GLASNIK MATEMATIČKI
Vol. 53(73)(2018), 449 – 475

ON THE EXISTENCE OF A SOLUTION OF A CLASS OF NON-STATIONARY FREE BOUNDARY PROBLEMS

Mahmoud Bousselsal, Abdeslem Lyaghfouri and Elmehdi Zaouche

Ecole Normale Supérieure, Vieux Kouba, Algeria, American University of Ras Al Khaimah, UAE and University of EL Oued, Algeria

Abstract. We consider a class of parabolic free boundary problems with heterogeneous coefficients including, from a physical point of view, the evolutionary dam problem. We establish existence of a solution for this problem. We use a regularized problem for which we prove existence of a solution by applying the Tychonoff fixed point theorem. Then we pass to the limit to get a solution of our problem. We also give a regularity result of the solutions.

1. Introduction and statement of the problem

A dam problem is a study of a fluid flow through a porous medium Ω, which is a bounded locally Lipschitz domain in $\mathbb{R}^n (n \geq 2)$. We are interested in the motion of compressible and incompressible fluids in Ω and in a time interval $[0, T]$ when we shall interest us with the problem of finding the pressure u of the fluid and the saturation χ of the wet part W of $Q := \Omega \times (0, T)$ which is unknown. The boundary Γ of Ω is divided in two parts. The impervious part Γ_1, and the part in contact with air or covered by fluid Γ_2 (see Figure 1), where we assume that Γ_2 is a nonempty relatively open subset of Γ. Let ϕ be a nonnegative Lipschitz continuous function defined in $\overline{Q}$, and let us set $\Sigma_1 = \Gamma_1 \times (0, T)$, $\Sigma_2 = \Gamma_2 \times (0, T)$, $\Sigma_3 = \Sigma_2 \cap \{\phi > 0\}$ and $\Sigma_4 = \Sigma_2 \cap \{\phi = 0\}$, where ϕ represents the assigned pressure on Σ_2. The velocity v and the pressure of the fluid in W are related to Darcy's law by

$$v = -a(x)\nabla(u + x_n), \tag{1.1}$$

2010 *Mathematics Subject Classification.* 76S05, 35B35.

Key words and phrases. Parabolic free boundary problems, evolution dam problem, existence, regularity.

where $x = (x_1, \dots, x_{n-1}, x_n) := (x', x_n) \in \mathbb{R}^n$ and $a(x)$ is an $n \times n$ matrix of regular functions, which represents the permeability of the porous medium.

Let us assume that the wet part $W = \{u > 0\}$ is given by

$$W = \{(x', x_n, t) \in Q/\, x_n < \Phi(x', t)\},$$

where Φ is a regular function on $\mathbb{R}^n$.

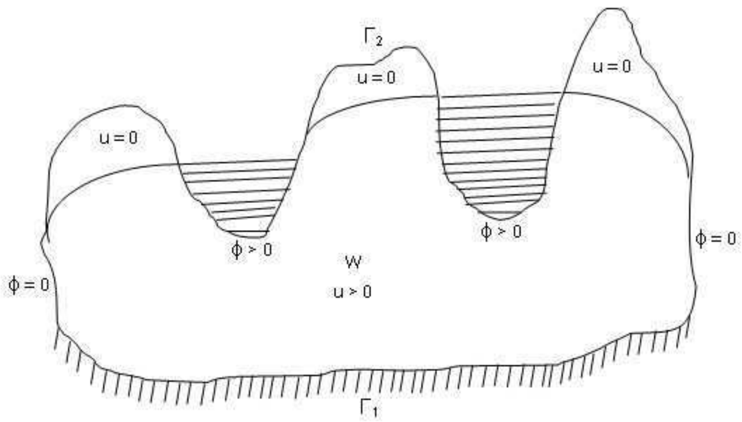

FIGURE 1. Dam

If we combine (1.1) with the mass conservation equation, we obtain

$$\alpha u_t - \operatorname{div}(a(x)\nabla(u + x_n)) = 0 \quad \text{in } W, \tag{1.2}$$

where α is a positive number, which refers to the state of the fluid, compressible ($\alpha > 0$) or incompressible ($\alpha = 0$). If we denote by ν the unit outward normal to Σ_1 and using the fact that no fluid flow can go through Σ_1, we obtain

$$v \cdot \nu = 0 \quad \text{on } \Sigma_1,$$

which can be written using (1.1) as

$$\frac{\partial}{\partial \nu_a}(u + x_n) = a(x)\nabla(u + x_n) \cdot \nu = 0 \quad \text{on } \Sigma_1.$$

The flow of fluid through Σ_4 can be written as

$$v \cdot \nu \geq 0 \quad \text{on } \Sigma_4$$

or, by (1.1),

$$a(x)\nabla(u + x_n) \cdot \nu \leq 0 \quad \text{on } \Sigma_4.$$

The pressure on Σ_2 is represented by ϕ, and thus

$$u = \phi \quad \text{on } \Sigma_2.$$

Let us assume that the free boundary is represented by the surface $\Sigma = \{x_n = \Phi(x', t)\} = \partial(\{u > 0\}) \cap Q$ and let us extend u outside $Q \backslash W$ and still denote by u this function which is supposed to be regular . Then, the outward unit normal to Σ is given by

$$\nu = (\nu_x, \nu_t) = -\frac{(\nabla_x u, u_t)}{\sqrt{|\nabla_x u|^2 + u_t^2}}.$$

Since $\nu \cdot (v, 1) = 0$ on Σ, we deduce from (1.1) that

$$\nu_t = a(x)\nabla(u + x_n) \cdot \nu_x \quad \text{on } \Sigma.$$

Thus we have, in the sense of distributions, for all $\xi \in \mathcal{D}(Q)$

$$\begin{aligned} \langle \operatorname{div}(a(x)\nabla u), \xi \rangle &= -\int_Q a(x)\nabla u \cdot \nabla \xi dxdt \\ &= -\int_{\{u>0\}} a(x)\nabla u \cdot \nabla \xi dxdt. \end{aligned} \tag{1.3}$$

Moreover, thanks to (1.2), we get

$$\begin{aligned} \int_{\{u>0\}} a(x)\nabla u \cdot \nabla \xi dxdt &= \int_\Sigma a(x)\nabla u \cdot \nu_x \xi d\sigma \\ &\quad - \int_{\{u>0\}} a(x)e \cdot \nabla \xi dxdt + \alpha \int_{\{u>0\}} u\,\xi_t dxdt \\ &= \int_\Sigma \nu_t \xi d\sigma - \int_{\{u>0\}} a(x)e \cdot \nabla \xi dxdt + \alpha \int_{\{u>0\}} u\,\xi_t dxdt \\ &= \int_{\{u>0\}} \xi_t dxdt - \int_{\{u>0\}} a(x)e \cdot \nabla \xi dxdt + \alpha \int_{\{u>0\}} u\,\xi_t dxdt, \end{aligned} \tag{1.4}$$

where $e = (0, \ldots, 0, 1) \in \mathbb{R}^n$. Using (1.3)-(1.4) and if we denote by $\chi_{\{u>0\}}$ the characteristic function of the set $\{u > 0\}$, we obtain

$$\langle \operatorname{div}(a(x)\nabla u), \xi \rangle = \int_Q \chi_{\{u>0\}} a(x)e.\nabla \xi dxdt - \int_Q (\alpha u + \chi_{\{u>0\}})\xi_t dxdt,$$

which leads to

$$\operatorname{div}(a(x)(\nabla u + \chi_{\{u>0\}} e)) - (\alpha u + \chi_{\{u>0\}})_t = 0 \quad \text{in } \mathcal{D}'(Q).$$

Now, if we add the initial condition and we consider a more general framework for the function $a(x)e$, which we denote by $H(x)$, we obtain the following strong formulation of a class of parabolic free boundary problems with heterogeneous coefficients $a(x)$ and $H(x)$: find $u, \chi : \overline{Q} \to \mathbb{R}$ such that

$$(1.5) \qquad \begin{cases} u \geq 0,\ 0 \leq \chi \leq 1,\ u(1-\chi) = 0 & \text{in } Q \\ \operatorname{div}(a(x)\nabla u + \chi H(x)) - (\alpha u + \chi)_t = 0 & \text{in } Q \\ u = \phi & \text{on } \Sigma_2 \\ (\alpha u + \chi)(\cdot, 0) = u_0 + \chi_0 & \text{in } \Omega \\ (a(x)\nabla u + \chi H(x)) \cdot \nu = 0 & \text{on } \Sigma_1 \\ (a(x)\nabla u + \chi H(x)) \cdot \nu \leq 0 & \text{on } \Sigma_4 \end{cases},$$

where, for a.e. $x \in \Omega,\ a(x) = (a_{ij}(x))_{ij}$ is an $n \times n$ matrix satisfying for two positive constants λ, Λ

$$(1.6) \qquad \forall \xi \in \mathbb{R}^n, \quad \text{a.e. } x \in \Omega : \ \ \lambda|\xi|^2 \leq a(x)\xi \cdot \xi,$$

$$(1.7) \qquad \forall \xi \in \mathbb{R}^n, \quad \text{a.e. } x \in \Omega : \ \ |a(x).\xi| \leq \Lambda|\xi|$$

and $H : \Omega \longrightarrow \mathbb{R}^n$ is a vector function satisfying for a positive constant $\overline{H}$

$$(1.8) \qquad |H(x)| \leq \overline{H} \quad \text{a.e.} x \in \Omega,$$

$$(1.9) \qquad \operatorname{div}(H(x)) \in L^2(\Omega).$$

Finally, u_0 and χ_0 are functions of the variable x such that we have for a positive constant U_0,

$$(1.10) \qquad 0 \leq \chi_0(x) \leq 1 \quad \text{a.e. } x \in \Omega,$$

$$(1.11) \qquad 0 \leq u_0(x) \leq U_0 \quad \text{a.e. } x \in \Omega.$$

To derive the weak formulation corresponding to (1.5), let us consider a regular function ξ. Then,

$$\int_Q \big[(a(x)\nabla u + \chi H(x)) \cdot \nabla \xi - (\alpha u + \chi)\xi_t\big] dxdt - \int_{\Sigma_2} (a(x)\nabla u + \chi H(x)) \cdot \nu \xi d\sigma$$
$$+ \int_\Omega (\alpha u + \chi)(x, T)\xi(x, T)dx - \int_\Omega (\alpha u_0(x) + \chi_0(x))\xi(x, 0)dx = 0$$

and if we assume that $\xi(\cdot, T) = 0$ in Ω, $\xi = 0$ on Σ_3, and $\xi \geq 0$ on Σ_4, we obtain

$$\int_Q \big[(a(x)\nabla u + \chi H(x)) \cdot \nabla \xi - (\alpha u + \chi)\xi_t\big] dxdt \leq \int_\Omega (\alpha u_0(x) + \chi_0(x))\xi(x, 0)dx.$$

This leads us to the following weak formulation

$$(1.12)\qquad \begin{cases} \text{Find } (u,\chi)\in L^2(0,T;H^1(\Omega))\times L^\infty(Q) \text{ such that :}\\ (i)\quad u\geq 0,\ 0\leq\chi\leq 1,\ u\cdot(1-\chi)=0 \quad \text{a.e. in } Q,\\ (ii)\quad u=\phi \quad \text{on } \Sigma_2,\\ (iii)\quad \displaystyle\int_Q \big[\big(a(x)\nabla u+\chi H(x)\big)\cdot\nabla\xi-(\alpha u+\chi)\xi_t\big]\,dx\,dt\\ \qquad\qquad \leq \displaystyle\int_\Omega (\chi_0(x)+\alpha u_0(x))\xi(x,0)\,dx\\ \qquad \forall\xi\in H^1(Q),\ \xi=0 \text{ on } \Sigma_3,\ \xi\geq 0 \text{ on } \Sigma_4,\\ \qquad\qquad \xi(x,T)=0 \ \text{ for a.e. } x\in\Omega. \end{cases}$$

From a physical point of view, this class contains the evolutionary dam problem. Thus, if $H(x)=a(x)e$, then (1.12) is the weak formulation of the evolutionary dam problem (see [21, 6, 20] for the evolutionary dam problem with homogeneous coefficients).

This work studies an expanded form of a class of parabolic free boundary problems including the evolutionary dam problem. Indeed, an existence result for a weak formulation of the evolution dam problem (with homogeneous coefficients) for an incompressible flow where $a(x)=I_n$ and $H(x)=I_n e$, and a domain with general geometry was established in [21], which was then extended in [6] to the compressible case. In [20] existence of a solution was given by a different method, both for compressible and incompressible fluids. For the problem with Neumann boundary condition we refer to [9, 18, 19, 26, 24]. For the problem with unified boundary condition and/or generalized nonlinear Darcy's law, we refer to [25] and [12] respectively for the stationary and evolutionary cases.

In [7] and [32], uniqueness was obtained by using the method of doubling variables respectively for a homogeneous porous medium with general geometry and for an incompressible fluid through a heterogeneous porous medium. Moreover, uniqueness has been proved in [20] and [28] by a different method for a rectangular dam wet at the bottom and dry near to the top, respectively, in homogeneous and heterogeneous domains. It is also difficult to adapt these methods to the general case.

In this paper, we establish an existence theorem of a solution for the class of parabolic free boundary problems (1.12) with heterogeneous coefficients $a(x)$ and $H(x)$, where $a(x)=(a_{ij}(x))_{ij}$ is an $n\times n$ matrix with variable coefficients satisfying (1.6)-(1.7) and $H(x)$ is a vector function satisfying (1.8)-(1.9). The method adopted in this study combines techniques from [21] and [6] by using the assumptions (1.6)-(1.9). Indeed, we start with a regularized problem (2.1) and we employ the uniform ellipticity of $a(x)$ and the boundedness of $a(x)$ and $H(x)$ for which we prove the comparison Lemma 2.1, and consequently, the uniqueness and that the solutions of (2.1) are uniformly bounded independently of ϵ. Thus, by applying the Tychonoff fixed point theorem we get existence of a unique solution of (2.1), denoted u_ϵ. Also, the hypotheses (1.6)-(1.9) leads to some a priori estimates as in Proposition 3.3, Lemma 3.4 and Lemma 4.2. These a priori estimates and the boundedness of u_ϵ will play important role in the proof of existence of a solution of our problem (1.12) by passing to the limit in (2.1) (see Theorem 4.1), and for the regularity result of the solutions (last section) including the regularity of u in $H^1_{loc}(Q)$ (see Proposition 5.1) where (u,χ) is a solution of the problem (1.12) obtained as the limit of the regularized problem (2.1).

2. A Regularized Problem

In order to establish the existence of a solution, we introduce the following approximated problem

$$\tag{2.1} \begin{cases} \text{Find } u_\epsilon \in H^1(Q) \text{ such that : } \quad u_\epsilon = \phi \text{ on } \Sigma_2 \\ \displaystyle\int_Q \big[\big(a(x)\nabla u_\epsilon + H_\epsilon(u_\epsilon)H(x)\big)\cdot\nabla\xi + \epsilon u_{\epsilon t}\xi_t - G_\epsilon(u_\epsilon)\xi_t\big]dx\,dt \\ \displaystyle\quad + \int_\Omega G_\epsilon(u_\epsilon(x,T))\xi(x,T)dx = \int_\Omega (\alpha u_0(x)+\chi_0(x))\xi(x,0)dx \\ \quad \forall\xi\in H^1(Q),\ \xi = 0 \text{ on } \Sigma_2, \end{cases}$$

where

$$H_\epsilon(r) = \begin{cases} 1 & \text{if } r\ge\epsilon \\ \dfrac{r}{\epsilon} & \text{if } 0\le r\le\epsilon \\ 0 & \text{if } r\le 0 \end{cases} \qquad \text{and} \qquad G_\epsilon(r) = \alpha r + H_\epsilon(r), \quad r\in\mathbb{R}.$$

For first, we establish the following lemma

Lemma 2.1. *Let $v_1, v_2 \in H^1(Q)$ such that $v_1 \ge v_2$ on Σ_2, $\delta > 0$, $f_\delta(s) = \frac{(s-\delta)^+}{s}\chi_{\{s>0\}}$, and $\xi_\delta = f_\delta(v_2 - v_1)$. Assume that we have for any $\delta > 0$,*

$$\tag{2.2} \begin{aligned} &\int_Q \Big[\Big(a(x)\nabla(v_2-v_1) + (H_\epsilon(v_2)-H_\epsilon(v_1))H(x)\Big)\cdot\nabla\xi_\delta \\ &\qquad + \epsilon(v_2-v_1)_t\xi_{\delta t} - (G_\epsilon(v_2)-G_\epsilon(v_1))\xi_{\delta t}\Big]dxdt \\ &\qquad + \int_\Omega (G_\epsilon(v_2(x,T)) - G_\epsilon(v_1(x,T)))\xi_{\delta t}(x,T)dx \le 0. \end{aligned}$$

Then we have

$$\tag{2.3} v_2 \le v_1 \quad a.e.\ in\ Q.$$

Proof. Since f_δ is Lipschitz continuous, we have $\xi_\delta \in H^1(Q)$ and we have for $z = x_1, \dots, x_n, t$

$$\frac{\partial \xi_\delta}{\partial z} = f'_\delta(v_2 - v_1)\frac{\partial(v_2 - v_1)}{\partial z} = \frac{\delta}{(v_2 - v_1)^2}\frac{\partial(v_2 - v_1)}{\partial z}\chi_{\{v_2 - v_1 > \delta\}}. \tag{2.4}$$

From (2.2) and (2.4), we have

$$\begin{aligned}&\delta\int_{\{v_2 - v_1 > \delta\}}\Big[\Big(a(x)\nabla(v_2 - v_1) + (H_\epsilon(v_2) - H_\epsilon(v_1))H(x)\Big).\frac{\nabla(v_2 - v_1)}{(v_2 - v_1)^2}\\ &+\epsilon\Big|\frac{(v_2 - v_1)_t}{v_2 - v_1}\Big|^2 - (G_\epsilon(v_2) - G_\epsilon(v_1))\frac{(v_2 - v_1)_t}{(v_2 - v_1)^2}\Big]dxdt+\\ &+\int_\Omega (G_\epsilon(v_2(x,T)) - G_\epsilon(v_1(x,T)))\frac{(v_2(x,T) - v_1(x,T) - \delta)^+}{v_2(x,T) - v_1(x,T)}dx \le 0.\end{aligned} \tag{2.5}$$

Since G_ϵ is nondecreasing, the last integral in the inequality (2.5) is nonnegative. Hence by (1.6), (1.8) and the Lipschitz continuity of H_ϵ, we get from (2.5)

$$\begin{aligned}&\int_{\{v_2 - v_1 > \delta\}}\lambda\Big|\frac{\nabla(v_2 - v_1)}{v_2 - v_1}\Big|^2 + \epsilon\Big|\frac{(v_2 - v_1)_t}{v_2 - v_1}\Big|^2 dxdt\\ &\qquad\le \int_{\{v_2 - v_1 > \delta\}}\frac{\overline{H}}{\epsilon}.\Big|\frac{\nabla(v_2 - v_1)}{v_2 - v_1}\Big| + \Big(\alpha + \frac{1}{\epsilon}\Big)\Big|\frac{(v_2 - v_1)_t}{v_2 - v_1}\Big|dxdt\end{aligned}$$

which leads by Young's inequality for some positive constant C independent of δ to

$$\int_{\{v_2 - v_1 > \delta\}}\Big|\frac{\nabla(v_2 - v_1)}{v_2 - v_1}\Big|^2 + \Big|\frac{(v_2 - v_1)_t}{v_2 - v_1}\Big|^2 dxdt \le C,$$

which in turn can be written as

$$\begin{aligned}&\int_Q\Big|\nabla\ln\Big(1 + \frac{(v_2 - v_1 - \delta)^+}{\delta}\Big)\Big|^2\\ &\qquad + \Big|\frac{\partial}{\partial t}\ln\Big(1 + \frac{(v_2 - v_1 - \delta)^+}{\delta}\Big)\Big|^2 dxdt \le C.\end{aligned} \tag{2.6}$$

By the Poincaré inequality, we obtain from (2.6) for another constant C' independent of δ,

$$\int_Q\Big|\ln\Big(1 + \frac{(v_2 - v_1 - \delta)^+}{\delta}\Big)\Big|^2 dxdt \le C'. \tag{2.7}$$

Letting $\delta \to 0$ in (2.7), we obtain (2.3). □

A first consequence of Lemma 2.1 is the uniqueness of the solution of (2.1).

Proposition 2.2. *There is at most one solution to problem (2.1).*

PROOF. Let us denote by $u_{\epsilon 1}$ and $u_{\epsilon 2}$ two solutions of (2.1). If we use $f_\delta(u_{\epsilon 2} - u_{\epsilon 1})$ as a test function for both solutions and subtract one equation from another, we see that (2.2) is satisfied with equality. It follows from Lemma 2.1 that $u_{\epsilon 2} \leq u_{\epsilon 1}$ a.e. in Q. Similarly, we obtain $u_{\epsilon 1} \leq u_{\epsilon 2}$ a.e. in Q. Therefore we have $u_{\epsilon 1} = u_{\epsilon 2}$ a.e. in Q. □

A second consequence of Lemma 2.1 is that any solution of (2.1) is uniformly bounded independently of ϵ.

PROPOSITION 2.3. *Let u_ϵ be a solution of (2.1). Then we have for some positive constant M independently of ϵ*

$$0 \leq u_\epsilon \leq M \quad a.e. \text{ in } Q. \tag{2.8}$$

PROOF. i) $\underline{u_\epsilon \geq 0 \text{ a.e. in } Q}$: We denote by $(\cdot)^-$ the negative part of a function. Since $\xi_\epsilon = (u_\epsilon)^-$ is a test function for (2.1), we obtain

$$\begin{aligned} &\int_Q \big[\big(a(x)\nabla u_\epsilon + H_\epsilon(u_\epsilon)H(x)\big)\cdot\nabla\xi_\epsilon + \epsilon u_{\epsilon t}\xi_{\epsilon t} - G_\epsilon(u_\epsilon)\xi_{\epsilon t}\big]dx\,dt \\ &\qquad + \int_\Omega G_\epsilon(u_\epsilon(x,T))\xi_\epsilon(x,T)dx = \int_\Omega (\alpha u_0(x) + \chi_0(x))\xi_\epsilon(x,0)dx. \end{aligned} \tag{2.9}$$

We have

$$\int_Q a(x)\nabla u_\epsilon \cdot \nabla\xi_\epsilon dxdt = -\int_{\{u_\epsilon \leq 0\}} a(x)\nabla u_\epsilon \cdot \nabla u_\epsilon dx \tag{2.10}$$

and

$$\int_Q \epsilon u_{\epsilon t}\xi_{\epsilon t} dxdt = -\int_{\{u_\epsilon \leq 0\}} \epsilon u_{\epsilon t}^2 dxdt. \tag{2.11}$$

Next, since $H_\epsilon(r) = 0$ if $r \leq 0$,

$$\int_Q H_\epsilon(u_\epsilon)H(x)\cdot\nabla\xi_\epsilon dxdt = 0. \tag{2.12}$$

Moreover, integrating on t and using the fact that $H'_\epsilon(r) = 0$ if $r < 0$, we get

$$\begin{aligned} &-\int_Q G_\epsilon(u_\epsilon)\xi_{\epsilon t} dxdt + \int_\Omega G_\epsilon(u_\epsilon(x,T))\xi_\epsilon(x,T)dx \\ &\qquad = -\int_{\{u_\epsilon(\cdot,0)\leq 0\}} \alpha u_\epsilon^2(x,0)dx + \int_Q \alpha u_{\epsilon t}\xi_\epsilon dxdt. \end{aligned} \tag{2.13}$$

But

$$\begin{aligned} \int_Q \alpha u_{\epsilon t}\xi_\epsilon dxdt &= \int_\Omega \alpha u_\epsilon(x,T)\xi_\epsilon(x,T)dx \\ &\quad - \int_\Omega \alpha u_\epsilon(x,0)\xi_\epsilon(x,0)dx - \int_Q \alpha u_\epsilon \xi_{\epsilon t} dxdt \\ &= -\int_{\{u_\epsilon(\cdot,T)\leq 0\}} \alpha u^2(x,T)dx + \int_{\{u_\epsilon(\cdot,0)\leq 0\}} \alpha u^2(x,0)dx - \int_{\{u_\epsilon\leq 0\}} \alpha u_{\epsilon t}u_\epsilon dxdt \end{aligned}$$

from which we deduce

$$\int_Q \alpha u_{\epsilon t}\xi_\epsilon dxdt = -\frac{\alpha}{2}\Big\{\int_{\{u_\epsilon(\cdot,T)\le 0\}} u^2(x,T)dx - \int_{\{u_\epsilon(\cdot,0)\le 0\}} u^2(x,0)dx\Big\}.$$

Then by (2.13), we get

$$\begin{aligned}(2.14)\qquad & -\int_Q G_\epsilon(u_\epsilon)\xi_{\epsilon t}dxdt + \int_\Omega G_\epsilon(u_\epsilon(x,T))\xi_\epsilon(x,T)dx \\ &= -\frac{\alpha}{2}\Big\{\int_{\{u_\epsilon(\cdot,T)\le 0\}} u^2(x,T)dx + \int_{\{u_\epsilon(\cdot,0)\le 0\}} u^2(x,0)dx\Big\}.\end{aligned}$$

Using (2.10)-(2.12), (2.14) and the fact that $\alpha u_0 + \chi_0 \ge 0,\ \xi_\epsilon(\cdot,0)\ge 0$ a.e. in Ω, we obtain from (2.9)

$$\int_{\{u_\epsilon\le 0\}} a(x)\nabla u_\epsilon\cdot\nabla u_\epsilon + \epsilon|u_{\epsilon t}|^2 dxdt \le 0.$$

Hence (1.6) leads to

$$\min(\lambda,\epsilon)\int_{\{u_\epsilon\le 0\}} |\nabla u_\epsilon|^2 + |u_{\epsilon t}|^2 dxdt \le 0.$$

Then we deduce that $u_\epsilon \ge 0$ a.e. in Q.

ii) $\underline{u_\epsilon \le M \text{ a.e. in } Q}$: Let v be the unique solution of the following problem

$$\begin{cases} v\in H^1(\Omega) \text{ such that:} \\ v = 1 \quad \text{on} \quad \Gamma_2, \\ \displaystyle\int_\Omega (a(x)\nabla v + H(x))\cdot\nabla\xi dx = 0, \\ \forall\xi\in H^1(\Omega),\ \xi = 0 \ \text{ on } \Gamma_2. \end{cases}$$

Applying Theorem 3 of [15] to $\pm v$, we obtain for two constants M_1 and M_2 depending on the data that

$$M_1 \le v \le M_2 \quad \text{a.e. in} \quad \Omega.$$

Setting $w = v - M_1 + \max(|\phi|_{L^\infty(Q)}, |u_0|_{L^\infty(\Omega)}, 1)$, we see that $w \ge 1$ a.e. in Ω, and consequently $\forall\epsilon\in(0,1]$, $H_\epsilon(w) = 1$ a.e. in Ω. Moreover we have $w \ge |\phi|_{L^\infty(Q)} \ge u_\epsilon$ on Γ_2. It follows that for each $\delta > 0$, the function

$\xi_\delta = f_\delta(u_\epsilon - w)$ vanishes on Σ_2. We deduce that

$$\begin{aligned}
&\int_Q \Big[(a(x)\nabla w + H_\epsilon(w)H(x))\cdot\nabla\xi_\delta + \epsilon w_t \xi_{\delta t} - G_\epsilon(w)\xi_{\delta t}\Big]dxdt \\
&\qquad + \int_\Omega G_\epsilon(w)\xi_\delta(x,T)dx \\
&\quad = \int_Q \Big[(a(x)\nabla w + H(x))\cdot\nabla\xi_\delta - (\alpha w+1)\xi_{\delta t}\Big]dxdt \\
&\qquad + \int_\Omega (\alpha w+1)\xi_\delta(x,T)dx \\
&\quad = \int_Q -(\alpha w+1)\xi_{\delta t}dxdt + \int_\Omega (\alpha w+1)\xi_\delta(x,T)dx \\
&\quad = \int_\Omega (\alpha w+1)\xi_\delta(x,0)dx.
\end{aligned} \tag{2.15}$$

Moreover, since ξ_δ is a test function for (2.1) we have

$$\begin{aligned}
&\int_Q \Big[(a(x)\nabla u_\epsilon + H_\epsilon(u_\epsilon)H(x))\cdot\nabla\xi_\delta + \epsilon u_{\epsilon t}\xi_{\delta t} - G_\epsilon(u_\epsilon)\xi_{\delta t}\Big]dxdt \\
&\qquad + \int_\Omega G_\epsilon(u_\epsilon(x,T))\xi_\delta(x,T)dx = \int_\Omega (\alpha u_0(x) + \chi_0(x))\xi_\delta(x,0)dx.
\end{aligned} \tag{2.16}$$

Subtracting (2.15) from and (2.16), we get

$$\begin{aligned}
&\int_Q \Big[\Big(a(x)\nabla(u_\epsilon - w) + (H_\epsilon(u_\epsilon) - H_\epsilon(w))H(x)\Big)\cdot\nabla\xi_\delta + \epsilon(u_\epsilon - w)_t\xi_{\delta t} \\
&\qquad - (G_\epsilon(u_\epsilon) - G_\epsilon(w))\xi_{\delta t}\Big]dxdt + \int_\Omega (G_\epsilon(u_\epsilon(x,T)) - G_\epsilon(w))\xi_\delta(x,T)dx \\
&\quad = \int_\Omega (\alpha(u_0(x) - w) + (\chi_0(x) - 1))\xi_\delta(x,0)dx \le 0
\end{aligned}$$

since $w \ge |u_0|_{L^\infty(\Omega)}$, $0 \le \chi_0 \le 1$ and $\xi_\delta(\cdot,0) \ge 0$ a.e. in Ω. Using Lemma 2.1, we obtain $u_\epsilon \le w$ a.e. in Q. In particular, we have $u_\epsilon \le |w|_{L^\infty(\Omega)} = M$ a.e. in Q. ∎

Remark 2.4. Let us define a truncation function of G_ϵ as follows

$$\overline{G}_\epsilon(r) = \begin{cases} G_\epsilon(M) & \text{if } \ r \ge M, \\ G_\epsilon(r) & \text{if } \ 0 \le r \le M, \\ 0 & \text{if } \ r \le 0. \end{cases}$$

It is easy to see that if u_ϵ is a solution of (2.1), then it also satisfies

$$\begin{aligned}&\int_Q \big[\big(a(x)\nabla u_\epsilon + H_\epsilon(u_\epsilon)H(x)\big)\cdot\nabla\xi + \epsilon u_{\epsilon t}\xi_t - \overline{G}_\epsilon(u_\epsilon)\xi_t\big]dx\,dt\\ &\qquad + \int_\Omega \overline{G}_\epsilon(u_\epsilon(x,T))\xi(x,T)dx = \int_\Omega(\alpha u_0(x)+\chi_0(x))\xi(x,0)dx\\ &\forall\xi\in H^1(Q),\ \xi=0 \text{ on } \Sigma_2.\end{aligned}\tag{2.17}$$

Conversely, if there exists a function $v_\epsilon \in H^1(Q)$ such that $v_\epsilon = \phi$ on Σ_2 and satisfies (2.17), then by arguing as in the proof of Proposition 2.2, we obtain $0 \le v_\epsilon \le M$, with the same positive constant M in Proposition 2.2. Hence v_ϵ is a solution of (2.1), and by uniqueness we have $v_\epsilon = u_\epsilon$.

Now, we shall deal with the question of existence of a solution to (P_ϵ).

Theorem 2.5. *The problem (2.1) has a solution.*

Proof. We observe that if we take into account Remark 2.4, then it is enough to prove the existence of a function $u_\epsilon \in H^1(Q)$ such that $u_\epsilon = \phi$ on Σ_2 and satisfies (2.17). We will give the proof in three steps

Step 1: We define

$$W = \{v\in H^1(Q)\ /\ v=0 \text{ on } \Sigma_2\},\quad K=\{v\in H^1(Q)/v=\phi \text{ on } \Sigma_2\},$$

and the mapping

$$\begin{aligned}&A : H^1(Q)\times H^1(Q)\longrightarrow\mathbb{R},\\ &(u,v)\longmapsto A(u,v)=\int_Q(a(x)\nabla u\cdot\nabla v+\epsilon u_t v_t)dxdt.\end{aligned}$$

Note that K is a nonempty closed convex subset of $H^1(Q)$. It is obvious from (1.7) that A is a bilinear continuous form on $H^1(Q)$. Thus, let us define $B : K \to (H^1(Q))'$ such that $\langle Bu, v\rangle = A(u, v)$ for all $v \in H^1(Q)$. Using (1.6) and the Poincaré inequality, we obtain for a positive constant $C > 0$

$$\langle Bu-Bv,u-v\rangle \ge C|u-v|^2_{1,2} > 0\quad \forall u,v\in K,\ u\ne v,$$

which implies that B is strictly monotone and coercive on K in the sense that there exists $v_0 = \phi \in K$ such that

$$\frac{\langle Bu-B\phi,u-\phi\rangle}{|u-\phi|_{1,2}}\to+\infty\quad \text{when } u\in K,\ |u|_{1,2}\to+\infty.$$

Now for $v \in H^1(Q)$, we consider the mapping

$$\begin{aligned}&f_v : H^1(Q)\longrightarrow\mathbb{R}\\ &\xi\longmapsto\int_Q \overline{G}_\epsilon(v)\xi_t - H_\epsilon(v)H(x)\cdot\nabla\xi dxdt\\ &\qquad+\int_\Omega(\alpha u_0(x)+\chi_0(x))\xi(x,0)dx-\int_\Omega\overline{G}_\epsilon(v(x,T))\xi(x,T)dx.\end{aligned}$$

It is clear that f_v is a linear form on $H^1(Q)$. Moreover, using (1.8), the continuity of the trace operator $H^1(Q) \to H^{1/2}(\Omega)$, and the fact that H_ϵ, $\overline{G}_\epsilon$, u_0 and χ_0 are bounded, it is not difficult to see that f_v is continuous on $H^1(Q)$. We conclude from [16, Theorem 1.10] that for each $v \in H^1(Q)$, there exists a unique $u_\epsilon \in K$ solution of the variational inequality

$$\langle Bu_\epsilon, w - u_\epsilon \rangle \geq \langle f_v, w - u_\epsilon \rangle \quad \forall w \in K.$$

If we choose in the above inequality $w = u_\epsilon \pm \xi$ with $\xi \in W$, then we get

$$A(u_\epsilon, \xi) = \langle f_v, \xi \rangle \quad \forall \xi \in W. \tag{2.18}$$

Step 2: We consider the mapping F_ϵ defined by

$$F_\epsilon : H^1(Q) \longrightarrow K, \quad v \longmapsto u_\epsilon.$$

If we denote by $\overline{B}(0,R)$ the closed ball in $H^1(Q)$ of center 0 and radius R, then we have the following statement.

Lemma 2.6. *There exists $R_\epsilon > 0$ such that*

i) $\exists R_\epsilon > 0 : \ F_\epsilon(\overline{B}(0,R_\epsilon)) \subset \overline{B}(0,R_\epsilon)$,

ii) $F_\epsilon : \overline{B}(0,R_\epsilon) \longrightarrow \overline{B}(0,R_\epsilon)$ *is weakly continuous.*

Proof. i) Since $u_\epsilon - \phi$ is a suitable test function for (2.1), we have

$$\begin{aligned}
&\int_Q \big[\big(a(x)\nabla u_\epsilon + H_\epsilon(v)H(x)\big) \cdot \nabla(u_\epsilon - \phi) \\
&\qquad + \epsilon u_{\epsilon t}(u_\epsilon - \phi)_t - \overline{G}_\epsilon(v)(u_\epsilon - \phi)_t\big] dx\, dt + \int_\Omega \overline{G}_\epsilon(v(x,T))(u_\epsilon - \phi)(x,T) dx \\
&= \int_\Omega (\alpha u_0(x) + \chi_0(x))(u_\epsilon - \phi)(x,0) dx,
\end{aligned}$$

which can be written as

$$\begin{aligned}
\int_Q a(x)\nabla u_\epsilon \cdot \nabla u_\epsilon + \epsilon u_{\epsilon t}^2 dxdt &= \int_Q a(x)\nabla u_\epsilon \cdot \nabla \phi dxdt \\
&- \int_Q H_\epsilon(v)H(x).\nabla u_\epsilon dxdt + \int_Q H_\epsilon(v)H(x).\nabla\phi dxdt \\
&+ \int_Q \epsilon u_{\epsilon t}\phi_t dxdt + \int_Q \overline{G}_\epsilon(v)u_{\epsilon t} dxdt - \int_Q \overline{G}_\epsilon(v)\phi_t dxdt \\
&- \int_\Omega \overline{G}_\epsilon(v(x,T))(u_\epsilon - \phi)(x,T) dx \\
&+ \int_\Omega (\alpha u_0(x) + \chi_0(x))(u_\epsilon - \phi)(x,0) dx.
\end{aligned} \tag{2.19}$$

Using (1.6)-(1.8), (1.10)-(1.11), Hölder's inequality, the fact that u_ϵ, $H_\epsilon(v)$ and $\overline{G}_\epsilon(v)$ are bounded, and that $\phi \in C^{0,1}(\overline{Q})$, and the continuity of the trace operator, we obtain from (2.19) the following estimate

$$|u_\epsilon|_{1,2}^2 \leq c_1(\epsilon)|u_\epsilon|_{1,2} + c_2(\epsilon),$$

which implies that we have for some positive constant R_ϵ depending on ϵ

$$|u_\epsilon|_{1,2} \leq R_\epsilon \quad \text{or} \quad |F_\epsilon(v)|_{1,2} \leq R_\epsilon.$$

Hence we have proved that $F_\epsilon(\overline{B}(0, R(\epsilon))) \subset \overline{B}(0, R(\epsilon))$.

ii) Let $(v_i)_{i\in I}$ be a generalized sequence in $C = \overline{B}(0, R(\epsilon))$ weakly converging to v in C for the weak topology of $H^1(Q)$. Set $u_\epsilon^i = F_\epsilon(v_i)$ and $u_\epsilon = F_\epsilon(v)$, and let us prove that $(u_\epsilon^i)_{i\in I}$ converges to u_ϵ weakly in C. Since C is compact with respect to the weak topology, it is enough to show that $(u_\epsilon^i)_{i\in I}$ has u_ϵ as unique limit point for that topology. So let u be a weak limit point for $(u_\epsilon^i)_{i\in I}$ in C. Using the following compact imbeddings $H^1(Q) \hookrightarrow L^2(Q)$ and $H^{1/2}(\Omega\times\{T\}) \hookrightarrow L^2(\Omega\times\{T\})$, one can construct two sequences $(v_{i_k})_{k\in\mathbb{N}}$ and $(u_\epsilon^{i_k})_{k\in\mathbb{N}}$ such that

$$v_\epsilon^{i_k} \to v \quad \text{strongly in} \quad L^2(Q), \tag{2.20}$$

$$v_\epsilon^{i_k}(\cdot, T) \to v(\cdot, T) \quad \text{strongly in} \quad L^2(\Omega\times\{T\}), \tag{2.21}$$

$$u_\epsilon^{i_k} \rightharpoonup u \quad \text{weakly in} \quad H^1(Q). \tag{2.22}$$

Writing (2.18) for $u_\epsilon^{i_k}$ and u_ϵ with $\xi = u_\epsilon^{i_k} - u_\epsilon$ and subtracting the two inequalities from each other, we obtain

$$\begin{aligned}
&\int_Q a(x)\nabla(u_\epsilon^{i_k} - u_\epsilon)\cdot\nabla(u_\epsilon^{i_k} - u_\epsilon) + \epsilon(u_\epsilon^{i_k} - u_\epsilon)_t^2\,dxdt \\
&\quad = \int_Q (\overline{G}_\epsilon(v_{i_k}) - \overline{G}_\epsilon(v))(u_\epsilon^{i_k} - u_\epsilon)_t\,dxdt \\
&\qquad - \int_Q (H_\epsilon(v_{i_k}) - H_\epsilon(v))H(x).\nabla(u_\epsilon^{i_k} - u_\epsilon)\,dxdt \\
&\qquad - \int_\Omega (\overline{G}_\epsilon(v_{i_k}(x,T)) - \overline{G}_\epsilon(v(x,T)))(u_\epsilon^{i_k} - u_\epsilon)(x,T)\,dx.
\end{aligned} \tag{2.23}$$

Using the Lipschitz continuity of $\overline{G}_\epsilon$ and H_ϵ, (1.8), Hölder's inequality, and Lemma 2.6 i), and the imbeddings $H^1(Q) \hookrightarrow L^2(Q)$, we obtain

$$\begin{aligned}
&\Big|\int_Q (\overline{G}_\epsilon(v_{i_k}) - \overline{G}_\epsilon(v))(u_\epsilon^{i_k} - u_\epsilon)_t\,dxdt\Big| \\
&\quad \leq \Big(\alpha + \frac{1}{\epsilon}\Big)|v_{i_k} - v|_2|u_\epsilon^{i_k} - u_\epsilon|_{1,2} \leq 2R_\epsilon\Big(\alpha + \frac{1}{\epsilon}\Big)|v_{i_k} - v|_{1,2},
\end{aligned} \tag{2.24}$$

$$\begin{aligned}
&\Big|\int_Q (H_\epsilon(v_{i_k}) - H_\epsilon(v))H(x)\cdot\nabla(u_\epsilon^{i_k} - u_\epsilon)\,dxdt\Big| \\
&\quad \leq \frac{1}{\epsilon}\overline{H}|v_{i_k} - v|_2.|u_\epsilon^{i_k} - u_\epsilon|_{1,2} \leq \frac{2R_\epsilon}{\epsilon}\overline{H}|v_{i_k} - v|_2.
\end{aligned} \tag{2.25}$$

Similarly we get by the Lipschitz continuity of $\overline{G}_\epsilon$, Hölder's inequality, and the continuity of the trace operator

$$\begin{aligned}&\Big|\int_\Omega(\overline{G}_\epsilon(v_{i_k}(x,T)) - \overline{G}_\epsilon(v(x,T)))(u_\epsilon^{i_k} - u_\epsilon)(x,T)dx\Big|\\ &\leq \Big(\alpha + \frac{1}{\epsilon}\Big)|(v_{i_k} - v)(x,T)|_{2,\Omega}|(u_\epsilon^{i_k} - u_\epsilon)(x,T)|_{2,\Omega}\\ &\leq cR_\epsilon\Big(\alpha + \frac{1}{\epsilon}\Big)|(v_{i_k} - v)(x,T)|_{2,\Omega}.\end{aligned}\tag{2.26}$$

Using (2.24)-(2.26) and (1.6), we deduce from (2.23)

$$|u_\epsilon^{i_k} - u_\epsilon|_{1,2} \leq c_\epsilon\big(|v_{i_k} - v|_2 + |(v_{i_k} - v)(x,T)|_{2,\Omega}\big)^{1/2}.\tag{2.27}$$

Combining (2.20)-(2.21) and (2.27), we get

$$u_\epsilon^{i_k} \to u_\epsilon \quad \text{strongly in } H^1(Q).$$

Taking into account (2.22), we obtain $u_\epsilon = u$ and therefore u_ϵ is the unique weak limit point of $(u_\epsilon^i)_{i\in I}$ in C. Thus we have

$$u_\epsilon^i = F_\epsilon(v^i) \rightharpoonup u_\epsilon = F_\epsilon(v) \quad \text{weakly in } C,$$

and the weak continuity of F_ϵ holds. ∎

<u>Step 3</u>: At this point, we can apply the Tychonoff fixed point theorem [30] to conclude that F_ϵ has a fixed point, which thanks to Remark 2.4 is a solution of problem (2.1). ∎

3. Regularity of the approximated solution

Proposition 3.1. *Assume that* Γ_2 *is of class* $C^{1,1}$, $\phi \in H^2(Q)$ *and* $a_{ij} \in C^{1,1}(\Omega \cup \Gamma_2)$. *Then we have* $u_\epsilon \in H^2_{loc}(Q \cup \Sigma_2)$.

Proof. Let $\xi \in \mathcal{D}(Q)$. Using $\pm\xi$ as test functions for (2.1), we obtain

$$\mathrm{div}(a(x)\nabla u_\epsilon + H_\epsilon(u_\epsilon)H(x)) + \epsilon u_{\epsilon tt} - G_\epsilon(u_\epsilon)_t = 0 \quad \text{in } \mathcal{D}'(Q).\tag{3.1}$$

We conclude from (3.1) and the fact that $u_\epsilon = \phi$ on Σ_2 [22, Lemma 9.16 p. 241]. ∎

Remark 3.2. If we use $\xi = \varphi\psi$, where $\varphi \in \mathcal{D}(\Omega)$, $\psi \in C^1(0,T]$, and $\psi(T) = 1$, as test functions for (2.1) and take into account (3.1), we obtain

$$u_{\epsilon t}(x,T) = 0 \;\text{ a.e. } x \in \Omega.\tag{3.2}$$

Proposition 3.3. *Assume that* $a(x)$ *is symmetric with* $a_{ij} \in C^{1,1}(\overline{\Omega})$. *Let* $\Omega' \subset\subset \Omega$ *be a nonempty open subset of* Ω *and* $\delta \in (0,T)$. *Then there exists* $\epsilon_0 > 0$ *small enough such that we have for some positive constant* $C(\Omega',\delta)$

$$\forall\epsilon \in (0,\epsilon_0): \quad \int_{\Omega'\times(\delta,T)}|u_{\epsilon t}|^2dxdt \leq C(\Omega',\delta).\tag{3.3}$$

We shall need the following lemma which will be useful for the existence proof of the solution.

LEMMA 3.4. *There exists $\epsilon_0 > 0$ small enough such that for any $\epsilon \in (0, \epsilon_0)$, we have*

$$\int_Q (\lambda|\nabla u_\epsilon|^2 + \epsilon|u_{\epsilon t}|^2)dxdt \le C, \tag{3.4}$$

where C is a constant independent of ϵ.

PROOF. Using $u_\epsilon - \phi$ as a test function for (2.1), we get

$$\begin{aligned}\int_Q \big[\big(a(x)\nabla u_\epsilon + H_\epsilon(u_\epsilon)H(x)\big)\cdot\nabla(u_\epsilon - \phi)\\ + \epsilon u_{\epsilon t}(u_\epsilon - \phi)_t - G_\epsilon(u_\epsilon)(u_\epsilon - \phi)_t\big]dx\,dt\\ + \int_\Omega G_\epsilon(u_\epsilon(x,T))(u_\epsilon - \phi)(x,T)dx\\ = \int_\Omega (\alpha u_0(x) + \chi_0(x))(u_\epsilon - \phi)(x,0)dx\end{aligned}$$

from which we deduce that

$$\begin{aligned}\int_Q a(x)\nabla u_\epsilon\cdot\nabla u_\epsilon + \epsilon|u_{\epsilon t}|^2dxdt &= \int_Q a(x)\nabla u_\epsilon\cdot\nabla\phi dxdt\\ &+ \int_Q \epsilon u_{\epsilon t}\phi_t dxdt - \int_Q H_\epsilon(u_\epsilon)H(x).\nabla u_\epsilon dxdt\\ &+ \int_Q G_\epsilon(u_\epsilon)u_{\epsilon t}dxdt + \int_Q H_\epsilon(u_\epsilon)H(x).\nabla\phi dxdt\\ &+ \int_Q G_\epsilon(u_\epsilon)\phi_t dxdt - \int_\Omega G_\epsilon(u_\epsilon(x,T))(u_\epsilon - \phi)(x,T)dx\\ &+ \int_\Omega (\alpha u_0(x) + \chi_0(x))(u_\epsilon - \phi)(x,0)dx.\end{aligned} \tag{3.5}$$

Using (1.7), (1.8), and the fact that H_ϵ is bounded, we obtain

$$\Big|\int_Q a(x)\nabla u_\epsilon\cdot\nabla\phi dxdt\Big| \le \Lambda\Big(\int_Q |\nabla u_\epsilon|^2dxdt\Big)^{1/2}\Big(\int_Q |\nabla\phi|^2dxdt\Big)^{1/2}, \tag{3.6}$$

$$\Big|\int_Q \epsilon u_{\epsilon t}\phi_t dxdt\Big| \le \epsilon_0^{1/2}\Big(\int_Q \epsilon|u_{\epsilon t}|^2dxdt\Big)^{1/2}.\Big(\int_Q |\phi_t|^2dxdt\Big)^{1/2}, \tag{3.7}$$

$$\Big|\int_Q H_\epsilon(u_\epsilon)H(x).\nabla u_\epsilon dxdt\Big| \le \overline{H}|Q|^{1/2}\Big(\int_Q |\nabla u_\epsilon|^2dxdt\Big)^{1/2}. \tag{3.8}$$

Setting $\widetilde{F}(r) = \int_0^r F(s)ds$ for any real function F of the real variable, integrating by parts, and using the fact that u_ϵ is bounded, we get

$$\begin{aligned}\int_Q G_\epsilon(u_\epsilon)u_{\epsilon t}dxdt &= \int_Q \frac{\partial}{\partial t}\widetilde{G}_\epsilon(u_\epsilon)dxdt \\ &= \int_\Omega \widetilde{G}_\epsilon(u_\epsilon(x,T))dx - \int_\Omega \widetilde{G}_\epsilon(u_\epsilon(x,0))dx \\ &\le \int_\Omega u_\epsilon(x,T)(\alpha u_\epsilon(x,T)+1)dx \le |\Omega|M(\alpha M+1) =: c.\end{aligned} \tag{3.9}$$

Using (3.5)-(3.9), (1.10)-(1.11), the fact that H_ϵ and u_ϵ are bounded, and $\phi \in C^{0,1}(\overline{Q})$, we obtain for some positive constant c_1,

$$\int_Q \lambda|\nabla u_\epsilon|^2 + \epsilon|u_{\epsilon t}|^2 dxdt \le c_1\Big\{1 + \Big(\int_Q \lambda|\nabla u_\epsilon|^2\Big)^{1/2} + \Big(\int_Q \epsilon|u_{\epsilon t}|^2\Big)^{1/2}\Big\},$$

which leads to $0 \le U_\epsilon \le C(1 + U_\epsilon^{1/2})$, where $U_\epsilon = \int_Q \lambda|\nabla u_\epsilon|^2 + \epsilon|u_{\epsilon t}|^2 dxdt$. Hence (3.4) holds. $\square$

Proof of Proposition 3.3. Let Ω' be as in the proposition. Let $\delta > 0$ and $\xi = \varphi\psi$ where $\varphi \in \mathcal{D}(\Omega)$ and $\psi \in C_0^1(0,T]$ such that $\varphi = 1$ in Ω', $\psi = 1$ in $(\delta, T]$. Multiplying (3.1) by $u_{\epsilon t}\xi^2$ and integrating over Q, we get

$$\begin{aligned}\alpha \int_Q u_{\epsilon t}^2\xi^2 dxdt &= \int_Q \operatorname{div}(a(x)\nabla u_\epsilon)u_{\epsilon t}\xi^2 dxdt \\ &+ \int_Q H_\epsilon'(u_\epsilon)\nabla u_\epsilon.H(x)u_{\epsilon t}\xi^2 dxdt + \int_Q H_\epsilon(u_\epsilon)\operatorname{div}(H(x))u_{\epsilon t}\xi^2 dxdt \\ &- \int_Q H_\epsilon'(u_\epsilon)u_{\epsilon t}^2\xi^2 dxdt + \int_Q \epsilon u_{\epsilon tt}u_{\epsilon t}\xi^2 dxdt.\end{aligned} \tag{3.10}$$

Using the symmetry of $a(x)$ and integrating by parts, we get

$$\begin{aligned}\int_Q \operatorname{div}(a(x)\nabla u_\epsilon).u_{\epsilon t}\xi^2 dxdt &= -\int_Q a(x)\nabla u_\epsilon \cdot \nabla(u_{\epsilon t}\xi^2)dxdt \\ &= -\frac{1}{2}\int_Q (a(x)\nabla u_\epsilon \cdot \nabla u_\epsilon)_t \xi^2 dxdt - 2\int_Q a(x)\nabla u_\epsilon \cdot \nabla\xi\ u_{\epsilon t}\xi dxdt \\ &= \int_Q a(x)\nabla u_\epsilon \cdot \nabla u_\epsilon\ \xi\xi_t dxdt - 2\int_Q a(x)\nabla u_\epsilon \cdot \nabla\xi\ u_{\epsilon t}\xi dxdt \\ &\quad - \frac{1}{2}\int_\Omega a(x)\nabla u_\epsilon(x,T)\cdot\nabla u_\epsilon(x,T)\xi^2(x,T)dx.\end{aligned} \tag{3.11}$$

Moreover we have by taking into account (3.2),

$$\int_Q \epsilon u_{\epsilon tt}u_{\epsilon t}\xi^2 dxdt = \frac{1}{2}\int_Q \epsilon(u_{\epsilon t}^2)_t\xi^2 dxdt = -\int_Q \epsilon u_{\epsilon t}^2\xi\xi_t dxdt. \tag{3.12}$$

Using (3.11)-(3.12) and (1.6), we deduce from (3.10) that

$$\begin{aligned}\alpha\int_Q u_{\epsilon t}^2\xi^2dxdt \le &\int_Q a(x)\nabla u_\epsilon\cdot\nabla u_\epsilon\ \xi\xi_t dxdt\\ &-2\int_Q a(x)\nabla u_\epsilon\cdot\nabla\xi\ u_{\epsilon t}\xi dxdt-\int_Q \epsilon u_{\epsilon t}^2\xi\xi_t dxdt\\ &+\int_Q H_\epsilon'(u_\epsilon)\xi^2(\nabla u_\epsilon.H(x)u_{\epsilon t}-u_{\epsilon t}^2)dxdt\\ &+\int_Q H_\epsilon(u_\epsilon)\operatorname{div}(H(x))u_{\epsilon t}\xi^2dxdt.\end{aligned}\tag{3.13}$$

Next we will prove that

$$\int_Q H_\epsilon'(u_\epsilon)\xi^2(\nabla u_\epsilon.H(x)u_{\epsilon t}-u_{\epsilon t}^2)dxdt\le c(\xi).\tag{3.14}$$

Multiplying (3.1) by $v_\epsilon\xi^2$ with $v_\epsilon=\min(u_\epsilon,\epsilon)$ and integrating over Q, we get

$$\begin{aligned}\int_Q \xi^2a(x)\nabla u_\epsilon\cdot\nabla v_\epsilon dxdt =& -2\int_Q v_\epsilon\xi a(x)\nabla u_\epsilon\cdot\nabla\xi dxdt\\ &-\int_Q \xi^2H_\epsilon(u_\epsilon)H(x)\cdot\nabla v_\epsilon dxdt-2\int_Q \xi v_\epsilon H_\epsilon(u_\epsilon)H(x)\cdot\nabla\xi dxdt\\ &-\int_Q \epsilon u_{\epsilon t}v_{\epsilon t}\xi^2dxdt-2\int_Q\epsilon u_{\epsilon t}v_\epsilon\xi\xi_t dxdt\\ &+\int_Q G_\epsilon(u_\epsilon)v_{\epsilon t}\xi^2dxdt+2\int_Q G_\epsilon(u_\epsilon)v_\epsilon\xi\xi_t dxdt\\ &-\int_\Omega G_\epsilon(u_\epsilon(x,T))v_\epsilon(x,T)\xi^2(x,T)dx.\end{aligned}\tag{3.15}$$

We have

$$-\int_Q \epsilon u_{\epsilon t}v_{\epsilon t}\xi^2dxdt=-\int_{\{u_\epsilon<\epsilon\}}\epsilon u_{\epsilon t}^2\xi^2dxdt\le 0,\tag{3.16}$$

$$\begin{aligned}&\int_Q G_\epsilon(u_\epsilon)v_{\epsilon t}\xi^2dxdt\\ &=\int_Q G_\epsilon(v_\epsilon)v_{\epsilon t}\xi^2dxdt=\int_Q\xi^2\frac{\partial}{\partial t}\widetilde{G}_\epsilon(v_\epsilon)dxdt\\ &=-2\int_Q\xi\xi_t\widetilde{G}_\epsilon(v_\epsilon)dxdt+\int_\Omega\xi^2(x,T)\widetilde{G}_\epsilon(v_\epsilon(x,T))dx\\ &\le 2\int_Q|\xi\xi_t|v_\epsilon G_\epsilon(v_\epsilon)dxdt+\int_\Omega\xi^2(x,T)v_\epsilon(x,T)G_\epsilon(v_\epsilon(x,T))dx.\end{aligned}\tag{3.17}$$

Moreover

$$\begin{aligned}
&-\int_Q \xi^2 H_\epsilon(u_\epsilon)H(x)\cdot\nabla v_\epsilon dxdt\\
&\qquad = -\int_Q \xi^2 H_\epsilon(v_\epsilon)H(x)\cdot\nabla v_\epsilon dxdt\\
&\qquad = -\int_Q \xi^2 H(x)\cdot\nabla(\widetilde{H}_\epsilon(v_\epsilon))dxdt\\
&\qquad = \int_Q \widetilde{H}_\epsilon(v_\epsilon)\operatorname{div}(\xi^2 H(x))dxdt\\
&\qquad \le \int_Q v_\epsilon H_\epsilon(v_\epsilon)|\operatorname{div}(\xi^2 H(x))|dxdt.
\end{aligned} \tag{3.18}$$

Using (1.7)-(1.9), (3.4), (3.16)-(3.18) and the fact that H_ϵ, G_ϵ are bounded, we deduce from (3.15) for a constant $c_1(\xi)$ independent of ϵ and for any $\epsilon \in (0,\epsilon_0)$

$$\int_Q \xi^2 a(x)\nabla u_\epsilon\cdot\nabla v_\epsilon dxdt \le \epsilon c_1(\xi). \tag{3.19}$$

Now, using (1.6) and (1.8), the fact that $H'_\epsilon(u_\epsilon)\ge 0$ and Young's inequality, we obtain

$$\begin{aligned}
&\int_Q H'_\epsilon(u_\epsilon)\nabla u_\epsilon.H(x)u_{\epsilon t}\xi^2 dxdt - \int_Q H'_\epsilon(u_\epsilon)u_{\epsilon t}^2\xi^2 dxdt\\
&\quad \le \frac{1}{2}\int_Q H'_\epsilon(u_\epsilon)\xi^2|\nabla u_\epsilon\cdot H(x)|^2 dxdt + \frac{1}{2}\int_Q H'_\epsilon(u_\epsilon)\xi^2 u_{\epsilon t}^2 dxdt\\
&\qquad - \int_Q H'_\epsilon(u_\epsilon)\xi^2 u_{\epsilon t}^2 dxdt\\
&\quad = \frac{1}{2}\int_Q H'_\epsilon(u_\epsilon)\xi^2|\nabla u_\epsilon.H(x)|^2 dxdt - \frac{1}{2}\int_Q H'_\epsilon(u_\epsilon)\xi^2 u_{\epsilon t}^2 dxdt\\
&\quad \le \int_Q H'_\epsilon(u_\epsilon)\xi^2|\nabla u_\epsilon.H(x)|^2 dxdt \le \overline{H}^2\int_Q H'_\epsilon(u_\epsilon)\xi^2|\nabla u_\epsilon|^2 dxdt\\
&\quad \le \frac{\overline{H}^2}{\lambda}\int_Q H'_\epsilon(u_\epsilon)\xi^2 a(x)\nabla u_\epsilon\cdot\nabla u_\epsilon dxdt\\
&\quad \le \frac{\overline{H}^2}{\lambda\epsilon}\int_Q \xi^2 a(x)\nabla u_\epsilon\cdot\nabla v_\epsilon dxdt.
\end{aligned} \tag{3.20}$$

Combining (3.19) and (3.20), we get (3.14).

Now, using (1.7), (3.4) and (3.14), we get from (3.13)

$$\begin{aligned}\alpha\int_Q u_{\epsilon t}^2\xi^2dxdt &\le \Lambda|\xi|_\infty|\xi_t|_\infty\int_Q|\nabla u_\epsilon|^2dxdt\\ &\quad+2\Lambda|\nabla\xi|_\infty\int_Q|\nabla u_\epsilon|.|u_{\epsilon t}|\xi dxdt\\ &\quad+|\xi|_\infty|\xi_t|_\infty\int_Q\epsilon u_{\epsilon t}^2dxdt+c(\xi)+\int_Q\xi|\operatorname{div}(H(x))|.|u_{\epsilon t}|\xi dxdt\\ &\le 2\Lambda|\nabla\xi|_\infty\int_Q|\nabla u_\epsilon|.|u_{\epsilon t}|\xi dxdt+\int_Q\xi|\operatorname{div}(H(x))|.|u_{\epsilon t}|\xi dxdt\\ &\quad+|\xi|_\infty|\xi_t|_\infty C\Big(\frac{\Lambda}{\lambda}+1\Big)+c(\xi).\end{aligned}\tag{3.21}$$

Applying Young's inequality and taking into account (1.9) and (3.4), we obtain from (3.21)

$$\begin{aligned}\alpha\int_Q u_{\epsilon t}^2\xi^2dxdt &\le \frac{\alpha}{4}\int_Q u_{\epsilon t}^2\xi^2dxdt+2\frac{\Lambda^2}{\alpha}|\nabla\xi|_\infty^2\int_Q|\nabla u_\epsilon|^2.\xi^2dxdt\\ &+\frac{\alpha}{4}\int_Q u_{\epsilon t}^2\xi^2dxdt+\frac{1}{\alpha}\int_Q\xi^2|\operatorname{div}(H(x))|^2dxdt+|\xi|_\infty|\xi_t|_\infty C\Big(\frac{\Lambda}{\lambda}+1\Big)+c(\xi)\end{aligned}$$

which leads to

$$\begin{aligned}\int_Q u_{\epsilon t}^2\xi^2dxdt &\le 4\frac{\Lambda^2C}{\lambda\alpha^2}|\nabla\xi|_\infty^2|\xi|_\infty^2+\frac{2}{\alpha^2}|\xi|_\infty^2\int_Q|\operatorname{div}(H(x))|^2dxdt\\ &+|\xi|_\infty|\xi_t|_\infty\frac{2C}{\alpha}\Big(\frac{\Lambda}{\lambda}+1\Big)+\frac{2c(\xi)}{\alpha}=:C(\xi).\end{aligned}$$

Since $\xi=1$ in $\Omega'\times(\delta,T)$, the estimate (3.3) holds from the last inequality. ∎

4. Existence of a solution

Theorem 4.1. *Assume that ϕ is a nonnegative Lipschitz continuous function, that (1.6)-(1.11) hold. Then there exists a solution (u,χ) to problem (1.12).*

The proof will consist in passing to the limit, when ϵ goes to 0, in (2.1). To do that we shall need a few preliminary lemmas.

Lemma 4.2. *Let u_ϵ be the solution of (2.1). Then we have*

$$\begin{aligned}&\int_Q\big[\big(a(x)\nabla u_\epsilon+H_\epsilon(u_\epsilon)H(x)\big)\cdot\nabla\xi+\epsilon u_{\epsilon t}\xi_t-G_\epsilon(u_\epsilon)\xi_t\big]dx\,dt\\ &\quad\le\int_\Omega(\alpha u_0(x)+\chi_0(x))\xi(x,0)dx\\ &\quad\forall\xi\in H^1(Q),\ \xi=0\ \text{on}\ \Sigma_3,\ \xi\ge0\ \text{on}\ \Sigma_4,\ \xi(x,T)=0\ \text{a.e. } x\in\Omega.\end{aligned}\tag{4.1}$$

PROOF. Let $\xi \in H^1(Q)$, $\xi = 0$ on Σ_3, $\xi \geq 0$ on Σ_4, $\xi(x,T) = 0$ a.e. $x \in \Omega$. Using for $\delta > 0$, $\min\big(\frac{u_\epsilon}{\delta}, \xi\big)$ as a test function for (2.1) and taking into account the fact that $\alpha u_0(x) + \chi_0(x) \geq 0$ a.e. $x \in \Omega$, and (1.6), we obtain

$$\begin{aligned}(4.2)\quad &\int_{\{u_\epsilon \geq \delta\xi\}} a(x)\nabla u_\epsilon \cdot \nabla\xi + \varepsilon u_{\epsilon t}\xi_t dxdt \\ &\qquad + \int_Q \Big[H_\epsilon(u_\epsilon)H(x)\cdot\nabla\Big(\min(\frac{u_\epsilon}{\delta},\xi)\Big) - G_\epsilon(u_\epsilon)\Big(\min(\frac{u_\epsilon}{\delta},\xi)\Big)_t\Big]dxdt \\ &\quad \leq \int_\Omega (\alpha u_0(x) + \chi_0(x))\xi(x,0)dx.\end{aligned}$$

Arguing as in [26], we can verify that

$$\begin{aligned}\lim_{\delta\to 0}\int_{\{u_\epsilon \geq \delta\xi\}} a(x)\nabla u_\epsilon \cdot \nabla\xi + \varepsilon u_{\epsilon t}\xi_t dxdt \\ = \int_Q a(x)\nabla u_\epsilon \cdot \nabla\xi + \varepsilon u_{\epsilon t}\xi_t dxdt,\end{aligned} \tag{4.3}$$

$$\lim_{\delta\to 0}\int_Q H_\epsilon(u_\epsilon)H(x)\cdot\nabla\min\big(\frac{u_\epsilon}{\delta},\xi\big)dxdt = \int_Q H_\epsilon(u_\epsilon)H(x)\cdot\nabla\xi dxdt, \tag{4.4}$$

$$\lim_{\delta\to 0}\int_Q G_\epsilon(u_\epsilon)\min\big(\frac{u_\epsilon}{\delta},\xi\big)_t dxdt = \int_Q G_\epsilon(u_\epsilon)\xi_t dxdt. \tag{4.5}$$

Letting $\delta \to 0$ in (4.2) and using (4.3)-(4.5), we obtain (4.1). ∎

LEMMA 4.3. *There exists a subsequence ϵ_k and $u \in L^2(0,T;H^1(\Omega))$, $\chi \in L^2(Q)$ such that*

$$u_{\epsilon_k} \rightharpoonup u \quad \text{weakly in } L^2(0,T;H^1(\Omega)), \tag{4.6}$$

$$H_{\epsilon_k}(u_{\epsilon_k}) \rightharpoonup \chi \quad \text{weakly in } L^2(Q). \tag{4.7}$$

Moreover, we have

$$u = \phi \quad \text{on } \Sigma_2, \quad u \geq 0 \quad \text{a.e. in } Q, \tag{4.8}$$

$$0 \leq \chi \leq 1 \quad \text{a.e. in } Q, \tag{4.9}$$

$$u\cdot(1-\chi) = 0 \quad \text{a.e. in } Q. \tag{4.10}$$

PROOF. First, (4.6)-(4.7) hold since $H_\epsilon(u_\epsilon)$ is uniformly bounded in Q and u_ϵ is bounded independently of ϵ in $L^2(0,T;H^1(\Omega))$ by (2.8) and (3.4). Next, the set

$$K_1 = \{v \in L^2(0,T;H^1(\Omega)) /\ v \geq 0 \quad \text{a.e. in } Q, \quad v = \phi \quad \text{on } \Sigma_2\}$$

is weakly closed in $L^2(0,T;H^1(\Omega))$, since it is closed and convex. Since $u_{\epsilon_k} \in K_1$, u is also in this set. So (4.8) holds. In the same way, the set

$$K_2 = \{v \in L^2(Q) /\ 0 \leq v \leq 1 \quad \text{a.e. in } Q\}$$

being closed and convex, it is weakly closed in $L^2(Q)$. Thus, since $H_{\epsilon_k}(u_{\epsilon_k}) \in K_2$, χ is in this set and (4.9) holds.

To prove (4.10), we need the following lemma.

Lemma 4.4.

$$G_{\epsilon_k}(u_{\epsilon_k}) \longrightarrow \alpha u + \chi \quad \textit{strongly in } L^2(0,T;V'), \tag{4.11}$$

where $V = \{v \in H^1(\Omega)\ ;\ v = 0 \text{ on } \Gamma_2\}$.

Proof. Define

$$w_{\epsilon_k} = -\epsilon_k u_{\epsilon_k t} + G_{\epsilon_k}(u_{\epsilon_k}). \tag{4.12}$$

Next, from (3.4), we have

$$\epsilon_k u_{\epsilon_k t} \longrightarrow 0 \quad \text{strongly in } L^2(Q) \tag{4.13}$$

since

$$\int_Q |\epsilon_k u_{\epsilon_k t}|^2 dxdt = \epsilon_k \int_Q \epsilon_k |u_{\epsilon_k t}|^2 dxdt \le \epsilon_k C.$$

Then from (4.6)-(4.7) and (4.12)-(4.13) we deduce that

$$w_{\epsilon_k} \rightharpoonup \alpha u + \chi \quad \text{weakly in } L^2(0,T;L^2(\Omega)). \tag{4.14}$$

We are going to prove that

$$w_{\epsilon_k} \rightharpoonup \alpha u + \chi \quad \text{weakly in } L^2(0,T;V'). \tag{4.15}$$

If we choose $\xi \in \mathcal{D}(0,T;V)$ as a test function for (P_{ϵ_k}), we have

$$\int_Q \big(a(x)\nabla u_{\epsilon_k} + H_{\epsilon_k}(u_{\epsilon_k})H(x)\big)\cdot\nabla\xi + \epsilon u_{\epsilon_k t}\xi_t - G_{\epsilon_k}(u_{\epsilon_k})\xi_t dxdt = 0,$$

which can be written as

$$\int_Q w_{\epsilon_k}\xi_t dxdt = \int_Q \big(a(x)\nabla u_{\epsilon_k} + H_{\epsilon_k}(u_{\epsilon_k})H(x)\big)\cdot\nabla\xi dxdt$$

and which leads by (1.7)-(1.8), (3.4) and Cauchy-Schwarz inequality to

$$\begin{aligned} &\Big|\int_Q w_{\epsilon_k}\xi_t dxdt\Big| \\ &\le \Lambda\Big(\Big(\int_Q |\nabla u_{\epsilon_k}|^2 dxdt\Big)^{1/2} + \overline{H}|Q|^{1/2}\Big)\cdot\Big(\int_Q |\nabla\xi|^2 dxdt\Big)^{1/2} \\ &\le C|\xi|_{L^2(0,T;V)}. \end{aligned} \tag{4.16}$$

Since (4.16) holds for any $\xi \in \mathcal{D}(0,T;V)$, we have proved that

$$|w_{\epsilon_k t}|_{L^2(0,T;V')} \le C, \tag{4.17}$$

i.e. that $w_{\epsilon_k t}$ is bounded in $L^2(0,T;V')$. At this point, we introduce the Banach vector space

$$Z = \{v \in L^2(0,T;L^2(\Omega))/v_t \in L^2(0,T;V')\}$$

under the norm

$$|v|_{L^2(0,T;L^2(\Omega))} + |v_t|_{L^2(0,T;V')}.$$

As explained in [26] the imbedding $Z \hookrightarrow L^2(0,T;V')$ is compact. Since by (2.8), (3.4) and (4.17), the sequence w_{ϵ_k} is bounded in Z, there exists a subsequence still denoted by ϵ_k, such that by (4.15),

$$w_{\epsilon_k} \to \alpha u + \chi \quad \text{strongly in } L^2(0,T;V').$$

which leads by (4.12)-(4.13) to (4.11). ∎

We now return to the proof of (4.10). We first observe that

$$0 \leq \int_Q u_{\epsilon_k}(1 - H_{\epsilon_k}(u_{\epsilon_k}))dxdt = \int_{Q\cap\{0\leq u_{\epsilon_k}\leq\epsilon_k\}} u_{\epsilon_k}(1 - H_{\epsilon_k}(u_{\epsilon_k}))dxdt \leq \epsilon_k|Q|$$

which leads to

$$\lim_{k\to+\infty}\int_Q u_{\epsilon_k}(1 - H_{\epsilon_k}(u_{\epsilon_k}))dxdt = 0. \tag{4.18}$$

We distinguish two cases

$\underline{* \ \alpha = 0}$: Using (4.6) and (4.11), we get

$$\begin{aligned}
&\lim_{k\to+\infty}\int_Q u_{\epsilon_k}(1 - H_{\epsilon_k}(u_{\epsilon_k}))dxdt\\
&\quad= \lim_{k\to+\infty}\int_Q (u_{\epsilon_k} - \phi)(1 - H_{\epsilon_k}(u_{\epsilon_k}))dxdt\\
&\qquad\quad + \lim_{k\to+\infty}\int_Q \phi(1 - H_{\epsilon_k}(u_{\epsilon_k}))dxdt\\
&\quad= \int_Q (u-\phi)(1-\chi)dxdt + \int_Q \phi(1-\chi)dxdt\\
&\quad= \int_Q u(1-\chi)dxdt.
\end{aligned} \tag{4.19}$$

It follows from (4.18)-(4.19) that we have

$$\int_Q u\cdot(1-\chi)dxdt = 0.$$

Since $u\cdot(1-\chi) \geq 0$ a.e. in Q, we obtain $u\cdot(1-\chi) = 0$ a.e. in Q.

$\underline{* \ \alpha > 0}$: Since $H_{\epsilon_k}(u_{\epsilon_k}) = G_{\epsilon_k}(u_{\epsilon_k}) - \alpha u_{\epsilon_k}$, we deduce from (4.18) that

$$\lim_{k\to+\infty}\int_Q u_{\epsilon_k}(1 - G_{\epsilon_k}(u_{\epsilon_k}) + \alpha u_{\epsilon_k})dxdt = 0,$$

which can be written, using (4.6) and (4.11), as

$$
\begin{aligned}
\lim_{k\to+\infty}\int_Q u_{\epsilon_k}^2 dxdt &= \frac{1}{\alpha}\lim_{k\to+\infty}\int_Q u_{\epsilon_k}(G_{\epsilon_k}(u_{\epsilon_k})-1)dxdt \\
&= -\frac{1}{\alpha}\lim_{k\to+\infty}\int_Q u_{\epsilon_k}dxdt+\frac{1}{\alpha}\lim_{k\to+\infty}\int_Q(u_{\epsilon_k}-\phi)G_{\epsilon_k}(u_{\epsilon_k})dxdt \\
&\quad +\frac{1}{\alpha}\lim_{k\to+\infty}\int_Q \phi\, G_{\epsilon_k}(u_{\epsilon_k})dxdt \\
&= -\frac{1}{\alpha}\int_Q udxdt+\frac{1}{\alpha}\int_Q(u-\phi)(\alpha u+\chi)dxdt+\frac{1}{\alpha}\int_Q\phi(\alpha u+\chi)dxdt \\
&= -\frac{1}{\alpha}\int_Q u(1-\chi)dxdt+\int_Q u^2dxdt.
\end{aligned}
\tag{4.20}
$$

Given that u_{ϵ_k} converges weakly to u in $L^2(Q)$, we have

$$
\int_Q u^2dxdt \le \lim_{k\to+\infty}\int_Q u_{\epsilon_k}^2dxdt. \tag{4.21}
$$

It follows from (4.20)-(4.21) that we have

$$
\int_Q u^2dxdt \le -\frac{1}{\alpha}\int_Q u(1-\chi)dxdt+\int_Q u^2dxdt \le \int_Q u^2dxdt,
$$

which leads to $\int_Q u(1-\chi)dxdt=0$. Since $u\cdot(1-\chi)\ge 0$ a.e. in Q, we obtain $u\cdot(1-\chi)=0$ a.e. in Q. ∎

Remark 4.5. We deduce from (4.20)-(4.21) that we have

$$
\lim_{k\to+\infty}\int_Q u_{\epsilon_k}^2dxdt=\int_Q u^2dxdt,
$$

which leads to

$$
u_{\epsilon_k}\longrightarrow u \quad \text{strongly in } L^2(Q).
$$

Proof of Theorem 4.1. It is clear that (1.12)(i) and (1.12) (ii) follow from Lemma 4.3. Let us establish (1.12)(iii). Let $\xi\in H^1(Q)$, $\xi=0$ on Σ_3, $\xi\ge 0$ on Σ_4 and $\xi(x,T)=0$ a.e. in Ω. Then we have by (4.1)

$$
\begin{aligned}
&\int_Q\big[\big(a(x)\nabla u_{\epsilon_k}+H_{\epsilon_k}(u_{\epsilon_k})H(x)\big)\cdot\nabla\xi+\epsilon_k u_{\epsilon_k t}\xi_t-G_{\epsilon_k}(u_{\epsilon_k})\xi_t\big]dx\,dt \\
&\qquad\le\int_\Omega(\alpha u_0(x)+\chi_0(x))\xi(x,0)dx.
\end{aligned}
\tag{4.22}
$$

Letting k go to ∞ in (4.22) and using (4.6)-(4.7), (4.11) and (4.13), we get

$$\int_Q \big[\big(a(x)\nabla u + \chi H(x)\big)\cdot\nabla\xi - (\alpha u+\chi)\xi_t\big]dx\,dt$$
$$\leq \int_\Omega (\alpha u_0(x) + \chi_0(x))\xi(x,0)dx,$$

which achieves the proof of Theorem 4.1 □

5. Regularity of the solution

In this section, we give two regularity results of the solutions of the problem (1.12). First, we have a restricted result.

Proposition 5.1. *Assume that* $\alpha > 0$ *and* $a(x)$ *is symmetric with* $a_{ij} \in C^{0,1}(\overline{\Omega})$. *Then there exists a solution* (u,χ) *of the problem (1.12) such that for any open set* $\Omega' \subset\subset \Omega$ *of* Ω *and any* $T > \delta > 0$, *we have* $u \in H^1(\Omega' \times (\delta, T))$.

Proof. Let (u,χ) be a solution of the problem (1.12) obtained as a limit of the sequence $(u_{\epsilon_k}, H_{\epsilon_k}(u_{\epsilon_k}))$, where u_{ϵ_k} is the solution of the problem (2.1) introduced in Sect. 2. Using the estimates (3.3)-(3.4), we see that u_{ϵ_k} is bounded in $H^1(\Omega'\times(\delta,T))$. Therefore there exists a subsequence still denoted by $(u_{\epsilon_k})_k$ such that

$$u_{\epsilon_k} \rightharpoonup u \quad \text{weakly in } \ H^1(\Omega' \times (\delta, T)).$$

□

Now we have a second regularity result of the solutions.

Proposition 5.2. *Let* (u,χ) *be a solution of problem (1.12). Then we have*

$$\alpha u + \chi \in C^0([0,T]; V').$$

We need a lemma.

Lemma 5.3. *We have* $(\alpha u + \chi)_t \in L^2(0,T;V')$.

Proof. Let $v \in \mathcal{D}(0,T;V)$. Since $\pm v$ are test functions for (1.12), we have

$$\int_Q (\alpha u + \chi)v_t dxdt = \int_Q \big(a(x)\nabla u + \chi H(x)\big)\cdot\nabla v dxdt,$$

which leads to

$$
\begin{aligned}
\Big|\int_Q(\alpha u+\chi)v_t dxdt\Big| &= \Big|\int_Q \big(a(x)\nabla u+\chi H(x)\big)\cdot\nabla v dxdt\Big| \\
&\le \int_Q \big(\Lambda|\nabla u|+\overline{H}\big)\cdot|\nabla v|dxdt \\
&\le \max(\Lambda,\overline{H})\Big|\int_Q \big(|\nabla u|+1\big).|\nabla v|dxdt\Big| \qquad (5.1)\\
&\le \sqrt{2}\max(\Lambda,\overline{H})\Big(\int_Q \big(|\nabla u|^2+1)dx\Big)^{1/2}.\Big(\int_Q |\nabla v|^2dxdt\Big)^{1/2} \\
&\le K|v|_{L^2(0,T;V)}
\end{aligned}
$$

where $K=\sqrt{2}\max(\Lambda,\overline{H})\Big(\int_Q \big(|\nabla u|^2+1)dx\Big)^{1/2}$. We deduce from (5.1) that the linear form

$$v\to\int_Q(\alpha u+\chi)v_t dxdt$$

is continuous on the dense subspace $\mathcal{D}(0,T;V)$ of the vector space $L^2(0,T;V)$ under its natural norm. It follows that it can be extended to a continuous linear form F up to $L^2(0,T;V)$. Since the distribution $(\alpha u+\chi)_t$ coincides with F on $\mathcal{D}(0,T;V)$, they coincide on $L^2(0,T;V)$. Hence $(\alpha u+\chi)_t\in L^2(0,T;V')$. ∎

Proof of Proposition 5.2. As a consequence of Lemma 5.3, we have $\alpha u+\chi\in H^1(0,T;V')$, which leads to the result by the Sobolev imbedding $H^1(0,T;V')\subset C^0([0,T];V')$. ∎

Remark 5.4. If $\alpha>0$ and $a(x)$ is symmetric and $a_{ij}\in C^{0,1}(\overline{\Omega})$, then there exists a solution (u,χ) of the problem (1.12) such that $\chi\in C^0((0,T];H^{-1}(\Omega'))$ for any nonempty open set $\Omega'\subset\subset\Omega$. Indeed, from Proposition 5.1, there exists a solution (u,χ) of the problem (1.12) such that for each $T>\delta>0$, $\Omega'\subset\subset\Omega$, $u\in H^1(\Omega'\times(\delta,T))$. We deduce from Lemma 5.3 that

$$\chi_t=(\chi+\alpha u)_t-\alpha u_t\in L^2(\delta,T;H^{-1}(\Omega')).$$

Hence $\chi\in C^0((\delta,T];H^{-1}(\Omega'))$.

References

[1] H. W. Alt, *Strömungen durch inhomogene poröse Medien mit freiem Rand,* J. Reine Angew. Math. **305** (1979), 89–115.

[2] C. Baiocchi, *Su un problema di frontiera libera connesso a questioni di idraulica,* Ann. Mat. Pura Appl. (4) **92** (1972), 107–127.

[3] C. Baiocchi, *Free boundary problems in fluid flows through porous media and variational inequalities*, in: Free boundary problems, Ist. Naz. Alta Mat. Francesco Severi, Rome 1980, 175–191.

[4] G. Bayada and M. Chambat, *Existence and uniqueness for a lubrication problem with nonregular conditions on the free boundary,* Boll. Un. Mat. Ital. B (6) **3** (1984), 543–557.

[5] A. Bermúdez, M. C. Muñiz and P. Quintela, *Existence and uniqueness for a free boundary problem in aluminum electrolysis,* J. Math. Anal. Appl. **191** (1995), 497–527.

[6] J. Carrillo, *An evolution free boundary problem: filtrations of a compressible fluid in a porous medium,* in: Contributions to nonlinear partial differential equations, Pitman, London, 1983, 97–110.

[7] J. Carrillo, *On the uniqueness of the solution of the evolution dam problem,* Nonlinear Anal. **22** (1994), 573–607.

[8] J. Carrillo and M. Chipot, *On the dam problem,* J. Differential Equations **45** (1982), 234–271.

[9] J. Carrillo and M. Chipot, *The dam problem with leaky boundary conditions,* Appl. Math. Optim. **28** (1993), 57–85.

[10] J. Carrillo and G. Gilardi, *La vitesse de propagation dans le problème de la digue,* Ann. Fac. Sc. de Toulouse Math. (5) **11** (1990), 7–28.

[11] J. Carrillo and A. Lyaghfouri, *The dam problem for nonlinear Darcy's law and Dirichlet boundary conditions,* Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4) **26** (1998), 453–505.

[12] J. Carrillo and A. Lyaghfouri, *A filtration problem with nonlinear Darcy's law and generalized boundary conditions,* Adv. Differential Equations **5** (2000), 515–555.

[13] S. Challal and A. Lyaghfouri, *A filtration problem through a heterogeneous porous medium,* Interfaces and Free Bound. **6** (2004), 55–79.

[14] S. Challal and A. Lyaghfouri, *On a class of free boundary problems of type* $div(a(X)\nabla u) = -\,div(H(X)\chi(u))$, Differential and Integral Equations **19** (2006), 481–516.

[15] M. Chicco and M. Venturino, *A priori inequalities for solutions of mixed boundary-value problems in unbounded domains,* Ann. Mat. Pura Appl. (4) **183** (2004), 241–259.

[16] M. Chipot, Variational inequalities and flow in porous media, Springer, Berlin, 1984.

[17] M. Chipot and M. C. Muñiz, *A free boundary problem modelling the electrolysis of aluminium,* Appl. Math. Optim. **47** (2003), 231–252.

[18] M. Chipot and A. Lyaghfouri, *The dam problem for non-linear Darcy's laws and non-linear leaky boundary conditions,* Math. Methods Appl. Sci. **20** (1997), 1045–1068.

[19] M. Chipot and A. Lyaghfouri, *The dam problem with linear Darcy's law and nonlinear leaky boundary conditions,* Adv. Differential Equations **3** (1998), 1–50.

[20] E. DiBenedetto, A. Friedman, *Periodic behaviour for the evolutionary dam problem and related free boundary problems,* Comm. Partial Differential Equations **11** (1986), 1297–1377.

[21] G. Gilardi, *A new approach to evolution free boundary problems,* Comm. Partial Differential Equations **4** (1979), 1099–1122; **5** (1980), 983–984.

[22] D. Gilbarg and N. S. Trudinger, Elliptic partial differential equations of second order, Springer, New York, 1983.

[23] A. Lyaghfouri, *The inhomogeneous dam problem with linear Darcy's law and Dirichlet boundary conditions,* Math. Models Methods Appl. Sci. **6** (1996), 1051–1077.

[24] A. Lyaghfouri, *The evolution dam problem for nonlinear Darcy's law and nonlinear leaky boundary conditions,* Ricerche di Matematica (2) **XLVII** (1998), 297–357.

[25] A. Lyaghfouri, *A unified formulation for the dam problem,* Riv. Mat. Univ. Parma (6) **1** (1998), 113–148.

[26] A. Lyaghfouri, *The evolution dam problem for nonlinear Darcy's law and Dirichlet boundary conditions,* Portugal. Math. **56** (1999), 1–37.

[27] A. Lyaghfouri, *A regularity result for a heterogeneous evolution dam problem,* Z. Anal. Anwendungen **24** (2005), 149–166.

[28] A. Lyaghfouri and E. Zaouche, *Uniqueness of solution of the unsteady filtration problem in heterogeneous porous media,* Rev. R. Acad. Cienc. Exactas Fs. Nat. Ser. A Math. RACSAM **112** (2018), 89–102.

[29] J. F. Rodrigues, *On the dam problem with leaky boundary condition,* Portugal. Math. **39** (1980), 399–411.

[30] D. R. Smart, *Fixed point theorems,* Cambridge University Press, Cambridge, 1974.

[31] A. Torelli, *Existence and uniqueness of the solution of a non steady free boundary problem,* Boll. Un. Mat. Ital. B (5) **14** (1977), 423–466.

[32] E. Zaouche, *Uniqueness of solution in a rectangular domain of an evolution dam problem with heterogeneous coefficients,* Electron. J. Differ. Equations (169) **2018** (2018), 1–17.

M. Bousselsal
Department of Mathematics
Labo. Part. Diff. Eq. & Hist. Maths
Ecole Normale Supérieure
16050 Vieux-Kouba Algiers
Algeria
E-mail: `bousselsal55@gmail.com`

A. Lyaghfouri
Department of Mathematics and Natural Sciences
American University of Ras Al Khaimah
Ras Al Khaimah
UAE
E-mail: `abdeslem.lyaghfouri@aurak.ac.ae`

E. Zaouche
Department of Mathematics
Labo. Oper. Theo. & Part. Diff. Eq.: Foundations and Applications
University of EL Oued
B. P. 789 El Oued 39000
Algeria
E-mail: `elmehdi-zaouche@univ-eloued.dz`

Received: 17.1.2018.
Revised: 14.10.2018.