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Jeřabek Hyperbola of a Triangle in an Isotropic Plane

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ABSTRACT

In this paper, we examine the Jeřabek hyperbola of an allowable triangle in an isotropic plane. We investigate different ways of generating this special hyperbola and derive its equation in the case of a standard triangle in an isotropic plane. We prove that some remarkable points of a triangle in an isotropic plane lie on that hyperbola whose centre is at the Feuerbach point of a triangle. We also explore other interesting properties of this hyperbola and its connection with some other significant elements of a triangle in an isotropic plane.

Key words: allowable triangle, standard triangle, Jeřabek hyperbola

MSC2010: 51N25

Jeřabekova hiperbola trokuta u izotropnoj ravnini

SAŽETAK

U radu proučavamo Jeřabekovu hiperbolu dopustivog trokuta u izotropnoj ravnini. Istražujemo različite načine generiranja ove specijalne hiperbole i izvodimo njenu jednadžbu u slučaju standardnog trokuta. Dokazujemo da neke značajne točke trokuta u izotropnoj ravnini leže na toj hiperboli čiji je centar u Feuerbachovoj točki trokuta. Proučavamo i neka druga zanimljiva svojstva ove hiperbole i njezinu vezu s nekim značajnim elementima trokuta u izotropnoj ravnini.

Ključne riječi: dopustivi trokut, standardni trokut, Jeřabekova hiperbola

In Euclidean geometry, the Jeřabek hyperbola of a triangle ABC is its circumscribed rectangular hyperbola, which is the isogonal image of the Euler line of this triangle. This hyperbola is generated by the centre of homology of the triangle ABC and the homothetic triangle to its tangential triangle $A_tB_tC_t$ with respect to the circumcentre of the triangle ABC. In Euclidean geometry, the Feuerbach hyperbola of a triangle ABC is its circumscribed rectangular hyperbola with the centre at the Feuerbach point Φ of this triangle. We will show that an analogous hyperbola exists in an isotropic plane, which unites the aforementioned properties and some other properties of these two hyperbolas. It is a special hyperbola obtained in [10], i.e., the Jeřabek hyperbola of an allowable triangle in an isotropic plane.

In an isotropic plane, a triangle is allowable if none of its sides is an isotropic line. Each allowable triangle in an isotropic plane can be set by a suitable choice of coordinates in the so–called *standard position*, where its circumscribed circle has the equation $y = x^2$, and its vertices are of the form $A = (a, a^2)$, $B = (b, b^2)$, $C = (c, c^2)$, while $a+b+c = 0$ (see [11]). By using the abbreviations

$$p = abc, \qquad q = bc+ca+ab,$$

we can get some useful expressions, e.g., $a^2+b^2+c^2 = -2q$, as well as, $q = bc-a^2$, $(c-a)(a-b) = 2q-3bc$, and identities derived therefrom by a cyclic permutation of a, b and c.

In [6], it is proved that the Brocard angle of a standard triangle ABC is given by the formula

$$\omega = -\frac{1}{3q}(b-c)(c-a)(a-b).$$

We will start with the following theorem:

Theorem 1 *Let $A_tB_tC_t$ be a tangential triangle of an allowable triangle and $A'B'C'$ the triangle obtained from the triangle $A_tB_tC_t$ by any translation into an isotropic direction. The triangles ABC and $A'B'C'$ are homologic, i.e., the lines AA', BB' and CC' pass through a point T, and the points $BC \cap B'C'$, $CA \cap C'A'$, $AB \cap A'B'$ lie on one line $\mathcal{T}$ (Figure 1).*

Proof. By [1], e.g. $A_t = \left(-\frac{a}{2}, bc\right)$, and then

$$A' = \left(-\frac{a}{2}, t+bc\right), \quad B' = \left(-\frac{b}{2}, t+ca\right),$$
$$C' = \left(-\frac{c}{2}, t+ab\right), \tag{1}$$

where t is a perimeter. The line with the equation

$$y = -\frac{2(t+q)}{3a}x + \frac{2t}{3} - \frac{q}{3} + bc$$

passes through the point $A = (a, a^2)$ and the point A' from (1) because we get

$$-\frac{2(t+q)}{3a} \cdot a + \frac{2t}{3} - \frac{q}{3} + bc = bc - q = a^2,$$
$$-\frac{2(t+q)}{3a} \cdot \left(-\frac{a}{2}\right) + \frac{2t}{3} - \frac{q}{3} + bc = t + bc,$$

and it is the line AA'. It passes through the point

$$T = \left(\frac{3p}{2(t+q)}, \frac{1}{3}(2t-q)\right) \tag{2}$$

because we get

$$-\frac{2(t+q)}{3a} \cdot \frac{3p}{2(t+q)} + \frac{2t}{3} - \frac{q}{3} + bc$$
$$= -\frac{p}{a} + bc + \frac{1}{3}(2t-q) = \frac{1}{3}(2t-q),$$

and analogously, the lines BB' and CC' pass through the point T. The line with the equation

$$y = 2ax + t - a^2 \tag{3}$$

passes through the points B' and C' from (1) because e.g. for the point B' we get

$$2a\left(-\frac{b}{2}\right) + t - a^2 = t + ca.$$

Therefore this is the line $B'C'$. From its equation (3) and the equation $y = -ax - bc$ of the line BC for the abscissa x of the point $BC \cap B'C'$ we get the equation $3ax = a^2 - t - bc$ with the solution $x = -\frac{1}{3a}(t+q)$, and then the equation of the line BC implies $y = \frac{1}{3}(t+q) - bc$. So we get

$$BC \cap B'C' = \left(-\frac{1}{3a}(t+q), \frac{1}{3}(t+q) - bc\right).$$

This point lies on the line $\mathcal{T}$ with the equation

$$\mathcal{T} \quad \ldots \quad y = \frac{3p}{t+q}x + \frac{t+q}{3} \tag{4}$$

because of

$$\frac{3p}{t+q} \cdot \left(-\frac{1}{3a}\right)(t+q) + \frac{t+q}{3} = \frac{1}{3}(t+q) - bc,$$

and the line $\mathcal{T}$ also passes through the analogous points $CA \cap C'A'$ and $AB \cap A'B'$. □

Corollary 1 *In the case of a standard triangle ABC, the point T and the line $\mathcal{T}$ are given by formulas (2) and (4).*

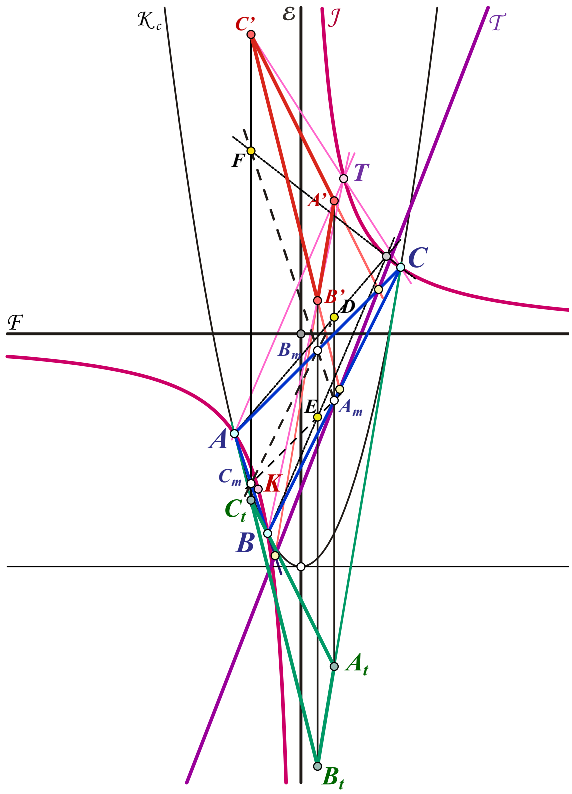

Figure 1.

Theorem 2 *The point T from Theorem 1 describes one special hyperbola $\mathcal{J}$ (Figure 1), which in the case of a standard triangle ABC has the equation*

$$xy + qx - p = 0. \tag{5}$$

Proof. The point T from (2) describes the curve $\mathcal{J}$ with the parametric equation

$$x = \frac{3p}{2(t+q)}, \quad y = \frac{1}{3}(2t-q). \tag{6}$$

This implies firstly $y + q = \frac{2}{3}(t+q)$, and then $x(y+q) = p$, i.e. the equation (5) is written in the form $y = \frac{p}{x} - q$, it implies that the curve $\mathcal{J}$ has an isotropic asymptote with the equation $x=0$, and by [11], it is the Euler line of a triangle ABC. Its nonisotropic asymptote is given by the equation $y=-q$, and by [1], it is the Feuerbach line of that triangle. The curve $\mathcal{J}$ is a special hyperbola with the centre $(0,-q)$, and by [1], it is the Feurbach point Φ of a triangle ABC. □

It is shown in [10] that a hyperbola with the equation (5) is the isogonal image of the Euler line of a triangle ABC and it is called the **Jeřabek hyperbola** of that triangle.

Corollary 2 *The Jeřabek hyperbola of an allowable triangle ABC is the isogonal image of its Euler line and this line is its isotropic asymptote, while its nonisotropic asymptote is the Feuerbach line of a triangle ABC, and its centre is the Feuerbach point of that triangle (Figure 1). The Jeřabek hyperbola of a standard triangle ABC has the equation (5).*

Inserting $y=x^2$ in (5), we get the equation $x^3+qx-p=0$ for the abscissa of the intersection of the hyperbola $\mathcal{J}$ with the circumscribed circle of the triangle ABC, as the abscissas a,b,c of the points A,B,C satisfy this equation, we obtain:

Corollary 3 *The Jeřabek hyperbola of a triangle is circumscribed to this triangle, i.e., it passes through its vertices (Figure 1).*

Inserting $t=0$ in (6) we get $x=\dfrac{3p}{2q}$, $y=-\dfrac{q}{3}$ while with $t=-\dfrac{3}{2}q$, $x=-\dfrac{3p}{q}$, $y=-\dfrac{4}{3}q$ is obtained. It follows that the hyperbola $\mathcal{J}$ passes through the points

$$K=\left(\frac{3p}{2q},-\frac{q}{3}\right),\quad \Gamma=\left(-\frac{3p}{q},-\frac{4}{3}q\right),$$

being, by [8] and [2], the symmedian centre and the Gergonne point of a triangle ABC. Further, with $t=-\dfrac{3}{2}q$ in (1) we get the point $A'=\left(-\dfrac{a}{2},-\dfrac{3}{2}q+bc\right)$. This point lies on the midline B_mC_m of a triangle ABC, which, by [11], has the equation $y=-ax+\dfrac{bc}{2}-q$ and

$$-a\left(-\frac{a}{2}\right)+\frac{bc}{2}-q=\frac{1}{2}(bc-q)+\frac{bc}{2}-q=bc-\frac{3}{2}q.$$

We have obtained:

Theorem 3 *The Jeřabek hyperbola of a triangle passes through its symmedian centre and its Gergonne point. If the points D,E,F are the intersections of the perpendicular bisectors of the sides BC,CA and AB of the triangle ABC with its corresponding midlines, then the lines AD,BE and CF pass through its Gergonne point Γ.*

The geometrical results of Theorems 1, 2 and 3 and Corollaries 1, 2 and 3 are analogous to the results in the Euclidean case (see [4] and [5]).

The following two statements are analogous to Theorems 1 and 2.

Theorem 4 *Let $\mathcal{A}'$, $\mathcal{B}'$ and $\mathcal{C}'$ be the lines parallel to the bisectors of the angles A,B and C and let the distances between them be proportional to the measure of these angles of the triangle ABC. These lines determine the triangle $A''B''C''$, which is homologic to the triangle ABC. In the case of a standard triangle ABC, the centre T'' of homology is the point*

$$T''=\left(-\frac{6pt}{q(2t-3\omega)},-q-\frac{q}{6t}(2t-3\omega)\right), \tag{7}$$

and the axis of homology $\mathcal{T}''$ is given by the equation

$$\mathcal{T}'' \;\ldots\; 3q(3\omega-2t)y=9ptx+3q\omega t^2+4q^2t-3q^2\omega. \tag{8}$$

Proof. By [7], the bisector of the angle A has the equation $y=\frac{a}{2}x+\frac{a^2}{2}$, and the line $\mathcal{A}'$ has the first of the three analogous equations

$$\begin{aligned}
\mathcal{A}' \quad &\ldots \quad y=\frac{a}{2}x+\frac{a^2}{2}+\frac{1}{2}(b-c)t,\\
\mathcal{B}' \quad &\ldots \quad y=\frac{b}{2}x+\frac{b^2}{2}+\frac{1}{2}(c-a)t,\\
\mathcal{C}' \quad &\ldots \quad y=\frac{c}{2}x+\frac{c^2}{2}+\frac{1}{2}(a-b)t,
\end{aligned} \tag{9}$$

where t is the perimeter. The point

$$A''=\left(a+\frac{3at}{b-c},\frac{qt}{b-c}-\frac{bc}{2}\right) \tag{10}$$

lies on the line $\mathcal{B}'$ and $\mathcal{C}'$ from (9) because e.g. for the line $\mathcal{B}'$ we get

$$\begin{aligned}
&\frac{b}{2}\left(a+\frac{3at}{b-c}\right)+\frac{b^2}{2}+\frac{1}{2}(c-a)t\\
&=\frac{t}{2(b-c)}[3ab+(b-c)(c-a)]+\frac{b}{2}(a+b)\\
&=\frac{t}{2(b-c)}\cdot 2q-\frac{bc}{2}=\frac{qt}{b-c}-\frac{bc}{2}.
\end{aligned}$$

Hence, $A''=\mathcal{B}'\cap\mathcal{C}'$. The line with the equation

$$y=\frac{q}{6at}(2t-3\omega)x+a^2-\frac{q}{6t}(2t-3\omega)$$

obviously passes through the point$A=(a,a^2)$, as well as through the points A'' and T'' from (10) and (7) because

$$\begin{aligned}&\frac{q}{6at}(2t-3\omega)\left(a+\frac{3at}{b-c}\right)+a^2-\frac{q}{6t}(2t-3\omega)\\&=\frac{q}{2(b-c)}(2t-3\omega)+a^2\\&=\frac{qt}{b-c}+\frac{1}{2(b-c)}\cdot(b-c)(c-a)(a-b)+a^2\\&=\frac{qt}{b-c}+\frac{1}{2}(2q-3bc)+bc-q=\frac{qt}{b-c}-\frac{bc}{2},\end{aligned}$$

$$\begin{aligned}&-\frac{q}{6at}(2t-3\omega)\cdot\frac{6pt}{q(2t-3\omega)}+a^2-\frac{q}{6t}(2t-3\omega)\\&=-bc+a^2-\frac{q}{6t}(2t-3\omega)=-q-\frac{q}{6t}(2t-3\omega).\end{aligned}$$

So, the point T'' lies on the line AA'', and analogously on the lines BB'' and CC''. The point

$$L=\left(\frac{q-3bc}{3a}-\frac{b-c}{3a}t,\frac{b-c}{3}t-\frac{q}{3}\right)$$

lies on the line BC with the equation $y=-ax-bc$ and on the line $\mathcal{A}'$ from (9) because

$$\begin{aligned}&-a\left(\frac{q-3bc}{3a}-\frac{b-c}{3a}t\right)-bc=bc-\frac{q}{3}+\frac{b-c}{3}t-bc\\&=\frac{b-c}{3}t-\frac{q}{3},\\&\frac{a}{2}\left(\frac{q-3bc}{3a}-\frac{b-c}{3a}t\right)+\frac{a^2}{2}+\frac{1}{2}(b-c)t\\&=\frac{1}{6}(q-3bc)-\frac{1}{6}(b-c)t+\frac{1}{2}(b-c)t+\frac{1}{2}(bc-q)\\&=\frac{1}{3}(b-c)t-\frac{q}{3},\end{aligned}$$

and then $L=BC\cap\mathcal{A}'$. Accordingly, the point L lies on the line $\mathcal{T}''$ with the equation (8)

$$\begin{aligned}&3q(3\omega-2t)\left(\frac{b-c}{3}t-\frac{q}{3}\right)-9pt\left(\frac{q-3bc}{3a}-\frac{b-c}{3a}t\right)\\&\quad-3q\omega t^2-4q^2t+3q^2\omega\\&=-2(b-c)qt^2+3(b-c)q\omega t+2q^2t-3q^2\omega-3bc(q-3bc)t\\&\quad+3bc(b-c)t^2+(b-c)(c-a)(a-b)t^2-4q^2t+3q^2\omega\\&=[(b-c)(3bc-2q)+(b-c)(c-a)(a-b)]t^2\\&\quad+(9b^2c^2-3bcq-2q^2)t-(b-c)^2(c-a)(a-b)t\\&=[9b^2c^2-3bcq-2q^2+(q+3bc)(2q-3bc)]t=0,\end{aligned}$$

and the analogous points $CA\cap\mathcal{B}''$ and $AB\cap\mathcal{C}''$ lie on the same line. □

Theorem 5 *The point T'' from Theorem 4 describes the Jeřabek hyperbola of a triangle ABC.*

Proof. The point T'' from (7) describes the curve with the parametric equation

$$x=-\frac{6pt}{q(2t-3\omega)},\quad y=-q-\frac{q}{6t}(2t-3\omega),$$

which immediately implies $x(y+q)=p$. □

The geometrical results of Theorems 4 and 5 are analogous to the results in the Euclidean case (see [3]).

Theorem 6 *Let $A_iB_iC_i$ be a contact triangle of a triangle ABC and $A'''B'''C'''$ the triangle obtained from the triangle $A_iB_iC_i$ by any translation into an isotropic direction. The triangles ABC and $A'''B'''C'''$ are homologic, and the centre T''' of this homology describes the Jeřabek hyperbola of a triangle ABC.*

Proof. By [1], let e.g. $A_i=(-2a,bc-2q)$; so then $A'''=(-2a,t+bc-2q)$. The line with the equation

$$y=-\frac{t-q}{3a}x+\frac{t}{3}-\frac{4}{3}q+bc$$

passes through the points $A=(a,a^2)$ and A''' because

$$-\frac{t-q}{3a}\cdot a+\frac{t}{3}-\frac{4}{3}q+bc=bc-q=a^2,$$

$$-\frac{t-q}{3a}(-2a)+\frac{t}{3}-\frac{4}{3}q+bc=t+bc-2q,$$

and also through the point

$$T'''=\left(\frac{3p}{t-q},\frac{1}{3}(t-4q)\right)$$

since

$$-\frac{t-q}{3a}\cdot\frac{3p}{t-q}+\frac{t}{3}-\frac{4}{3}q+bc=-bc+\frac{t}{3}-\frac{4}{3}q+bc=\frac{1}{3}(t-4q).$$

The point T''' describes the curve with the parametric equation

$$x=\frac{3p}{t-q},\quad y=\frac{1}{3}(t-q)-q,$$

which implies $x(y+q)=p$. □

The Jeřabek hyperbola is the isogonal image of the Euler line ([10]). The following property also holds.

Theorem 7 *The Jeřabek hyperbola of an allowable triangle is the reciprocal image of the line which is anticomplementary to its Brocard diameter, i.e., the isotropic line, which passes through the Gergonne point of that triangle.*

Proof. By [13], the reciprocity with respect to the standard triangle ABC is the mapping $(x,y) \longmapsto (x',y')$, where

$$x' = -\frac{3pqx^2+4q^2xy-9py^2+9p^2x+4q^3x-12pqy-4pq^2}{q^2x^2-9pxy-3qy^2-6pqx-4q^2y+9p^2}.$$

It therefore maps the line with the equation $x' = -\frac{3p}{q}$ to the curve with the equation

$$q(3pqx^2+4q^2xy-9py^2+9p^2x+4q^3x-12pqy-4pq^2) \\ -3p(q^2x^2-9pxy-3qy^2-6pqx-4q^2y+9p^2)=0,$$

and it is the equation

$$(4q^3+27p^2)(xy+qx-p)=0,$$

i.e. equation (5) since $4q^3+27p^2 \neq 0$. □

Theorem 8 *If A_0,B_0,C_0 are the intersections of the corresponding sides of the orthic triangle $A_hB_hC_h$ and the complementary triangle $A_mB_mC_m$ of an allowable triangle ABC, then the lines A_hA_0, B_hB_0 and C_hC_0 pass through the centre Φ of the Jeřabek hyperbola of a triangle ABC.*

Proof. According to [11], the lines B_hC_h and B_mC_m have the equations

$$y=2ax+2bc-q, \quad y=-ax+\frac{bc}{2}-q$$

and because of $bc-q=a^2$, they pass through the point

$$A_0=\left(-\frac{bc}{2a},a^2\right).$$

By [11], the point A_h is of the form $A_h=(a,q-2bc)$. The line with the equation

$$y=-2ax-q$$

passes through the points A_h and A_0 because of

$$-2a^2-q=-2(bc-q)-q=q-2bc,$$

$$-2a\left(-\frac{bc}{2a}\right)-q=bc-q=a^2,$$

which is the line A_hA_0. However, this line obviously passes through the point $\Phi=(0,-q)$. □

An analogous statement of Theorem 8 in the Euclidean case is given in [12].

Now we will use the parametric equation (6) of the hyperbola $\mathcal{J}$.

Theorem 9 *The points T_1 and T_2 of the Jeřabek hyperbola with the parametric equation (6), which correspond to the values t_1 and t_2 of the parameter t, have the joint line with the equation*

$$y=-\frac{4}{9p}(t_1+q)(t_2+q)x+\frac{1}{3}(2t_1+2t_2+q). \tag{11}$$

Proof. The point T from (2) with $t=t_1$ lies on the line (11) since we get

$$-\frac{4}{9p}(t_1+q)(t_2+q)\cdot\frac{3p}{2(t_1+q)}+\frac{1}{3}(2t_1+2t_2+q) \\ =-\frac{2}{3}(t_2+q)+\frac{1}{3}(2t_1+2t_2+q)=\frac{1}{3}(2t_1-q).$$ □

Corollary 4 *At the point T given by equality (2), the Jeřabek hyperbola with the equations (6) has the tangent with the equation*

$$y=-\frac{4}{9p}(t+q)^2x+\frac{1}{3}(4t+q). \tag{12}$$

Theorem 10 *The point T given by equality (2) is isogonal, with respect to a triangle ABC, to the point*

$$T'=\left(0,-\frac{2}{3}(t+q)\right). \tag{13}$$

Proof. With $x'=0$, $y'=-\frac{2}{3}(t+q)$ we get

$$y'-x'^2=-\frac{2}{3}(t+q), \qquad x'y'+qx'-p=-p,$$

$$px'-qy'-y'^2=\frac{2}{3}(t+q)\cdot\frac{1}{3}(q-2t),$$

and by [10], the point isogonal to the point T' has the coordinates

$$x=\frac{x'y'+qy'-p}{y'-x'^2}=\frac{3p}{2(t+q)},$$

$$y=\frac{px'-qy'-y'^2}{y'-x'^2}=\frac{1}{3}(2t-q),$$

i.e., that is the point T from (2). □

Theorem 11 *Let T_1, T_2 and T_3 be the points on the Jeřabek hyperbola with the equations (6), which correspond to the values t_1,t_2 and t_3 of the perimeter t, and let T_3' be the point isogonal to the point T_3 with respect to the triangle ABC. The points T_1, T_2 and T_3' are collinear if and only if*

$$t_1+t_2+t_3=-\frac{3}{2}q. \tag{14}$$

Proof. The point T_3' given by the equality (13) with $t=t_3$ lies on the line T_1T_2 with equation (11) supposing that

$$-\frac{2}{3}(t_3+q)=\frac{1}{3}(2t_1+2t_2+q),$$

that is the condition (14). □

Symmetry by t_1, t_2 and t_3 of the condition (14) gives the following statement.

Corollary 5 *Let T_1, T_2 and T_3 be the points on the Jeřabek hyperbola with the equations (6), which correspond to the values t_1, t_2 and t_3 of the perimeter t so that the equality (14) holds, and let T_1', T_2' and T_3' be the points on the Euler line of the triangle ABC isogonal, with respect to that triangle, to the points T_1, T_2 and T_3. Then the points T_1', T_2, T_3; T_1, T_2', T_3; T_1, T_2, T_3' are the triples of the collinear points, i.e., T_1, T_1'; T_2, T_2'; T_3, T_3' are the pairs of the opposite vertices of one complete quadrilateral.*

The points $\Phi = (0, -q)$ and $G = (0, -\frac{2}{3}q)$ lie on the line (11) under the condition

$$\frac{1}{3}(2t_1 + 2t_2 + q) = -q, \qquad \frac{1}{3}(2t_1 + 2t_2 + q) = -\frac{2}{3}q$$

respectively.
Hence, we get:

Corollary 6 *Points T_1 and T_2 on the Jeřabek hyperbola with the equations (6), which correspond to the values t_1 and t_2 of the perimeter t, are diametrically opposite on this hyperbola supposing that $t_1 + t_2 = -2q$, and they are collinear with the centroid G of the triangle ABC under the condition $t_1 + t_2 = -\frac{3}{2}q$.*

The value of the perimeter t, which is associated with the point A on the hyperbola (6) follows from the equality

$$\frac{3p}{2(t+q)} = a,$$

i.e., the equality $2(t+q) = 3bc$, and then $t = \frac{3}{2}bc - q$. With this value of t the right-hand side of the equation (12) gets the form

$$-\frac{4}{9p}\left(\frac{3}{2}bc\right)^2 x + \frac{1}{3}(6bc - 3q) = -\frac{bc}{a}x + 2bc - q,$$

and the following statement follows.

Theorem 12 *Tangents $\mathcal{A}, \mathcal{B}$ and $\mathcal{C}$ of the Jeřabek hyperbola of a standard triangle ABC at its vertices A, B and C have the equations*

$$y = -\frac{bc}{a}x + 2bc - q, \qquad y = -\frac{ca}{b}x + 2ca - q,$$
$$y = -\frac{ab}{c}x + 2ab - q. \tag{15}$$

Theorem 13 *If $A_iB_iC_i$ and $A_hB_hC_h$ are the contact triangle and the orthic triangle of an allowable triangle ABC, respectively, then the points $D = B_iC_i \cap B_hC_h$, $E = C_iA_i \cap C_hA_h$ and $F = A_iB_i \cap A_hB_h$ are the poles of the lines BC, CA, and AB with respect to the Jeřabek hyperbola of a triangle ABC.*

Proof. The point

$$D = \left(-\frac{2bc}{a}, -q - 2bc\right)$$

lies on the lines $\mathcal{B}$ and $\mathcal{C}$ because e.g. for the line $\mathcal{B}$ with the second equation (15) we get

$$-\frac{ca}{b}\left(-\frac{2bc}{a}\right) + 2ca - q = 2c^2 + 2ca - q = -2bc - q.$$

Therefore the point D is the pole of the line BC. According to [11] and [1], the lines B_hC_h and B_iC_i have the equations

$$y = 2ax + 2bc - q, \quad y = \frac{a}{2}x - q - bc.$$

The point D lies on these lines because of the following:

$$2a\left(-\frac{2bc}{a}\right) + 2bc - q = -q - 2bc,$$
$$\frac{a}{2}\left(-\frac{2bc}{a}\right) - bc - q = -q - 2bc. \qquad \square$$

With $t = 0$ from (12), we get the equation $y = -\frac{4q^2}{9p}x + \frac{q}{3}$ of the tangent of the Jeřabek hyperbola $\mathcal{J}$ at the symmedian centre K of a triangle ABC. This tangent obviously passes through the point $G_t = \left(0, \frac{q}{3}\right)$, which is, by [1], the centroid of a tangential triangle $A_tB_tC_t$ of a triangle ABC.
With $t = -\frac{3}{2}q$ from (12), we get the equation $y = -\frac{q^2}{9p}x - \frac{5}{3}q$ of the tangent of a hyperbola $\mathcal{J}$ at the Gergonne point Γ of a triangle ABC. This tangent obviously passes through the point $G_t = \left(0, -\frac{5}{3}q\right)$, which is, by [2], the centroid of the contact triangle $A_iB_iC_i$ of a triangle ABC, i.e., the Cevian triangle of a point Γ for a triangle ABC.

Theorem 14 *Lines parallel with the lines AP, BP and CP through the vertices A_t, B_t and C_t of the tangential triangle $A_tB_tC_t$ of a triangle ABC pass through one point P' if and only if the point P lies on the Jeřabek hyperbola $\mathcal{J}$ of a triangle ABC.*

Proof. According to [1], we get e.g. $A_t = \left(-\frac{a}{2}, bc\right)$. Let $P = (u, v)$. The line AP has the slope $(v - a^2) : (u - a)$. Its parallel line given by the equation

$$2(u-a)y = 2(v - a^2)x + 2bcu + av + aq - 3p. \tag{16}$$

goes through the point A_t because of

$$\begin{aligned}&(v - a^2)(-a) + 2bcu + av + aq - 3p\\ &\quad = a(a^2 + q) + 2bcu - 3p = abc + 2bcu - 3p\\ &\quad = 2(u-a)bc.\end{aligned}$$

The line from the equation (16) and two more analogous lines pass through one point under the condition

$$\begin{vmatrix} u-a & v-a^2 & 2bcu + av + aq - 3p \\ u-b & v-b^2 & 2cau + bv + bq - 3p \\ u-c & v-c^2 & 2abu + cv + cq - 3p \end{vmatrix} = 0.$$

As e.g.

$$\begin{aligned}(u-b)(v-c^2)&-(u-c)(v-b^2)\\ &=(b^2-c^2)u-(b-c)v-bc(b-c)\\ &=-(au+v+bc)(b-c),\end{aligned}$$

this condition can be written in the form

$$\Sigma(au+v+bc)(2bcu+av+aq-3p)(b-c)=0,$$

where Σ represents the sum of three addends, where one is always written, and the other two are obtained therefrom by a cyclic permutation of the letters a,b and c. The same condition can also be written as follows:

$$\begin{aligned}2pu^2\Sigma(b-c)&+2uv\Sigma bc(b-c)+2u\Sigma b^2c^2(b-c)\\ &+uv\Sigma a^2(b-c)+v^2\Sigma a(b-c)+pv\Sigma(b-c)\\ &+qu\Sigma a^2(b-c)+qv\Sigma a(b-c)+pq\Sigma(b-c)\\ &-3pu\Sigma a(b-c)-3pv\Sigma(b-c)-3p\Sigma bc(b-c)=0.\end{aligned}$$

As we have $\Sigma(b-c)=0$, $\Sigma a(b-c)=0$, $\Sigma a^2(b-c)=-(b-c)(c-a)(a-b)$, $\Sigma bc(b-c)=-(b-c)(c-a)(a-b)$, we obtain

$$\begin{aligned}\Sigma b^2c^2(b-c)&=\Sigma bc(a^2+q)(b-c)\\ &=p\Sigma a(b-c)+q\Sigma bc(b-c)=-q(b-c)(c-a)(a-b).\end{aligned}$$

Then the last condition, without the factor $(b-c)(c-a)(a-b)$, has the form $-2uv-2qu-uv-qu+3p=0$, i.e., in the end it has the form $uv+qu-p=0$. It means that the point P lies on the hyperbola $\mathcal{J}$. □

Theorem 15 *With the labels from Theorem 14, the point P' describes the Jeřabek hyperbola $\mathcal{J}_t$ of the tangential triangle $A_tB_tC_t$ of a triangle ABC, which in the case of the standard triangle ABC has the equation*

$$2xy+p=0. \tag{17}$$

Proof. Let $P'=(u,v)$. The line A_tP' has the slope

$$\frac{v-bc}{u+\frac{a}{2}}=\frac{2(v-bc)}{2u+a}.$$

Its parallel line given by the equation

$$(2u+a)y=2(v-bc)x+2a^2u-2av-aq+3p \tag{18}$$

passes through the point $A=(a,a^2)$ because of

$$\begin{aligned}2(v-bc)a+2a^2u-2av-aq+3p&=2a^2u-aq+abc\\ &=2a^2u+a\cdot a^2=(2u+a)a^2.\end{aligned}$$

The line with the equation (18) and two more analogous lines pass through one point supposing that

$$\begin{vmatrix}2u+a & v-bc & 2a^2u-2av-aq+3p\\ 2u+b & v-ca & 2b^2u-2bv-bq+3p\\ 2u+c & v-ab & 2c^2u-2cv-cq+3p\end{vmatrix}=0.$$

As e.g.

$$\begin{aligned}(2u+b)(v-ab)&-(2u+c)(v-ca)\\ &=-2au(b-c)+v(b-c)-a(b^2-c^2)\\ &=-(2au-v-a^2)(b-c),\end{aligned}$$

this condition can also be written in the form

$$\begin{aligned}&\Sigma(2au-v-a^2)(2a^2u-2av-aq+3p)(b-c)=0,\\ &4u^2\Sigma a^3(b-c)-2uv\Sigma a^2(b-c)-2u\Sigma a^4(b-c)\\ &\quad-4uv\Sigma a^2(b-c)+2v^2\Sigma a(b-c)+2v\Sigma a^3(b-c)\\ &\quad-2qu\Sigma a^2(b-c)+qv\Sigma a(b-c)+q\Sigma a^3(b-c)\\ &\quad+6pu\Sigma a(b-c)-3pv\Sigma(b-c)-3p\Sigma a^2(b-c)=0.\end{aligned}$$

In addition to the aforementioned equations from the proof of Theorem 14, the following equations also hold:

$$\begin{aligned}&\Sigma a^3(b-c)=\Sigma a(bc-q)(b-c)=p\Sigma(b-c)-q\Sigma a(b-c)=0,\\ &\Sigma a^4(b-c)=\Sigma a^2(bc-q)(b-c)\\ &\quad=p\Sigma a(b-c)-q\Sigma a^2(b-c)=q(b-c)(c-a)(a-b),\end{aligned}$$

where, without the factor $(b-c)(c-a)(a-b)$, the last condition gets the form $2uv-2qu+4uv+2qu+3p=0$, i.e., we have $2uv+p=0$, meaning that the point P' lies on the curve $\mathcal{J}_t$ with the equation (17). This curve is a special hyperbola with the asymptotes $x=0$ and $y=0$ and its centre is at $(0,0)$, which is, by [1], the Feuerbach point Φ_t of a triangle $A_tB_tC_t$. By [1], the circumscribed circle of that triangle has the equation $y=4x^2+q$. From this equation and the equation (17) we get the equation $8x^3+2qx+p=0$ for the abscissa of the intersection of these two curves. The solutions of this equation are the abscissas $-\frac{a}{2}$, $-\frac{b}{2}$, $-\frac{c}{2}$ of the points A_t,B_t,C_t because of

$$-\frac{a}{2}-\frac{b}{2}-\frac{c}{2}=0,\quad \frac{1}{4}(bc+ca+ab)=\frac{1}{4}q,\quad -\frac{1}{8}abc=-\frac{1}{8}p.$$

So the hyperbola $\mathcal{J}_t$ is circumscribed to a triangle $A_tB_tC_t$, and since it has a centre Φ_t, it is the Jeřabek hyperbola of that triangle. □

The symmedian centre $K=\left(\frac{3p}{2q},-\frac{q}{3}\right)$ of a triangle ABC lies on the hyperbola (17), which is in line with the fact that K is the Gergonne point of a triangle $A_tB_tC_t$.

At its point (x_0,y_0), hyperbola (17) has the tangent with the equation $x_0y+y_0x+p=0$, which in the case of the point

$K = \left(\frac{3p}{2q}, -\frac{q}{3}\right)$ gets the form

$$\frac{3p}{2q}y - \frac{q}{3}x + p = 0, \qquad \text{i.e.,} \qquad y = \frac{2q^2}{9p}x - \frac{2}{3}q,$$

and it obviously passes through the point $G = \left(0, -\frac{2}{3}q\right)$ of a triangle ABC. So we get:

Theorem 16 *The Jeřabek hyperbola of a tangential triangle of an allowable triangle passes through its symmedian centre and at this point it touches its joint line with the centroid of the given triangle.*

On the basis of Corollary 2, hyperbola $\mathcal{J}_t$ has the Euler line and the Feuerbach line of a triangle $A_tB_tC_t$ as asymptotes, i.e., the lines with the equations $x = 0$ and $y = 0$, by [9], the Euler and the dual Feuerbach line of a triangle ABC. Therefore

Corollary 7 *The Jeřabek hyperbola of a tangential triangle of an allowable triangle has the Euler and the dual Feuerbach line of that triangle as asymptotes.*

Theorem 17 *The Jeřabek hyperbola of a tangential triangle of an allowable triangle is circumscribed to its symmetrical triangle.*

Proof. According to [7], the symmetrical triangle $A_sB_sC_s$ of a triangle ABC has e.g. the vertex $A_s = \left(a, -\frac{bc}{2}\right)$, which obviously satisfies equation (17). ☐

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