günaltili
Osmangazi University, Departmant of Mathematics
26480 Eskişehir-Türkiye
e-mail:igunalti@ogu.edu.tr