GLASNIK MATEMATIČKI
Vol. 53(73)(2018), 221 – 228

ON THE RAMANUJAN-NAGELL TYPE DIOPHANTINE EQUATION $x^2 + Ak^n = B$, II

Zhongfeng Zhang and Alain Togbé
Zhaoqing University, China and Purdue University Northwest, USA

Abstract. Let A, B be positive integers and q a prime. In this paper, we prove that the Ramanujan-Nagell type Diophantine equation $x^2 + Aq^n = B$ has at most four nonnegative integer solutions (x, n) for $q^2 \nmid B$ and $B \geq C$ where C is some constant depending of A. We also prove that the equation $x^2 + 3 \times 2^n = B$ has at most four nonnegative integer solutions (x, n). Therefore, we partially confirm a conjecture of Ulas ([4]).

1. Introduction

It is well-known that the Diophantine equation

$$x^2 + 7 = 2^{n+2} \tag{1.1}$$

is called *the Ramanujan-Nagell equation*. In 1960, Nagell ([2]) proved that only integer solutions to the Diophantine equation (1.1) are

$$(x, n) = (1, 1),\ (3, 2),\ (5, 3),\ (11, 5),\ (181, 13).$$

A generalized Ramanujan-Nagell equation is the Diophantine equation

$$x^2 + D = k^n \ \text{ in integers } x \geq 1,\ \ n \geq 1. \tag{1.2}$$

This Diophantine equation has a very rich literature. For examples, see the references in [6]. One aspect of the study of equation (1.2) is to determine the integer solutions (x, k, n). Diophantine equation (1.2) was studied for fixed values of D or when $D = \prod_i p_i^{a_i}$ with fixed prime numbers p_i.

2010 *Mathematics Subject Classification.* 11D41.
Key words and phrases. Diophantine equations.

Recently, many mathematicians have been interested in a more generalized Ramanujan-Nagell type equation of the form

$$x^2 = Ak^n + B, \quad k \in \mathbb{Z}_{\geq 2}, \quad n \geq 0, \quad A, B \in \mathbb{Z} \setminus \{0\}. \tag{1.3}$$

In 1996, Stiller ([3]) considered the equation

$$x^2 + 119 = 15 \cdot 2^n, \quad n \geq 0$$

and proved that this equation has exactly 6 solutions. His result motivated Ulas ([4]) to consider finding examples. He proved that for each $k \in \mathbb{Z} \setminus \{-1, 0, 1\}$ there are infinitely many pairs of integers A, B such that $\gcd(A, B)$ is square-free and Diophantine equation (1.3) has at least four solutions in non-negative integers. He was also able to solve some equations of the type (1.3) having five or more solutions. Besides proving many results, he also set many conjectures. In [5], we completely proved his Conjectures 4.2 and 4.3. Meng Bai and the first author ([1]) confirmed Conjecture 4.4 for $k = 2$, i.e. they proved that for any positive integer B, the Diophantine equation

$$x^2 + 2^n = B$$

has at most 3 solutions in nonnegative integers (x, n). The authors ([6]) partially confirmed Conjectures 4.4 and 4.5, i.e. they proved that for any positive integer B, the Diophantine equation

$$x^2 + Ak^n = B$$

has at most 3 solutions in nonnegative integers (x, n) for $A = 1, 2, 4$, and k an odd prime.

In this paper, we continue to consider the following conjecture:

Conjecture 1.1. *(Conjecture 4.5 in [4]) The Diophantine equation*

$$x^2 + Ak^n = B \tag{1.4}$$

has at most four nonnegative integer solutions (x, n)*, for any given integers* $k \geq 2$*,* $A \geq 1$*, and* $B \geq 1$*.*

We partially deal with the above conjecture by proving the following results.

Theorem 1.2. *Let* A, B *be positive integers such that* $4 \nmid B$*. Then the Diophantine equation*

$$x^2 + 2^n A = B \tag{1.5}$$

has at most four nonnegative integer solutions (x, n) *when* $B \geq B_0$*, where*

$$B_0 = 4A^4 - 6A^3 + \frac{17}{4}A^2 - \frac{1}{2}A + \frac{1}{4}.$$

THEOREM 1.3. *Let p be an odd prime, A, B positive integers such that $p^2 \nmid B$. Then the Diophantine equation*

$$x^2 + Ap^n = B \tag{1.6}$$

has at most four nonnegative integer solutions (x, n) when $B \geq B_1$, where

$$B_1 = \frac{A^4}{256} - \frac{A^3}{16} + \frac{3}{8}A^2 + 1.$$

THEOREM 1.4. *Let B be a positive integer. Then the Diophantine equation*

$$x^2 + 3 \times 2^n = B \tag{1.7}$$

has at most four nonnegative integer solutions (x, n).

REMARK 1.5. In contrast to Theorem 1.2, there is no condition on B in Theorem 1.4. When $B = 2212$, then equation (1.7) has exactly four nonnegative integer solutions

$$(x, n) = (26, 9), (38, 8), (46, 5), (47, 0).$$

2. Proof of Theorem 1.2

If $B < 2^4A = 16A$, then $n \leq 3$ and therefore equation (1.5) has at most four nonnegative integer solutions (x, n). If $2|B$, then $n \leq 1$ since $4 \nmid B$, and therefore equation (1.5) has at most two nonnegative integer solutions (x, n). Thus, for the remainder of the proof, we assume that $B \geq 16A$ and $2 \nmid B$. We divide the proof into three parts using the following lemmas.

LEMMA 2.1. *There is at most one nonnegative integer solution (x, n) satisfying*

$$2^nA < 2\sqrt{B - A} + A - 1.$$

PROOF. Suppose the contrary by assuming that (x_1, n_1) and (x_2, n_2) are two distinct integer solutions to equation (1.5) satisfying $2^{n_1}A < 2^{n_2}A < 2\sqrt{B - A} + A - 1$, so that $x_1 > x_2 \geq 0$. Thus, we get

$$x_1^2 - x_2^2 = A(2^{n_2} - 2^{n_1}) \leq A(2^{n_2} - 1)$$

and

$$x_1^2 - x_2^2 = (x_1 + x_2)(x_1 - x_2) \geq x_1 + x_2 \geq 2x_2 + 1.$$

This means that $2^{n_2}A - (A + 1) \geq 2x_2$, which yields

$$A^2 2^{2n_2} - 2A(A + 1)2^{n_2} + (A + 1)^2 \geq 4x_2^2 = 4(B - 2^{n_2}A).$$

Therefore, we obtain

$$A^2 2^{2n_2} - 2A(A - 1)2^{n_2} + (A - 1)^2 + 4A \geq 4B,$$

i.e.

$$(2^{n_2}A - (A - 1))^2 \geq 4(B - A),$$

which yields $2^{n_2}A \geq 2\sqrt{B - A} + A - 1$. This leads to a contradiction. ∎

Lemma 2.2. *There is at most one nonnegative integer solution* (x, n) *satisfying*

$$2^n A > 4A\sqrt{B + 4A^2 + 4A} - 4A(2A + 1).$$

Proof. Assume that (x_1, n_1) and (x_2, n_2) are two distinct integer solutions of equation (1.5) satisfying $x_1 > x_2 \geq 0$, $2^{n_2}A > 2^{n_1}A > 4A\sqrt{B + 4A^2 + 4A} - 4A(2A + 1)$. Then, we get

$$x_1^2 - x_2^2 = 2^{n_2}A - 2^{n_1}A = 2^{n_1}(2^{n_2 - n_1} - 1)A.$$

One can see that $2 \nmid x_i x_j$ since $2 \nmid B$ and $n_1 > 0$. Therefore, we have $n_1 > 2$ and $4 \nmid \gcd(x_1 + x_2, x_1 - x_2)$. We deduce that $2^{n_1 - 1} | x_1 + x_2$ or $2^{n_1 - 1} | x_1 - x_2$, so we get

$$2x_1 - 2 \geq x_1 + x_2 \geq 2^{n_1 - 1}.$$

This implies that

$$B - 2^{n_1}A = x_1^2 \geq (2^{n_1 - 2} + 1)^2 = 2^{2n_1 - 4} + 2^{n_1 - 1} + 1.$$

Thus, we deduce that

$$B + 4A^2 + 4A \geq (2^{n_1 - 2} + 2A + 1)^2,$$

which yields

$$2^{n_1}A \leq 4A\sqrt{B + 4A^2 + 4A} - 4A(2A + 1).$$

This leads to a contradiction. ∎

Lemma 2.3. *There are at most two nonnegative integer solutions* (x, n) *satisfying*

$$2\sqrt{B - A} + A - 1 \leq 2^n A \leq 4A\sqrt{B + 4A^2 + 4A} - 4A(2A + 1),$$

when $B \geq 4A^4 - 6A^3 + \frac{17}{4}A^2 - \frac{1}{2}A + \frac{1}{4} = B_0$.

Proof. Indeed, assume that (x_1, n_1), (x_2, n_2) and (x_3, n_3) are three distinct integer solutions of equation (1.5) satisfying $x_1 > x_2 > x_3 \geq 0$ and

$$2\sqrt{B - A} + A - 1 \leq 2^{n_1}A < 2^{n_2}A < A2^{n_3} \leq 4A\sqrt{B + 4A^2 + 4A} - 4A(2A + 1).$$

Then, we get

$$x_i^2 - x_j^2 = A2^{n_j} - A2^{n_i} = 2^{n_i}(2^{n_j - n_i} - 1)A$$

with $1 \leq i < j \leq 3$. It is obvious that $n_3 > n_2 > n_1 \geq 1$ and so $2 \nmid x_i x_j$ since $2 \nmid B$. Therefore, we have $4 \nmid \gcd(x_i + x_j, x_i - x_j)$, and so we obtain $2^{n_i - 1} | x_i + x_j$ or $2^{n_i - 1} | x_i - x_j$. Notice that $x_1 - x_3 = x_1 + x_2 - (x_2 + x_3)$, so we have $2^{n_1 - 1} | x_1 - x_2$ or $2^{n_2 - 1} | x_2 - x_3$ or $2^{n_1 - 1} | x_1 - x_3$. That is, there exist $1 \leq i_0 < j_0 \leq 3$ satisfying $2^{n_{i_0} - 1} | x_{i_0} - x_{j_0}$, therefore we obtain

$$x_{i_0} - x_{j_0} \geq 2^{n_{i_0} - 1}. \tag{2.1}$$

On the other hand, we have

$$x_i^2 = B - A2^{n_i} \leq B - 2\sqrt{B - A} - A + 1 = (\sqrt{B - A} - 1)^2,$$

that is $x_i \le \sqrt{B-A} - 1$. We also have

$$x_i^2 = B - A2^{n_i} \ge B - 4A\sqrt{B + 4A^2 + 4A} + 4A(2A+1)$$
$$= (\sqrt{B + 4A^2 + 4A} - 2A)^2,$$

that is $x_i \ge \sqrt{B + 4A^2 + 4A} - 2A$. So we get

$$x_{i_0} - x_{j_0} \le \sqrt{B-A} - 1 - \sqrt{B + 4A^2 + 4A} + 2A < 2A - 1. \tag{2.2}$$

We combine equations (2.1) and (2.2) to obtain $2A - 1 > 2^{n_{i_0}-1}$. This implies that

$$4A^2 - 2A > A2^{n_{i_0}} \ge 2\sqrt{B-A} + A - 1,$$

and therefore we get

$$B < 4A^4 - 6A^3 + \frac{17}{4}A^2 - \frac{1}{2}A + \frac{1}{4} = B_0,$$

which contradicts the assumption that $B \ge B_0$. □

Combining the above three lemmas, we complete the proof of Theorem 1.2.

3. Proof of Theorem 1.3

Before the proof of Theorem 1.3, we need the following lemma, which can be obtained from the proofs of Claim 1 and Claim 2 in Section 3 of [6].

Lemma 3.1. *Let p be an odd prime, A, B positive integers with $p \nmid B$ and $B \ge Ap^4$. Then the Diophantine equation* (1.6) *has at most one nonnegative integer solution (x, n) satisfying*

$$Ap^n < 4\sqrt{B-A} + A - 4$$

and at most one nonnegative integer solution (x, n) satisfying

$$Ap^n > A\sqrt{B + A + \frac{A^2}{4}} - A\left(1 + \frac{A}{2}\right).$$

We begin the proof of Theorem 1.3. If $B < Ap^4$, then $n \le 3$ and therefore equation (1.6) has at most four nonnegative integer solutions (x, n). If $p|B$, then $n \le 1$ since $p^2 \nmid B$. Hence, equation (1.6) has at most two nonnegative integer solutions (x, n). Thus, for the remainder of the proof, we assume that $B \ge Ap^4$ and $p \nmid B$. By Lemma 3.1, we only need to prove that there are at most two nonnegative integer solutions (x, n) for

$$4\sqrt{B-A} + A - 4 \le Ap^n \le A\sqrt{B + A + \frac{A^2}{4}} - A\left(1 + \frac{A}{2}\right).$$

Assume that (x_1, n_1), (x_2, n_2) and (x_3, n_3) are three distinct integer solutions of equation (1.6) satisfying $x_1 > x_2 > x_3 \geq 0$ and

$$4\sqrt{B-A}+A-4 \leq Ap^{n_1} < Ap^{n_2} < Ap^{n_3} \leq A\sqrt{B+A+\frac{A^2}{4}} - A\left(1+\frac{A}{2}\right).$$

Then, we get

$$x_i^2 - x_j^2 = Ap^{n_j} - Ap^{n_i} = Ap^{n_i}(p^{n_j - n_i} - 1)$$

with $1 \leq i < j \leq 3$. It is obvious that $n_3 > n_2 > n_1 \geq 1$ and so $p \nmid x_i x_j$ since $p \nmid B$. Since p is an odd prime, $p^{n_j-n_i} - 1 \equiv 0 \pmod 2$, thus $2 | x_i^2 - x_j^2$, so $2 | (x_i \pm x_j)$. Therefore, we obtain $2p^{n_i} | x_i + x_j$ or $2p^{n_i} | x_i - x_j$. Notice that $x_1 - x_3 = x_1 + x_2 - (x_2 + x_3)$, so we have $2p^{n_1} | x_1 - x_2$ or $2p^{n_2} | x_2 - x_3$ or $2p^{n_1} | x_1 - x_3$. This means that there exist $1 \leq i_0 < j_0 \leq 3$ satisfying $2p^{n_{i_0}} | x_{i_0} - x_{j_0}$. Thus, one sees that

$$x_{i_0} - x_{j_0} \geq 2p^{n_{i_0}}. \tag{3.1}$$

On the other hand, we have

$$x_i^2 = B - Ap^{n_i} \leq B - 4\sqrt{B-A} - A + 4 = (\sqrt{B-A} - 2)^2.$$

So $x_i \leq \sqrt{B-A} - 2$. We also have

$$x_i^2 = B - Ap^{n_i} \geq B - A\sqrt{B+A+\frac{A^2}{4}} + A\left(1+\frac{A}{2}\right) = (\sqrt{B+A+\frac{A^2}{4}} - \frac{A}{2})^2.$$

We deduce that $x_i \geq \sqrt{B+A+\frac{A^2}{4}} - \frac{A}{2}$. So we get

$$x_{i_0} - x_{j_0} \leq \sqrt{B-A} - 2 - \sqrt{B+A+\frac{A^2}{4}} + \frac{A}{2} < \frac{A}{2} - 2. \tag{3.2}$$

We combine (3.1) and (3.2) to obtain $\frac{A}{2} - 2 > 2p^{n_{i_0}}$. This implies that

$$\frac{A^2}{4} - A > Ap^{n_{i_0}} \geq 4\sqrt{B-A} + A - 4$$

and then

$$B < \frac{A^4}{256} - \frac{A^3}{16} + \frac{3}{8}A^2 + 1 = B_1.$$

Therefore, the proof of Theorem 1.3 is complete.

4. Proof of Theorem 1.4

If $A = 3$, then by Theorem 1.2, equation (1.7) has at most four nonnegative integer solutions (x, n) when $B \geq 199$ and $4 \nmid B$. A direct calculation by PARI/GP shows that equation (1.7) has at most three nonnegative integer solutions (x, n) when $B < 199$. We only need to prove the theorem for $B \geq 199$ and $4 | B$. We will use the following lemma.

LEMMA 4.1. *If $B = a^2 + a + 1$, for some positive integer a, then equation (1.7) has at most three nonnegative integer solutions (x, n).*

PROOF. It is obvious that $2 \nmid B$. We can assume $B \geq 199$, that is $a \geq 14$.

By the proof of Theorem 1.2, if there is no nonnegative integer solution (x, n) satisfying $3 \times 2^n < 2\sqrt{B-3}+2$, then there are at most three nonnegative integer solutions (x, n) for (1.7). Thus we only need to consider B when there is a solution (x, t) satisfying $3 \times 2^t < 2\sqrt{B-3} + 2$. In this case, we have

$$x^2 = B - 3 \times 2^t > B - 2\sqrt{B-3} - 2 = (\sqrt{B-3} - 1)^2,$$

that is

$$x > \sqrt{B-3} - 1 = \sqrt{a^2 + a - 2} - 1 > a - 1.$$

Then we obtain $x = a$ since $x < \sqrt{B} < a + 1$. Therefore, $3 \times 2^t = B - x^2 = a^2 + a + 1 - a^2 = a + 1$, that is $a = 3 \times 2^t - 1$. And then $B = (3 \times 2^t - 1)^2 + 3 \times 2^t - 1 + 1 = 9 \times 2^{2t} - 3 \times 2^t + 1$. Since $a \geq 14$, we can assume $t \geq 3$. Now we need to prove that there are at most three nonnegative integer solutions (x, n) for equation

$$x^2 + 3 \times 2^n = 9 \times 2^{2t} - 3 \times 2^t + 1, \quad t \geq 3. \tag{4.1}$$

Since $(x, n) = (3 \times 2^t - 1, t)$ is the only nonnegative integer solution with $3 \times 2^t < 2\sqrt{B-3}+2$, we know that $n \geq t$. We continue by proving that equation (4.1) has two nonnegative integer solutions for $t = 3$ and no nonnegative integer solutions for $t \geq 4$ when $n > t$.

Assume $n > t$. Since $3 \times 2^n \leq 9 \times 2^{2t} - 3 \times 2^t + 1 < 9 \times 2^{2t}$, we get $n \leq 2t + 1$, and so

$$x^2 \geq 9 \times 2^{2t} - 3 \times 2^t + 1 - 3 \times 2^{2t+1} = 3 \times 2^{2t} - 3 \times 2^t + 1 > 9 \times 2^{2t-2} - 3 \times 2^t + 1.$$

We deduce that $x^2 > (3 \times 2^{t-1} - 1)^2$ and obtain $x > 3 \times 2^{t-1} - 1$. We also have $x < 3 \times 2^t - 1$ since $n > t$.

From equation (4.1), we get $x^2 \equiv 1 \pmod{2^t}$, so $x = 2^{t-1}u \pm 1$. Using equation (4.1), we also have $x^2 \equiv 2^t + 1 \pmod{2^{t+1}}$ since $n > t$, that is

$$x^2 = 2^{2t-2}u^2 \pm 2^t u + 1 \equiv 2^t + 1 \pmod{2^{t+1}}.$$

Then we obtain $2 \nmid u$ because $t \geq 3$. Since $3 \times 2^{t-1} - 1 < x < 3 \times 2^t - 1$, we have $x = 3 \times 2^{t-1} + 1$ or $5 \times 2^{t-1} \pm 1$.

(i) Substituting $x = 3 \times 2^{t-1} + 1$ into equation (4.1) we get $2^n = 2^{t+1}(9 \times 2^{t-3} - 1)$. Therefore, $t = 3$ and $n = t + 4 = 7$.

(ii) We substitute $x = 5 \times 2^{t-1} - 1$ into equation (4.1) to obtain $3 \times 2^n = 2^{t+1}(11 \times 2^{t-3} + 1)$. Hence, $t = 3$ and $n = t + 3 = 6$.

(iii) The substitution of $x = 5 \times 2^{t-1} + 1$ into equation (4.1) gives $3 \times 2^n = 2^{t+1}(11 \times 2^{t-3} - 2^2)$ and then $3 \times 2^{n-t-1} = 11 \times 2^{t-3} - 2^2$. A direct calculation shows there is no solution for $t = 3, 4$. When $t \geq 5$, we have $3 \times 2^{n-t-3} = 11 \times 2^{t-5} - 1$, then $t - 5 = 0$, and $10 = 3 \times 2^{n-t-3}$, which is impossible. This completes the proof of Lemma 4.1. ∎

We continue to prove Theorem 1.4 for $B \geq 199$ and $4|B$ by induction. It is obvious that $n = 0$ or $n \geq 2$.

If $n = 0$, then $B = b^2 + 3$ for some b with $2 \nmid b$ since $4|B$. When $B = b^2 + 3$ and $n \geq 1$, then $n \geq 2$. Let $b = 2a + 1$, $n_1 = n - 2$, $x = 2x_1$, substitute into equation (1.7) we get $x_1^2 + 3 \times 2^{n_1} = a^2 + a + 1$, and by Lemma 4.1, there are at most three nonnegative integer solutions. Thus equation (1.7) has at most four nonnegative integer solutions for $B = b^2 + 3$.

If $B \neq b^2 + 3$ with $2 \nmid b$, then $n \geq 2$, and we have $2|x$ since $4|B$. Then equation (1.7) becomes $(\frac{x}{2})^2 + 3 \times 2^{n-2} = \frac{B}{4} < B$. By induction, there at most four nonnegative integer solutions. This completes the proof of Theorem 1.4.

5. Acknowledgements

The authors are grateful to the referees for the useful comments to improve the first version of this paper. The first author was supported by NSF of China (No. 11601476) and the Guangdong Provincial Natural Science Foundation (No. 2016A030313013).

References

[1] M. Bai and Z. Zhang, *On the Ramanujan-Nagell type equation $x^2 + 2^n = B$*, Journal of Southwest China Normal University (Natural Science Edition), **42 (6)** (2017), 5-7.

[2] T. Nagell, *The Diophantine equation $x^2 + 7 = 2^n$*, Ark. Math. **4** (1961), 185–187.

[3] J. Stiller, *The Diophantine equation $x^2 + 119 = 15 \cdot 2^n$ has exactly six solutions*, Rocky Mountain J. Math. **26** (1996), 295–298.

[4] M. Ulas, *Some experiments with Ramanujan-Nagell type Diophantine equations*, Glas. Mat. Ser. III **49(69)** (2014), 287–302.

[5] Z. Zhang and A. Togbé, *On two Diophantine equations of Ramanujan-Nagell type*, Glas. Mat. Ser. III **51(71)** (2016), 17–22.

[6] Z. Zhang and A. Togbé, *On the Ramanujan-Nagell type diophantine equation $x^2 + Ak^n = B$*, Glas. Mat. Ser. III **53(73)** (2018), 43–50.

Z. Zhang
School of Mathematics and Statistics
Zhaoqing University
Zhaoqing 526061
China
E-mail: bee2357@163.com

A. Togbé
Department of Mathematics, Statistics, and Computer Science
Purdue University Northwest
1401 S. U.S. 421 Westville, IN 46391
USA
E-mail: atogbe@pnw.edu

Received: 12.10.2018.
Revised: 19.11.2018.