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FRIEDRICH MANHART

Affine Geometry of Minkowski Minimal Surfaces in $\mathbb{R}^3_1$

Dedicated to Professor Hellmuth Stachel on occasion of his 65^{th} birthday

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ABSTRACT

On the one hand we give results concerning affine invariants of the focal surfaces of Minkowski minimal surfaces. Secondly we investigate the behavior of affine invariants in case of association of Minkowski minimal surfaces.

Key words: Minkowski minimal surfaces, focal surfaces, associated surfaces

MSC 2000: 53A15, 53B30

Afina geometrija minimalnih ploha Minkowskog u $\mathbb{R}^3_1$

SAŽETAK

S jedne strane mi dajemo rezultate vezane uz afine invarijante žarišnih ploha minimalnih ploha Minkowskog, dok s druge istražujemo karakteristike afinih invarijanata u slučaju pridruživanja kod minimalnih ploha Minkowskog.

Ključne riječi: minimalne plohe Minkowskog, žarišne plohe, pridružene plohe

There are many results on affine geometry of Euclidean minimal surfaces in $\mathbb{R}^3$. Several results concern the *focal surfaces* of a minimal surface. Denoting K_e the Gauss-curvature, in case of minimal surfaces the principal curvatures (eigenvalues of the shape operator) are $\varphi(-K_e)^{1/2}$ ($\varphi = \pm 1$).

Theorem 1 *(P. FRANCK [5]). Let $f : U \subseteq \mathbb{R}^2 \to f(U) =: \Phi \subset \mathbb{R}^3$ be a minimal immersion with $K_e \neq 0$ in U. The two sets of focal points Ψ_1 and Ψ_{-1} of $f(U) = \Phi$ are parametrized by $z = f + \varphi(-K_e)^{-1/2} n_e$, where n_e denotes the normal vector and Ψ_1 and Ψ_{-1} correspond to $\varphi = 1$ and $\varphi = -1$ respectively. Then the following holds for $\varphi \in \{1, -1\}$:*

(a) If Ψ_φ is a regular surface, then $K_e(\Psi_\varphi) = -1/4 K_e$, where $K_e(\Psi_\varphi)$ denotes the Gauss-curvature of Ψ_φ.

(b) If Ψ_φ is a regular surface the affine normal of Ψ_φ intersects the affine normal of Φ orthogonally.

(c) If Ψ_φ is a regular surface then it is an affine minimal surface.

(d) If Ψ_1 and Ψ_{-1} are both regular surfaces, then $K_a(\Psi_1) : K_a(\Psi_{-1}) = H_e(\Psi_1)^4 : H_e(\Psi_{-1})^4$, where $K_a(\Psi_1)$, $K_a(\Psi_{-1})$ and $H_e(\Psi_1)$, $H_e(\Psi_{-1})$ are the affine Gauss-curvature and the Euclidean mean curvature of Ψ_1 and Ψ_{-1} respectively.

Results concerning the behavior of affine quantities by association are contained in

Theorem 2 *(P. FRANCK [5], F.MANHART [9]). Let Φ be a regular minimal surface with $K_e \neq 0$ and Ψ_1 and Ψ_{-1} the sets of focal points as above. Denoting ${}^{(\lambda)}\Phi$ the pencil of associated minimal surfaces to Φ the following holds:*

(a) The affine normal vector $n_a(\Phi)$ of Φ is invariant: $n_a({}^{(\lambda)}\Phi) = n_a(\Phi)$.

(b) The affine Gauss-curvature of Φ is invariant: $K_a({}^{(\lambda)}\Phi) = K_a(\Phi)$.

(c) If Ψ_1 and Ψ_{-1} are regular surfaces then denoting by $n_a(\Psi_1)$ and $n_a(\Psi_{-1})$ the affine normal vector of Ψ_1 and Ψ_{-1} respectively, the figure of the three affine normals spanned by $n_a(\Phi)$, $n_a(\Psi_1)$, $n_a(\Psi_{-1})$ is invariant by translation along the orbit (ellipse).

In the present paper we will prove analogous results for minimal surfaces in Minkowski space and give some examples.

1 Preliminaries

A *Minkowski* (or *Lorentz*) 3-space $\mathbb{R}^3_1$ is $(\mathbb{R}^3, \langle x,y\rangle)$, where $\langle x,y\rangle$ is the scalar product

$$\langle x,y\rangle := x_1y_1 + x_2y_2 - x_3y_3,\, x = (x_1,x_2,x_3). \tag{1}$$

A vector $x \in \mathbb{R}^3_1$ is called

$$\begin{aligned} spacelike &\iff \langle x,x\rangle > 0,\\ timelike &\iff \langle x,x\rangle < 0,\\ isotropic(lightlike) &\iff \langle x,x\rangle = 0, \quad x \neq 0. \end{aligned}$$

The *(Minkowski-) length* of a vector x is defined by

$$\|x\| := \sqrt[+]{|\langle x,x\rangle|} \geq 0. \tag{2}$$

The *(Minkowski-) crossproduct* is

$$x \times y \; with \; \langle x \times y, z\rangle = det(x,y,z). \tag{3}$$

A surface Φ in $\mathbb{R}^3_1$ is locally parametrized by $f : U \subseteq \mathbb{R}^2 \to \mathbb{R}^3$. The local coordinates are denoted by (u,v) or $(u^1 := u, u^2 := v)$. Partial derivatives of a function $b : \mathbb{R}^2 \rightarrowtail \mathbb{R}$ or $b : \mathbb{R}^2 \rightarrowtail \mathbb{R}^3$ are denoted by

$$b_{,j} := \frac{\partial}{\partial u^j} b,\, b_{,jk} := \frac{\partial^2}{\partial u^j \partial u^k} b.$$

The scalar product $\langle\cdot,\cdot\rangle$ in $\mathbb{R}^3_1$ induces a (pseudo-)Riemannian metric on U, the *first fundamental form (I)* with components

$$g_{jk} := \langle f_{,j}, f_{,k}\rangle : U \to \mathbb{R}. \tag{4}$$

Denoting $\Delta := det(g_{jk})$ a surface $\Phi = f(U)$ is called

$$\begin{aligned} spacelike &\iff \Delta > 0 \quad in\, U,\\ timelike &\iff \Delta < 0 \quad in\, U. \end{aligned}$$

Points with $\Delta = 0$ are excluded. For $T \subset U$ a Jordan measurable set the *Minkowski surface area* is

$$O(f(T)) := \int_T \sqrt{|\Delta|}\, du\, dv. \tag{5}$$

The normal vector is

$$n := \frac{f_{,1} \times f_{,2}}{\|f_{,1} \times f_{,2}\|} = \frac{f_{,1} \times f_{,2}}{\sqrt{|\Delta|}}. \tag{6}$$

Because of $\Delta \neq 0$, n is a well defined non null vector in U. In the following we denote $\varepsilon := \langle n,n\rangle$. So in case of spacelike and timelike surfaces we have $\varepsilon = -1$ and $\varepsilon = 1$ and the *spherical image* $n(U)$ is part of the two-sheet hyperboloid $\langle x,x\rangle = -1$ and the one-sheet hyperboloid $\langle x,x\rangle = 1$, respectively. The *second fundamental form (II)* and the *shape operator* S are related by $(II)(f_{,j}, f_{,k}) = \varepsilon\langle S(f_{,j}), f_{,k}\rangle$. The components of (II) and S are

$$\begin{aligned} h_{jk} &:= (II)(f_j, f_k) = \varepsilon\langle n, f_{,jk}\rangle = \\ &= \frac{\varepsilon}{\sqrt{|\Delta|}} det(f_{,1}, f_{,2}, f_{jk}), \end{aligned} \tag{7}$$

$$S(f_{,j}) := -n_{,j} =: h^s_j f_{,s} \quad \text{with} \quad h^s_j = \varepsilon h_{jk} g^{ks}. \tag{8}$$

Mean curvature and *Gauss-curvature* are

$$H := \frac{1}{2}\mathrm{tr}S = \frac{\varepsilon}{2\Delta}(h_{11}g_{22} - 2h_{12}g_{12} + h_{22}g_{11}), \tag{9}$$

$$K := \varepsilon \det S = \varepsilon \frac{det(h_{jk})}{\Delta}. \tag{10}$$

In case of K we use the sign convention used in [10], [4] and [8]. The eigenvalues of S (principal curvatures of Φ) are

$$k_{1,2} = H \pm \sqrt{H^2 - \varepsilon K}. \tag{11}$$

The integrability conditions of Codazzi and the Theorema egregium read as usual:

$$Co \; : \; h_{js,k} - h_{jk,s} = \Gamma^p_{jk} h_{ps} - \Gamma^p_{js} h_{pk}, \tag{12}$$

$$Ga \; : \; R_{pjks} = R^t_{jks} g_{pt} = h_{jk}h_{sp} - h_{js}h_{kp}. \tag{13}$$

Remark 1 From (7) and (10) an easy calculation gives $\Delta^2 K = -\Delta_e^2 K_e$ where Δ_e is the determinant of the components of the Euclidean metric. Thus Euclidean and Minkowski Gauss-curvature have different sign.

We need some basics from affine differential geometry. For details see for instance [13], [2]. From affine point of view a surface $f(U)$ is *nondegenerate* if

$$D := \det(D_{jk}) \neq 0,\, D_{jk} := \det(f_{,1}, f_{,2}, f_{,jk}). \tag{14}$$

By (7) and (10) we have $D \neq 0 \iff K \neq 0$. Assuming f to be regular and nondegererate, it is said to be an *equiaffine immersion*. Then the components of the *affine metric* of $f(U)$ are

$$G_{jk} := |D|^{-1/4} D_{jk}, \tag{15}$$

and the *affine normal vector* of $\Phi = f(U)$ is

$$n_a := (1/2)\Delta_G f, \tag{16}$$

where Δ_G is the Laplacian with respect to the affine metric. The *affine shape operator* B defined by $B(f_{,j}) = -n_{a,j}$ has components B^k_j defined by

$$n_{a,j} =: -B^k_j f_{,k}. \tag{17}$$

Affine curvature K_a and *affine mean curvature* H_a are defined by

$$K_a := \det(B),\ H_a := (1/2)\mathrm{tr}(B). \tag{18}$$

In case of a constant affine normal vector the surface Φ is called an *improper affine sphere* and we have $K_a = H_a = 0$.

2 Minimal Surfaces and their focal surfaces

A regular surface $\Phi = f(U) \subset \mathbb{R}^3_1$ is called a *(Minkowski-) minimal surface* iff $H = 0$ in U. There are many investigations on these surfaces, for instance [1], [3], [4], [6], [7], [10], [11], [12], [14], [15]. Although a spacelike surface locally maximizes the surface area defined by (5), as E.CALABI proved in [3], and timelike minimal surfaces neither maximize nor minimize surface area (see [11]), we speak of minimal surfaces.
As we want to study properties of affine geometry too, we exclude points with $K = 0$ on Φ. Denoting by $k_{1,2}$ the eigenvalues of the shape operator S of Φ, the *focal surfaces* of $\Phi = f(U)$ are parametrized by $z = f + (1/k_{1,2})n$. So the focal points are real iff $k_{1,2}$ are real. From (11) the eigenvalues are in case of $H = 0$

$$k_{1,2} = \pm\sqrt{-\varepsilon K}. \tag{19}$$

In case of a spacelike minimal surface ($\varepsilon = -1$) the Gauss-curvature is positive (cf. [7, p. 298], [4, p. 518]), so we have two different real eigenvalues $\pm\sqrt{K}$. A timelike minimal surface ($\varepsilon = 1$) has real focal surfaces iff $K < 0$ that means by Remark 1 Φ is locally strongly convex. So in both cases locally we can take the lines of curvature as parametric lines.

Lemma 1 *(T. WEINSTEIN [14, p. 160]). Let Φ be a minimal surface in $\mathbb{R}^3_1$ with $K \neq 0$ and real focal surfaces. Then locally there is a parametrization $f : U \to \mathbb{R}^3_1$, $f(U) = \Phi$, so that*

$$g := g_{11} > 0, g_{22} = -\varepsilon g, g_{12} = 0,$$
$$h_{11} = 1, h_{22} = \varepsilon, h_{12} = 0,$$

where $\varepsilon = -1$ and $\varepsilon = 1$ refers to spacelike and timelike surfaces respectively.

The coordinate functions of (I) and (II) in Lemma 1 fulfil the Codazzi condition *Co* (12). The Theorema egregium (13) reads

$$g(g'' - \varepsilon\ddot{g} + 2) = {g'}^2 - \varepsilon\dot{g}^2, \tag{20}$$

where here as in the following the derivatives of $g : U \subseteq \mathbb{R}^2 \to \mathbb{R}$ are denoted by

$$\dot{g} := \frac{\partial g}{du},\ \ddot{g} := \frac{\partial^2 g}{du^2},\ g' := \frac{\partial g}{dv},\ g'' := \frac{\partial^2 g}{dv^2},\ \dot{g}' := \frac{\partial^2 g}{du\,dv}.$$

Theorem 3 *Let $f(U) = \Phi \subset \mathbb{R}^3_1$ be a regular minimal surface with $K \neq 0$ in U. The two sets of focal points Ψ_1 and Ψ_{-1} of $\Phi = f(U)$ are parametrized by $z = f + \varphi(-\varepsilon K)^{-1/2}n$, where n denotes the normal vector and Ψ_1 and Ψ_{-1} correspond to $\varphi = 1$ and $\varphi = -1$ respectively. Then the following holds for $\varphi \in \{1, -1\}$:*

(a) If Ψ_φ is a regular surface, then $K(\Psi_\varphi) = -1/4K$, where $K(\Psi_\varphi)$ denotes the Gauss-curvature of Ψ_φ.

(b) If Ψ_φ is a regular surface, then it is non degenerate and the affine normal of Ψ_φ intersects the affine normal of Φ orthogonally.

(c) If Ψ_φ is a regular surface then it is an affine minimal surface.

(d) If Ψ_1 and Ψ_{-1} are both regular surfaces, then

$$K_a(\Psi_1) : K_a(\Psi_{-1}) = (-\varepsilon)H(\Psi_1)^4 : H(\Psi_{-1})^4,$$

where $K_a(\Psi_\varphi)$ and $H(\Psi_\varphi)$ are the affine Gauss-curvature and the Minkowski mean curvature of Ψ_φ respectively.

Proof
(a) Using the parameters of Lemma 1 we calculate the Gauss-curvature

$$K = \frac{-\varepsilon}{g^2}, \tag{21}$$

and the Gauss equations

$$\begin{aligned} f_{,11} &= \gamma_1 f_{,1} + \varepsilon\gamma_2 f_{,2} + n, \\ f_{,12} &= \gamma_2 f_{,1} + \gamma_1 f_{,2}, \\ f_{,22} &= \varepsilon\gamma_1 f_{,1} + \gamma_2 f_{,2} + \varepsilon n, \end{aligned} \tag{22}$$

where

$$\gamma_1 := \frac{\dot{g}}{2g},\ \gamma_2 := \frac{g'}{2g}.$$

The Weingarten equations are

$$n_{,1} = \frac{-\varepsilon}{g} f_{,1},\ n_{,2} = \frac{\varepsilon}{g} f_{,2}. \tag{23}$$

The parametrizations of the focal surfaces (sets of focal points) are

$$z = f + \varphi g\, n,\ \varphi \in \{1, -1\}. \tag{24}$$

Expressing the derivatives of z by the derivatives of f and n we get

$$\begin{aligned} z_{,1} &= (1-\varepsilon\varphi)f_{,1}+\varphi\dot{g}n, \quad (25)\\ z_{,2} &= (1+\varepsilon\varphi)f_{,2}+\varphi g'n,\\ z_{,11} &= (1-3\varepsilon\varphi)\gamma_1 f_{,1}+(\varepsilon-\varphi)\gamma_2 f_{,2}+(1-\varepsilon\varphi+\varphi\ddot{g})n,\\ z_{,12} &= (1-\varepsilon\varphi)\gamma_2 f_{,1}+(1+\varepsilon\varphi)\gamma_1 f_{,2}+\varphi\dot{g}'n,\\ z_{,22} &= (\varepsilon+\varphi)\gamma_1 f_{,1}+(1+3\varepsilon\varphi)\gamma_2 f_{,2}+(\varepsilon+\varphi+\varphi g'')n. \end{aligned}$$

Using (25) we calculate the metric $g^*_{jk} := \langle z_{,j}, z_{,k}\rangle$ of the focal surfaces

$$\begin{aligned} g^*_{11} &= 2g(1-\varepsilon\varphi)+\varepsilon\dot{g}^2,\\ g^*_{12} &= \varepsilon\dot{g}g', \quad (26)\\ g^*_{22} &= -2\varepsilon g(1+\varepsilon\varphi)+\varepsilon g'^2,\\ \Delta^* &= det(g^*_{jk}) = 2g[\varepsilon g'^2(1-\varepsilon\varphi)-\dot{g}^2(1+\varepsilon\varphi)]. \end{aligned}$$

The determinants $D^*_{jk} := \det(z_{,1}, z_{,2}, z_{,jk})$ (according to (14)) for the focal surfaces are

$$\begin{aligned} D^*_{11} &= (1+\varepsilon\varphi)\dot{g}^2+(\varepsilon-\varphi)g'^2\\ D^*_{12} &= 0, \quad (27)\\ D^*_{22} &= (\varepsilon\varphi-1)g'^2-(\varepsilon+\varphi)\dot{g}^2\\ D^* &= det(D^*_{jk}) = -2(\varepsilon+\varphi)\dot{g}^4-2(\varepsilon-\varphi)g'^4. \end{aligned}$$

From (26) and (27) a focal surface is non degenerate iff it is regular. In case of $\varepsilon\varphi = 1$ or $\varepsilon\varphi = -1$ this requires $\dot{g} \neq 0$ or $g' \neq 0$ respectively. From (7),(10) and (14) the Gauss-curvature of a focal surface $K(\Psi_\varphi)$ is

$$K(\Psi_\varphi) = \varepsilon^*\frac{det(h^*_{jk})}{\Delta^*} = \varepsilon^*\frac{D^*}{|\Delta^*|\Delta^*} = -\varepsilon^*\frac{D^*}{\Delta^{*2}}, \quad (28)$$

because of $sgn(\Delta^*) = -\langle n^*, n^*\rangle = -\varepsilon^*$, where n^* is the normal vector of the focal surface. Calculating $K(\Psi_\varphi)$ from (28) using (26) and (27) gives $K(\Psi_\varphi) = -1/4K$ from (21).

(b) From (15) the components of the affine metric of Φ are

$$G_{11} = \varepsilon\sqrt{g},\ G_{12} = 0,\ G_{22} = \sqrt{g}. \quad (29)$$

Using (29) the affine normal vector of Φ is from (16)

$$n_a(\Phi) = \frac{1}{2g^{3/2}}(\varepsilon\dot{g}f_{,1}+g'f_{,2}+2\varepsilon g n). \quad (30)$$

In case of spacelike surfaces ($\varepsilon = -1$) we calculate the affine metric of the focal surfaces Ψ_φ

$$\begin{aligned} \varphi = 1 &: \quad G^*_{11} = G^*_{22} = -\sqrt{2}|g'|,\ G^*_{12} = 0,\\ \varphi = -1 &: \quad G^*_{11} = G^*_{22} = \sqrt{2}|\dot{g}|,\ G^*_{12} = 0 \end{aligned}$$

and the affine normal

$$\begin{aligned} \varphi = 1 &: \quad n_a(\Psi_1) = \frac{1}{|g'|}\left(\frac{-1}{\sqrt{2g}}\right)[\dot{g}f_{,1}-g'f_{,2}+\frac{1}{2}(\dot{g}^2+g'^2)n],\\ \varphi = -1 &: \quad n_a(\Psi_{-1}) = \frac{1}{|\dot{g}|}\left(\frac{-1}{\sqrt{2g}}\right)[\dot{g}f_{,1}-g'f_{,2}+\frac{1}{2}(\dot{g}^2+g'^2)n]. \end{aligned} \quad (31)$$

In case of timelike surfaces ($\varepsilon = 1$) we get

$$\begin{aligned} \varphi = 1 &: \quad G^*_{11} = \sqrt{2}|\dot{g}|,\ G^*_{12} = 0,\ G^*_{22} = -\sqrt{2}|\dot{g}|,\\ \varphi = -1 &: \quad G^*_{11} = \sqrt{2}|g'|,\ G^*_{12} = 0,\ G^*_{22} = -\sqrt{2}|g'| \end{aligned}$$

and the affine normal

$$\begin{aligned} \varphi = 1 &: \quad n_a(\Psi_1) = \frac{1}{|\dot{g}|}\left(\frac{-1}{\sqrt{2g}}\right)[\dot{g}f_{,1}+g'f_{,2}-\frac{1}{2}(\dot{g}^2-g'^2)n],\\ \varphi = -1 &: \quad n_a(\Psi_{-1}) = \frac{-1}{|g'|}\left(\frac{-1}{\sqrt{2g}}\right)[\dot{g}f_{,1}+g'f_{,2}-\frac{1}{2}(\dot{g}^2-g'^2)n]. \end{aligned} \quad (32)$$

In both cases we have used (20). From (31) and (32) the affine normals of Ψ_1 and Ψ_{-1} are parallel and using (30),(31) and (32) it follows $\langle n_a(\Phi), n_a(\Psi_\varphi)\rangle = 0$. So (b) is proved.

(c) From (31) and (32), using (22) and (23) a straightforward calculation gives the components B^{*k}_j of the affine shape operator B^* of the focal surfaces Ψ_φ.
In case of spacelike surfaces ($\varepsilon = -1$) we get

$$\varphi = 1 : \quad \left(B^{*k}_j\right) = \frac{sgn(g')}{2\sqrt{2g}}\begin{pmatrix} \dot{P} & P' \\ P' & -\dot{P}\end{pmatrix},$$

$$P := \frac{\dot{g}}{g'},\ (g' \neq 0), \quad (33)$$

$$\varphi = -1 : \quad \left(B^{*k}_j\right) = \frac{sgn(\dot{g})}{2\sqrt{2g}}\begin{pmatrix} Q' & -\dot{Q} \\ -\dot{Q} & -Q'\end{pmatrix},$$

$$Q := 1/P,\ (\dot{g} \neq 0). \quad (34)$$

In case of timelike surfaces ($\varepsilon = 1$) we get with P, Q as above

$$\varphi = 1 : \quad \left(B^{*k}_j\right) = -\frac{sgn(\dot{g})}{2\sqrt{2g}}\begin{pmatrix} Q' & \dot{Q} \\ -\dot{Q} & -Q'\end{pmatrix}, \quad (35)$$

$$\varphi = -1 : \quad \left(B^{*k}_j\right) = -\frac{sgn(g')}{2\sqrt{2g}}\begin{pmatrix} \dot{P} & P' \\ -P' & -\dot{P}\end{pmatrix}. \quad (36)$$

From (33)-(36) the focal surfaces Ψ_φ are affine minimal surfaces.

(d) From (33)-(36) the affine Gauss-curvatures of the focal surfaces are related by

$$\begin{aligned} \varepsilon = -1 &: \quad K_a(\Psi_1) = \left(\frac{\dot{g}}{g'}\right)^4 K_a(\Psi_{-1}) \quad (37)\\ \varepsilon = 1 &: \quad K_a(\Psi_1) = -\left(\frac{g'}{\dot{g}}\right)^4 K_a(\Psi_{-1}). \quad (38) \end{aligned}$$

The components of the second fundamental form of the focal surfaces are

$$\varepsilon=-1,\varphi=1 \;:\; h^*_{11}=h^*_{22}=-\frac{|g'|}{\sqrt{g}}, h^*_{12}=0, \quad (39)$$

$$\varepsilon=-1,\varphi=-1 \;:\; h^*_{11}=h^*_{22}=\frac{|\dot{g}|}{\sqrt{g}}, h^*_{12}=0, \quad (40)$$

$$\varepsilon=1,\varphi=1 \;:\; h^*_{11}=-h^*_{22}=\frac{|\dot{g}|}{\sqrt{g}}, h^*_{12}=0, \quad (41)$$

$$\varepsilon=1,\varphi=-1 \;:\; h^*_{11}=-h^*_{22}=\frac{|g'|}{\sqrt{g}}, h^*_{12}=0. \quad (42)$$

Calculating the mean curvature of the focal surfaces from (39)-(42) using (26) together with (37) and (38) proves (d). □

3 Associated spacelike minimal surfaces

On a spacelike minimal surface in $\Phi\subset\mathbb{R}^3_1$ there are always global (I)-isothermal coordinates (see [14, p.184]), for instance the normalized ones of Lemma 1:

$$g:=g_{11}=g_{22}>0, g_{12}=0,$$
$$h_{11}=1, h_{22}=-1, h_{12}=0.$$

Consequently we have

$$H=0\Longleftrightarrow h_{11}+h_{22}=0\Longleftrightarrow f_{,11}+f_{,22}=0\Longleftrightarrow \Delta_{(I)}f=0, \quad (43)$$

so the coordinate functions $f^\alpha: U\subseteq\mathbb{R}^2\to\mathbb{R}$, $(\alpha=1,2,3)$ are harmonic. Then the conjugate harmonic functions are

$$\bar{f}^\alpha: U\subseteq\mathbb{R}^2\to\mathbb{R}, (\alpha=1,2,3),$$

related to f^α by

$$f^\alpha_{,1}=\bar{f}^\alpha_{,2}, \quad f^\alpha_{,2}=-\bar{f}^\alpha_{,1}\ (\alpha=1,2,3). \quad (44)$$

Then the one parameter family ${}^{(\lambda)}\Phi$ of associated minimal surfaces is parametrized by

$${}^{(\lambda)}f(u,v):=\cos\lambda\, f(u,v)-\sin\lambda\,\bar{f}(u,v), \lambda\in\mathbb{R}. \quad (45)$$

If $|\lambda_2-\lambda_1|=\pi/2$ the surfaces ${}^{(\lambda_1)}\Phi$ and ${}^{(\lambda_2)}\Phi$ are called *adjoined*. It is well known that the surfaces of the pencil ${}^{(\lambda)}\Phi$ share metric (I), normal vector and Gauss-curvature as in the Euclidean situation (see [14, p.184]).

Theorem 4 *Let Φ be a spacelike minimal surface in $\mathbb{R}^3_1$ with $K\neq 0$ and ${}^{(\lambda)}\Phi$ the family of the associated minimal surfaces. Then the following holds.*

(a) The affine normal vector $n_a(\Phi)$ of Φ is invariant: $n_a({}^{(\lambda)}\Phi)=n_a(\Phi)$,

(b) The affine Gauss-curvature of Φ is invariant: $K_a({}^{(\lambda)}\Phi)=K_a(\Phi)$

(c) Assuming the focal surfaces Ψ_1 and Ψ_{-1} of Φ not to be degenerate and denoting by $n_a(\Psi_1)$ and $n_a(\Psi_{-1})$ the affine normal vectors of Ψ_1 and Ψ_{-1} respectively, the figure of the three affine normals spanned by $n_a(\Phi)$, $n_a(\Psi_1)$, $n_a(\Psi_{-1})$ is invariant by translation along the orbit.

Proof

(a) We use the coordinates according to Lemma 1. From (45) we calculate the derivatives of ${}^{(\lambda)}f$ and from this

$${}^{(\lambda)}g_{jk} = g_{jk}, \quad (46)$$

$${}^{(\lambda)}D_{11}=-g\cos\lambda, {}^{(\lambda)}D_{12} = g\sin\lambda, {}^{(\lambda)}D_{22}=g\cos\lambda, \quad (47)$$

where (46) expresses the well known isometry. Further we get the components of the affine metric

$${}^{(\lambda)}G_{11}=-\sqrt{g}\cos\lambda, {}^{(\lambda)}G_{12}=\sqrt{g}\sin\lambda, {}^{(\lambda)}G_{22}=\sqrt{g}\cos\lambda. \quad (48)$$

Using this the affine normal is

$$n_a({}^{(\lambda)}\Phi)=\frac{1}{2g^{3/2}}(-\dot{g}f_{,1}+g'f_{,2}-2gn). \quad (49)$$

Because of $\varepsilon=-1$, comparison with (30) proves (a).

(b) Denoting the components of the affine shape operator of f and ${}^{(\lambda)}f$ by B^k_j and ${}^{(\lambda)}B^k_j$ respectively, we get

$$\left({}^{(\lambda)}B^k_j\right)=\begin{pmatrix} B^1_1\cos\lambda+B^2_1\sin\lambda & B^1_2\cos\lambda+B^2_2\sin\lambda \\ -B^1_1\sin\lambda+B^2_1\cos\lambda & B^2_2\cos\lambda-B^1_2\sin\lambda\end{pmatrix}. \quad (50)$$

From this we get

$$K_a({}^{(\lambda)}\Phi)=\det\left({}^{(\lambda)}B^k_j\right)=K_a(\Phi).$$

(c) From (46) we have ${}^{(\lambda)}K=K$. Together with the invariance of the normal $n(\Phi)$ and the affine normals $n_a(\Phi)$ and $n_a(\Psi_1)$, $n_a(\Psi_{-1})$ this gives the invariance of the figure of the three affine normals. □

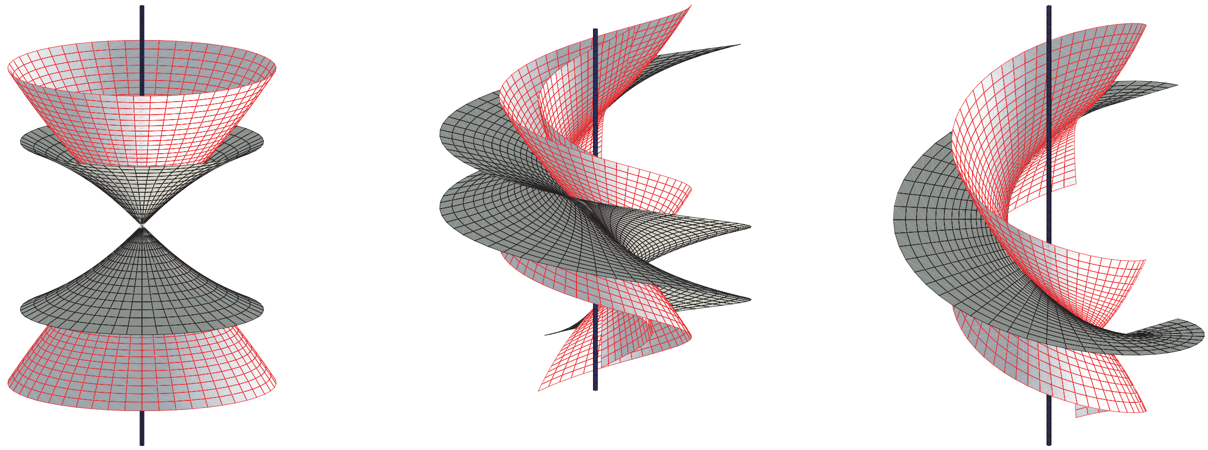

Figure 1a Figure 1b Figure 1c

Figure 1. *Associated spacelike surfaces and their focal surfaces (in red color).*

In figure 1 the dark surfaces are associated minimal surfaces (to the so called *elliptic catenoid* (62)). Starting with the elliptic catenoid (left figure) the adjoined surface is the spacelike portion of a right helicoid (right figure). It is well known, that the surfaces are *screw surfaces* in $\mathbb{R}^3_1$ ((see [12]). The red surfaces are the focal surfaces. The focal surface on the left is the surface (69), which is an improper affine sphere. Because of Theorem 4 the focal surfaces of every surface of the pencil is an improper affine sphere.

4 Associated timelike minimal surfaces

A timelike minimal surface in $\Phi \subset \mathbb{R}^3_1$ admits locally a representation in isotropic coordinates

$$f(u,v) = g(u) + h(v), \quad g : I \in \mathbb{R} \to \mathbb{R}^3_1, h : J \in \mathbb{R} \to \mathbb{R}^3_1, \tag{51}$$

where $g(I)$ and $h(J)$ are isotropic curves

$$\langle \dot{g}, \dot{g} \rangle = \langle h', h' \rangle = 0; \; \dot{g} := \frac{dg}{du}, h' := \frac{dh}{dv}.$$

(see [14, p.184] or [6, p.338]). So it is

$$g_{11} = g_{22} = 0, \; g_{12} = \langle \dot{g}, h' \rangle \neq 0. \tag{52}$$

This means a timelike minimal surface is locally a surface of translation with isotropic generating curves. The conjugate minimal surface $\bar{\Phi}$ is locally parame-trized by

$$\bar{f}(u,v) = g(u) - h(v), \tag{53}$$

and the family of associated minimal surfaces ${}^{(\lambda)}\Phi$ is given by

$${}^{(\lambda)}f(u,v) = \cosh f(u,v) + \sinh \bar{f}(u,v), \lambda \in \mathbb{R}. \tag{54}$$

Remark 2 : It is ${}^{(0)}f = f$. Obviously the surface $\bar{f}(U)$ does not to belong to the family of associated surfaces ([6, p.338]).

Analogous to Theorem 4 we have

Theorem 5 *Let Φ be a timelike minimal surface in $\mathbb{R}^3_1$ with $K \neq 0$ and ${}^{(\lambda)}\Phi$ the family of the associated minimal surfaces. Then the following holds.*

(a) The affine normal vector $n_a(\Phi)$ of Φ is invariant: $n_a({}^{(\lambda)}\Phi) = n_a(\Phi)$,

(b) The affine Gauss-curvature of Φ is invariant: $K_a({}^{(\lambda)}\Phi) = K_a(\Phi)$

(c) Assuming the focal surfaces Ψ_1 and Ψ_{-1} of Φ not to be degenerate and denoting by $n_a(\Psi_1)$ and $n_a(\Psi_{-1})$ the affine normal vectors of Ψ_1 and Ψ_{-1} respectively, the figure of the three affine normals spanned by $n_a(\Phi)$, $n_a(\Psi_1)$, $n_a(\Psi_{-1})$ is invariant by translation along the orbit.

Proof

(a) We use the isotropic coordinates from above. From (54) we calculate the derivatives of ${}^{(\lambda)}f$ and from this

$$ {}^{(\lambda)}g_{jk} = g_{jk}, \tag{55}$$

$$\begin{aligned} {}^{(\lambda)}D_{11} &= (\cosh\lambda + \sinh\lambda) D_{11}, \\ {}^{(\lambda)}D_{12} &= 0, \\ {}^{(\lambda)}D_{22} &= (\cosh\lambda - \sinh\lambda) D_{22}, \end{aligned} \tag{56}$$

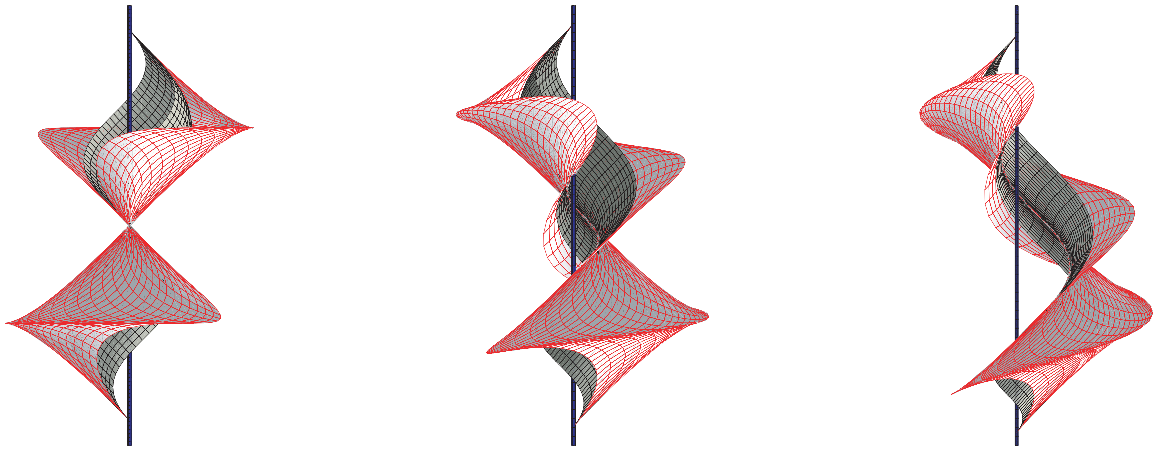

Figure 2a Figure 2b Figure 2c

Figure 2. *Associated timelike surfaces to the surface of rotation (65) and their focal surfaces (in red color).*

where (55) expresses the well known isometry. The components of the affine metric are

$$\begin{aligned} {}^{(\lambda)}G_{11} &= (\cosh\lambda+\sinh\lambda)G_{11}, \\ {}^{(\lambda)}G_{12} &= 0, \\ {}^{(\lambda)}G_{22} &= (\cosh\lambda-\sinh\lambda)G_{22}. \end{aligned} \tag{57}$$

Calculating the affine normal by (16) yields $n_a({}^{\lambda}\Phi) = n_a(\Phi)$.

(b) Denoting the components of the affine shape operator of f and ${}^{(\lambda)}f$ by B^k_j and ${}^{(\lambda)}B^k_j$ respectively, we get

$$\left({}^{(\lambda)}B^k_j\right) = \begin{pmatrix} e^{-\lambda}B^1_1 & e^{-\lambda}B^1_2 \\ e^{\lambda}B^2_1 & e^{\lambda}B^2_2 \end{pmatrix}. \tag{58}$$

Thus it is

$$K_a({}^{(\lambda)}\Phi) = \det\left({}^{(\lambda)}B^k_j\right) = K_a(\Phi).$$

(c) The argument is as in the proof of Theorem 4. □

5 Further Examples

A well known class of minimal surfaces in Minkowski space is that of rotation surfaces, that means surfaces admitting a one parameter family of isometries in $\mathbb{R}^3_1$ fixing the points of a straight line. If the axis is timelike (x^3-axis) or spacelike (x^1-axis) or isotropic ($x^1=x^3, x^2=0$) representations of this rotations are

$$\begin{pmatrix} x^1\\x^2\\x^3\end{pmatrix} \mapsto \begin{pmatrix} \cos v & -\sin v & 0\\ \sin v & \cos v & 0\\ 0 & 0 & 1\end{pmatrix}\begin{pmatrix} x^1\\x^2\\x^3\end{pmatrix}, \tag{59}$$

$$\begin{pmatrix} x^1\\x^2\\x^3\end{pmatrix} \mapsto \begin{pmatrix} 1 & 0 & 0\\ 0 & \cosh v & \sinh v\\ 0 & \sinh v & \cosh v\end{pmatrix}\begin{pmatrix} x^1\\x^2\\x^3\end{pmatrix}, \tag{60}$$

$$\begin{pmatrix} x^1\\x^2\\x^3\end{pmatrix} \mapsto \begin{pmatrix} 1-\frac{v^2}{2} & v & \frac{v^2}{2}\\ -v & 1 & v\\ -\frac{v^2}{2} & v & 1+\frac{v^2}{2}\end{pmatrix}\begin{pmatrix} x^1\\x^2\\x^3\end{pmatrix}. \tag{61}$$

There are seven types of minimal surfaces with rotational symmetry (see [1], [6], [7], [10], [15]).
The following surfaces are spacelike ($\varepsilon=-1$)

$$\begin{aligned} f(u,v) &= (\sinh u\cos v, \sinh u \sin v, u), & (62)\\ f(u,v) &= (u, \sin u \sinh v, \sin u\cosh v), & (63)\\ f(u,v) &= (u^3+u-uv^2, -2uv, u^3-u-uv^2). & (64) \end{aligned}$$

where the rotation axis is timelike, spacelike or isotropic respectively.
In case of timelike surfaces we have

$$\begin{aligned} f(u,v) &= (\sin u\cos v, \sin u\sin v, u), & (65)\\ f(u,v) &= (u, \sinh u\sinh v, \sinh u\cosh v), & (66)\\ f(u,v) &= (u, \pm\cosh u\cosh v, \cosh u\sinh v), & (67)\\ f(u,v) &= (-u^3+u-uv^2, -2uv, -u^3-u-uv^2). & (68) \end{aligned}$$

where in case of surfaces (65) and (68) the rotation axis is timelike and isotropic respectively. Surfaces (66) and (67) have a spacelike rotation axis. Clearly we have to exclude discrete values of u in order to have regular surfaces.

Calculation of the focal surfaces gives in case of surfaces (62), (63) and (64)

$$f(u,v) = (2\sinh u\cos v, 2\sinh u\sin v, u+\sinh u\cosh u), \quad (69)$$
$$f(u,v) = (u+\cos u\sin u, 2\sin u\sinh v, 2\sin u\cosh v), \quad (70)$$
$$f(u,v) = (-2u^3+2u-2uv^2, -4uv, -2u^3-2u-2uv^2). \quad (71)$$

and in case of surfaces (65), (66), (67) and (68)

$$f(u,v) = (2\sin u\cos v, 2\sin u\sin v, u+\cos u\sin u), \quad (72)$$
$$f(u,v) = (u+\cosh u\sinh u, 2\sinh u\sinh v, 2\sinh u\cosh v), \quad (73)$$
$$f(u,v) = (u-\cosh u\sinh u, \pm 2\cosh u\cosh v, 2\cosh u\sinh v), \quad (74)$$
$$f(u,v) = (2u^3+2u-2uv^2, -4uv, 2u^3-2u-2uv^2). \quad (75)$$

According to Theorem 3 the focal surfaces (69)-(75) are affine minimal surfaces, in fact we have improper affine spheres the affine normals of which are parallel to the axis of rotation.

Remark 3 In case of surfaces (64) and (68) the focal surfaces (71) and (75) respectively, coincide (up to a scaling factor 2) with the surfaces (68) and (64) respectively. That means: The (non degenerate) focal surface of a spacelike minimal surface of rotation with isotropic axis is a timelike minimal surface of rotation with isotropic axis and vice versa (figure 3).

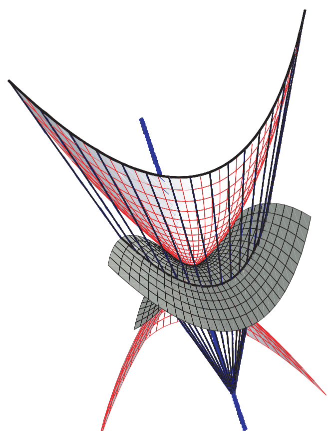

Figure 3a

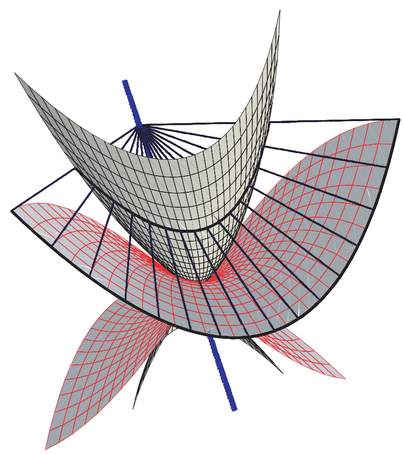

Figure 3b

Figure 3. *Surfaces of rotation with isotropic axis: Timelike (spacelike) surface as focal surface (in red color) of a spacelike (timelike) surface.*

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Friedrich Manhart
Technische Universität
Wiedner Hauptstraße 8–10/1131, A–1040 Wien
Austria
e-mail: manhart@geometrie.tuwien.ac.at