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The Universal Parabola

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ABSTRACT

We develop classical properties, as well as some novel facts, for the parabola using the more general framework of rational trigonometry. This extends the study of this conic to general fields.

Key words: parabola, rational trigonometry, conic

MSC2010: 51N20, 14H50

Univerzalna parabola

SAŽETAK

Dokazujemo neka klasična svojstva kao i neke nove činjenice o paraboli koristeći okvir racionalne trigonometrije. Proširujemo proučavanje konika na opća polja.

Ključne riječi: parabola, racionalna trigonometrija, konika

1 Introduction

Next to the circle, the parabola is perhaps the most accessible conic section. It was studied by Menaechmus, who used it to duplicate the volume of a cube. Apollonius gave it its name, and deduced many important properties. Archimedes studied areas of parabolic arcs, Euclid mentions the parabola, and Pappus investigated the focus and directrix. Galileo showed that projectiles follow parabolic arcs. The reflective property was studied by Gregory and Newton. The parabola appears in car headlights, solar ovens, telescopes, astronomical radio dishes, the orbits of comets, architecture and whenever one variable is proportional to the square of another.

Classical geometry considers the parabola to be an element of Euclidean geometry over the field of decimal or ‘real’ numbers. Treatises which establish some of these properties include [3], [4], [6], [7], [8], [9], [10], [11], [12], [13] and [14]. From our point of view, this traditional aspect is but a shadow of the *true parabola*, which is an object that properly lives in *universal geometry*, a form of Euclidean metrical geometry that is valid over a *general field*. There are parabolas defined over finite fields, over the complex numbers, and over the p-adic numbers. With universal geometry we may investigate properties of parabolas that are shared in these different contexts, in other words that hold in complete generality.

This may well strike the reader as curious. One of the most familiar properties of a parabola is the reflective property—any light beam coming in parallel to the axis and reflected off the parabola so as to make equal angles with the tangent line passes through the focus. How is one to even state such a fact over say the finite field \mathbb{F}_{11} where angles make no sense?

The answer is to free oneself from the straightjacket of traditional geometric thought. Distance and angle are not really the mathematically fundamental concepts that we like to believe. Euclid carefully avoided mentioning these metrical notions because of their attendant irrationalities. Make the shift to *quadrance* and *spread*, and you have an entirely new and simplified way of thinking about metrical geometry, which allows you to study parabolas and other conic sections in the universal setting, as well as much else besides, as shown in the recent book [16]. This can then be extended also to hyperbolic geometry, as in [1] and [2].

This paper derives numerous properties of the universal parabola. Some of the theorems are extensions of familiar and classical results, suitably restated in the new language to hold in an arbitrary field. Others are new even in the familiar setting. The diagrams mostly illustrate the situation in the familiar domain of the rational numbers, or numerically the decimal number field.

2 Universal geometry

Universal geometry, introduced in [16], is a form of Euclidean geometry that holds over any field, *characteristic two (and sometimes three) excluded*. Distance and angle are replaced by algebraic analogs; the separation of points is measured using *quadrance*, and the separation of lines is measured using *spread*. The following definitions and results are taken from [16].

Fix a field \mathbb{F} whose elements are called **numbers**. We will throughout assume that the characteristic of this field is *not two*. That means that the number 2 is always invertible in \mathbb{F} . A **point** is an ordered pair of numbers, denoted $A \equiv [x, y]$. A **side** $\overline{A_1A_2}$ is a set consisting of two points A_1 and A_2 . The **midpoint** of the side $\overline{A_1A_2}$, where $A_1 = [x_1, y_1]$ and $A_2 = [x_2, y_2]$, is the point

$$M = \left[\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2} \right].$$

A **line** is a proportion $(a : b : c)$ of numbers with at least one of a and b non-zero. The point $A \equiv [x, y]$ **lies on** the line $l \equiv (a : b : c)$ precisely when $ax + by + c = 0$, which is called the **equation** of the line. Equivalently we say l **passes through** A .

A line $(a : b : c)$ is a **null line** precisely when $a^2 + b^2 = 0$. Null lines do not occur over the rational or decimal number fields, but they occur whenever -1 is a square. In fact if $i^2 = -1$ then any null line has the form $(1 : \pm i : c)$ for some c .

Theorem 1 (Collinear points) *The points $[x_1, y_1]$, $[x_2, y_2]$ and $[x_3, y_3]$ are collinear (meaning they lie on the same line) precisely when*

$$x_1y_2 - x_1y_3 + x_2y_3 - x_3y_2 + x_3y_1 - x_2y_1 = 0.$$

Theorem 2 (Concurrent lines) *If the lines $(a_1 : b_1 : c_1)$, $(a_2 : b_2 : c_2)$ and $(a_3 : b_3 : c_3)$ are concurrent (meaning they pass through the same point) then*

$$a_1b_2c_3 - a_1b_3c_2 + a_2b_3c_1 - a_3b_2c_1 + a_3b_1c_2 - a_2b_1c_3 = 0.$$

Definition 1 *The lines $l_1 = (a_1 : b_1 : c_1)$ and $l_2 \equiv (a_2 : b_2 : c_2)$ are **parallel** precisely when*

$$a_1b_2 - a_2b_1 = 0.$$

Definition 2 *The lines $l_1 = (a_1 : b_1 : c_1)$ and $l_2 \equiv (a_2 : b_2 : c_2)$ are **perpendicular** precisely when*

$$a_1a_2 + b_1b_2 = 0.$$

Definition 3 *The **quadrance** $Q(A_1, A_2)$ between the points $A_1 \equiv [x_1, y_1]$ and $A_2 \equiv [x_2, y_2]$ is the number*

$$Q(A_1, A_2) \equiv (x_2 - x_1)^2 + (y_2 - y_1)^2.$$

Definition 4 *The **spread** $s(l_1, l_2)$ between the non-null lines $l_1 = (a_1 : b_1 : c_1)$ and $l_2 \equiv (a_2 : b_2 : c_2)$ is the number*

$$s(l_1, l_2) \equiv \frac{(a_1b_2 - a_2b_1)^2}{(a_1^2 + b_1^2)(a_2^2 + b_2^2)}.$$

Two lines l_1 and l_2 are perpendicular precisely when $s(l_1, l_2) = 1$. If $\overline{A_1A_2A_3}$ is a right triangle with A_1A_3 perpendicular to A_2A_3 , then the Spread ratio theorem asserts that

$$s(A_1A_3, A_1A_2) = \frac{Q_1}{Q_3}.$$

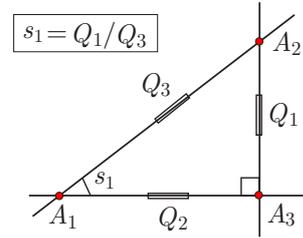


Figure 1: *Spread as ratio*

For any point $A \equiv [x, y]$ and any line $l \equiv (a : b : c)$ there is a unique line n , called the **altitude from A to l** , which passes through A and is perpendicular to l , namely

$$n = (-b : a : bx - ay).$$

If l is non-null, then this altitude intersects l at the point

$$N \equiv \left[\frac{b^2x - aby - ac}{a^2 + b^2}, \frac{-abx + a^2y - bc}{a^2 + b^2} \right]$$

called the **foot** of the altitude. The **quadrance** $Q(A, l)$ between the point A and the line l is then defined to be $Q(A, N)$. This turns out to be

$$Q(A, l) = \frac{(ax + by + c)^2}{a^2 + b^2}.$$

3 Isometries and similarities

Definition 5 *An **isometry** σ is a function that inputs and outputs points, such that for any points A_1 and A_2*

$$Q(A_1, A_2) = Q(\sigma(A_1), \sigma(A_2)).$$

Definition 6 *A **similarity** τ is a function that inputs and outputs points, such that for any points A_1, A_2 and A_3 ,*

$$Q(A_1, A_2) Q(\tau(A_2), \tau(A_3)) = Q(A_2, A_3) Q(\tau(A_1), \tau(A_2)).$$

Every isometry is a similarity, but scalings are similarities which are not generally isometries. A similarity preserves the ratio of quadrances, so it preserves the spread between lines.

Theorem 3 For any numbers a and b , the function

$$\tau([x, y]) = [ax + by, bx - ay]$$

is a similarity.

Proof. Suppose that $A_1 = [x_1, y_1]$, $A_2 = [x_2, y_2]$. Then

$$\begin{aligned} Q(\tau(A_1), \tau(A_2)) &= Q([ax_1 + by_1, bx_1 - ay_1], [ax_2 + by_2, bx_2 - ay_2]) \\ &= (ax_2 + by_2 - ax_1 - by_1)^2 + (bx_2 - ay_2 - bx_1 + ay_1)^2 \\ &= (a^2 + b^2)(x_1^2 - 2x_1x_2 + x_2^2 + y_1^2 - 2y_1y_2 + y_2^2) \\ &= (a^2 + b^2)Q(A_1, A_2) \end{aligned}$$

from which the result follows. \square

The transformation τ above is affine, implying that lines are sent to lines. The point $[x, y]$ lies on the line $ax + by + c = 0$ precisely when $\tau([x, y])$ lies on the line $x + c = 0$. The inverse transformation to τ is defined only if $a^2 + b^2 \neq 0$ and is

$$\tau^{-1}([x, y]) = \left[\frac{ax + by}{a^2 + b^2}, \frac{bx - ay}{a^2 + b^2} \right].$$

4 Parabolas

A conic is given by a polynomial equation in x and y of degree exactly two. A **circle** is a conic whose equation in $X \equiv [x, y]$ has the form $Q(X, C) = R$ for some point C and some number R . Some basic facts about circles will be taken from [16], such as the fact that if A and B lie on the circle, then the spread $s(AX, BX)$ is constant for all X lying on the circle, and is equal to 1 precisely when AB is a diameter of the circle (i.e. passes through the center C).

Definition 7 A **parabola** p is a conic whose equation in $X \equiv [x, y]$ has the form

$$Q(X, F) = Q(X, l)$$

for some fixed point F and some fixed non-null line l not passing through F .

A point $A = [x, y]$ **lies on** p precisely when x and y satisfy the equation of p . It is a theorem in [16] that the point F and the line l are determined by p . The point F is the **focus** and the line l the **directrix** of p . The **axis** of the parabola p is the altitude from F to l . The **vertex** of the parabola is the midpoint of the side \overline{FP} where P is the foot of the altitude from F to the directrix l . It lies on the axis and also on the parabola p .

Example 1 If $\lambda \neq 0$ then the parabola p with focus $F \equiv [\lambda, 0]$ and directrix $x + \lambda = 0$ has equation

$$(x - \lambda)^2 + y^2 = (x + \lambda)^2 \quad \text{or} \quad y^2 = 4\lambda x.$$

Every point lying on this parabola has the form $[\lambda a^2, 2\lambda a]$ for some number a , and every such point lies on the parabola.

To prove theorems about parabolas, it is very convenient to work with this simple equation. The following useful result shows how this can be done.

Theorem 4 Any parabola can be transformed by a similarity to one of the form $y^2 = 4\lambda x$ for some number λ .

Proof. If the directrix of the parabola p is $ax + by + c = 0$ then the similarity

$$\tau([x, y]) = [ax + by, bx - ay]$$

takes this line to $x + c = 0$. This transformation is invertible since the directrix of a parabola is a non-null line, so that $a^2 + b^2 \neq 0$. A translation (in the y direction) now moves the focus to the form $[\beta, 0]$ and leaves the directrix unchanged. Then the translation (in the x direction)

$$[x, y] \rightarrow [x - (\beta - c)/2, y]$$

takes the focus to $[(\beta + c)/2, 0]$ and the directrix to $x + (\beta + c)/2 = 0$ which has the required form, with $\lambda = (\beta + c)/2$. \square

Although the transformation used here is a similarity, not in general an isometry, it is sufficient to prove all the theorems in this paper, which are ultimately not about individual quadrances, but always only about the *proportions* between quadrances.

The parabola p with equation $y^2 = 4\lambda x$ has focus $F = [\lambda, 0]$, directrix $(1 : 0 : \lambda)$, axis the line $(0 : 1 : 0)$ and vertex the point $[0, 0]$. If A and B are two distinct points lying on p , then the line AB is a **chord** of the parabola, while the side \overline{AB} is a **side** of the parabola. A chord passing through the focus F is a **focal chord**. The chord of a parabola through its focus which is perpendicular to its axis is called the **latus rectum**. A **null point of** p is a point N lying on p such that $Q(N, F) = 0$.

Tangent lines to conics are best defined algebraically, since limiting procedures are not available for general fields. Given a conic which passes through the origin with equation such as $x^2 - 3xy + 5y^2 + 3x - 2y = 0$, the **tangent line** at the origin is defined to be just the linear part of this expression, that is the line $3x - 2y = 0$. To define the tangent at a general point of a general conic, first translate so that the point is at the origin, then take the linear part, and translate back. This is explained in more detail in [16], which also contains the following result, generalizing the most well-known property of a parabola to the universal case.

Theorem 5 Suppose k is the tangent line to a parabola p at a point A lying on it. Then $s(k, m) = s(k, n)$ where $m = AF$ with F the focus of p , and n is the altitude from A to the directrix of p .

The general point on $y^2 = 4\lambda x$ is $[x_1, y_1] = [\lambda t^2, 2\lambda t]$ in terms of a parameter t . The **tangent** to p at this point is the line $(2\lambda : -y_1 : 2\lambda x_1) = (1 : -t : \lambda t^2)$. A **null tangent** to a parabola is a null line that is tangent to the parabola, clearly this occurs precisely when $t^2 = -1$. For two different values of t , say $t = a$ and $t = b$ we get two points $A \equiv [\lambda a^2, 2\lambda a]$ and $B \equiv [\lambda b^2, 2\lambda b]$, which determine the chord $(2 : -(a+b) : 2\lambda ab)$. The tangents to these points intersect at the point

$$X = [x_0, y_0] = [\lambda ab, \lambda(a+b)]$$

which is the **external point** of the chord AB . The side \overline{AB} is the **side** determined by X , and the chord AB is the **chord of contact** of X ; it has the form $(2\lambda : -y_0 : 2\lambda x_0)$.

Summary of formulae Let p be the parabola with equation $y^2 = 4\lambda x$. Let $[x_1, y_1]$ denote a general point on the parabola, and $A \equiv [\lambda a^2, 2\lambda a]$ and $B \equiv [\lambda b^2, 2\lambda b]$ two specific points on the parabola. Let $[x_0, y_0]$ be the external point of the chord AB . Let $[\lambda t^2, 2\lambda t]$ be the parametric form of p .

The tangent to p at $[x_1, y_1]$ in Cartesian form

$$(2\lambda : -y_1 : 2\lambda x_1) \quad (1)$$

The tangent to p at $A = [\lambda a^2, 2\lambda a]$ in parametric form

$$(1 : -t : \lambda t^2) \quad (2)$$

The chord AB in Cartesian form

$$(2\lambda : -y_0 : 2\lambda x_0) \quad (3)$$

The chord AB in parametric form

$$(2 : -(a+b) : 2\lambda ab) \quad (4)$$

The external point of the chord AB in parametric form

$$[\lambda ab, \lambda(a+b)]. \quad (5)$$

5 Tangents and external points

Theorem 6 No two tangents to a parabola are parallel.

Proof. Let the equation of the parabola be

$$y^2 = 4\lambda x$$

for some $\lambda \neq 0$. Then the tangents to the parabola at two distinct points $[x_1, y_1], [x_2, y_2]$ are

$$(2\lambda : -y_1 : 2\lambda x_1) \quad \text{and} \quad (2\lambda : -y_2 : 2\lambda x_2)$$

respectively. But since $y_1 \neq y_2$, these lines are not parallel. \square

Theorem 7 No three tangents to a parabola are concurrent.

Proof. Since affine transformations preserve concurrence of lines, it suffices to prove this for the parabola

$$y^2 = 4\lambda x.$$

Let

$$A \equiv [\lambda a^2, 2\lambda a] \quad B \equiv [\lambda b^2, 2\lambda b] \quad \text{and} \quad C \equiv [\lambda c^2, 2\lambda c].$$

The tangents to the parabola at these points are

$$(1 : -a : \lambda a^2) \quad (1 : -b : \lambda b^2) \quad \text{and} \quad (1 : -c : \lambda c^2)$$

respectively. Suppose the three tangents are concurrent, then by the Concurrent lines theorem

$$\begin{vmatrix} 1 & -a & \lambda a^2 \\ 1 & -b & \lambda b^2 \\ 1 & -c & \lambda c^2 \end{vmatrix} = 0.$$

Expand the determinant to get

$$\lambda(a-b)(c-a)(c-b) = 0 \quad (6)$$

Since A, B and C are all distinct points, a, b and c are also distinct. This implies $\lambda = 0$, which is impossible. \square

Theorem 8 There is a unique pair of tangents to a parabola through an external point.

Proof. By Theorem 7 no three tangents are concurrent, hence there is no external point with two distinct pairs of tangents to the parabola. \square

6 Chords of a parabola

Theorem 9 Suppose A and B are two points on a parabola. Then the following are equivalent.

1. AB is a focal chord
2. The tangents at A and B are perpendicular
3. The tangents at A and B intersect on the directrix.

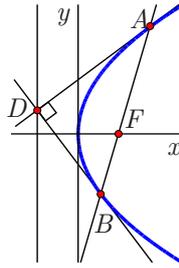


Figure 2.

Proof. $1 \Rightarrow 2)$ Let $A \equiv [\lambda a^2, 2\lambda a]$ and $B \equiv [\lambda b^2, 2\lambda b]$ and $[x_0, y_0]$ be the external point of the chord AB . By (2) the tangents to p at A and B are

$$(1 : -a : \lambda a^2) \quad \text{and} \quad (1 : -b : \lambda b^2)$$

respectively. Since the line AB passes through the focus $[\lambda, 0]$, from (4) we deduce that

$$ab = -1.$$

Hence the two tangents are perpendicular.

$2 \Rightarrow 3)$ If the tangents are perpendicular, then from (2) $ab = -1$. By comparing (4) with (3), $x_0 = -\lambda$. Therefore the external point of the chord AB lies on the directrix. Hence the two tangents intersect on the directrix.

$3 \Rightarrow 1)$ Suppose two tangents to the parabola intersect on the directrix i.e. $x_0 = -\lambda$. Then from (3) the chord AB is

$$(2\lambda : -y_0 : -2\lambda^2)$$

which passes through the point $[\lambda, 0]$. Hence AB is a focal chord. \square

Theorem 10 Any circle whose diameter is a focal chord of a parabola touches the directrix.

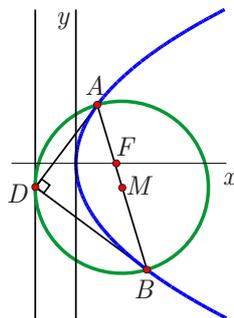


Figure 3.

Proof. Since the tangents to p at $A = [\lambda a^2, 2\lambda a]$ and $B = [\lambda b^2, 2\lambda b]$ intersect at

$$D \equiv [\lambda ab, \lambda(a+b)]$$

we may compute that

$$AD = (1 : -a : a^2\lambda) \quad \text{and} \quad BD = (1 : -b : b^2\lambda).$$

If AB is a focal chord then from the proof of the previous proposition, $ab = -1$, so that AD and BD are perpendicular. Then D lies on the circle with AB as its diameter. In particular, the midpoint

$$M \equiv \left[\frac{\lambda(a^2 + b^2)}{2}, \lambda(a+b) \right]$$

of the side \overline{AB} is the center of the circle. The equation of the line MD is

$$(0 : 1 : -\lambda(a+b))$$

which is perpendicular to the directrix. Therefore the directrix is tangent to the circle. \square

Theorem 11 Let A be a point on a parabola with focus F . Let D be the foot of the altitude from A to the directrix, and T be the point where the tangent at A meets the axis of the parabola. Then the side \overline{FD} is perpendicular to the side \overline{AT} and they share a common midpoint.

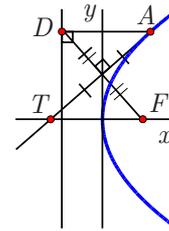


Figure 4.

Proof. Let $A \equiv [\lambda a^2, 2\lambda a]$. By (2) the tangent at A is

$$(1 : -a : \lambda a^2)$$

which meets the axis at

$$T = [-\lambda a^2, 0].$$

By the foot of an altitude formula

$$D = [-\lambda, 2\lambda a].$$

So the line AT and the line

$$DF = (a : 1 : -\lambda a)$$

are perpendicular. Moreover the midpoints of the side \overline{AT} and \overline{DF} are both

$$[0, \lambda a].$$

Theorem 12 Let AB be a chord of a parabola which is parallel to the directrix. Let E be a third point on the parabola. Suppose the lines EA and EB intersect the axis of the parabola at C and D respectively. Then the midpoint of the side \overline{CD} is the vertex of the parabola.

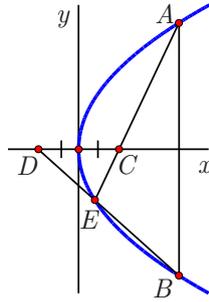


Figure 5.

Proof. Let

$$A \equiv [\lambda a^2, 2\lambda a] \quad B \equiv [\lambda b^2, 2\lambda b] \quad \text{and} \quad E \equiv [\lambda e^2, 2\lambda e].$$

Since the line AB is parallel to the directrix by (4)

$$a + b = 0.$$

From (4) the line EA intersects the axis of parabola at

$$C \equiv [-\lambda ea, 0].$$

Similarly the line EB intersects the axis of the parabola at

$$D \equiv [-\lambda eb, 0].$$

Therefore the midpoint of \overline{CD} is

$$\left[\frac{-\lambda e(a+b)}{2}, 0 \right] = [0, 0]$$

which is the vertex of the parabola. □

Theorem 13 Let \overline{AB} be the side of an external point X to a parabola. Let M be the midpoint of the side \overline{AB} . Then

1. MX is parallel to the axis of the parabola.
2. The midpoint T of the side \overline{MX} lies on the parabola.
3. The tangent at T is parallel to the chord AB .

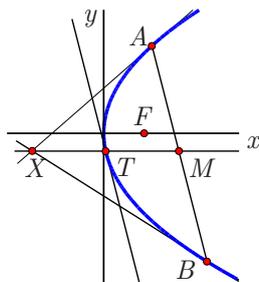


Figure 6.

Proof. 1.) Let $A \equiv [\lambda a^2, 2\lambda a]$ and $B \equiv [\lambda b^2, 2\lambda b]$. Then the midpoint of the side \overline{AB} is

$$M = \left[\frac{\lambda(a^2 + b^2)}{2}, \lambda(a + b) \right].$$

By (5) the external point of \overline{AB} is

$$X = [\lambda ab, \lambda(a + b)]$$

which has the same y component as M . Hence MX is parallel to the axis $(0 : 1 : 0)$ of the parabola.

2.) The midpoint T of the side \overline{MX} is

$$\left[\frac{\lambda(a + b)^2}{4}, \lambda(a + b) \right]$$

which lies on the parabola, with parameter $t = (a + b)/2$.

3.) By (2) the tangent at T is

$$(4 : -2(a + b) : \lambda(a + b)^2).$$

By (4) the line containing the chord is

$$(2 : -(a + b) : 2\lambda ab)$$

which is parallel to the tangent at T . □

Theorem 14 Let AB be the chord from an external point X to a parabola with focus F . Suppose the line AB meets the axis at I , and that the line through X parallel to the axis meets the parabola at N . Then

1. The midpoint of the side \overline{XI} lies on the tangent at the vertex of parabola.
2. The tangent at N is parallel to the chord AB .

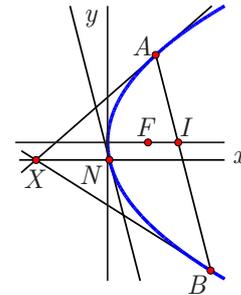


Figure 7.

Proof. Let $A \equiv [\lambda a^2, 2\lambda a]$ and $B \equiv [\lambda b^2, 2\lambda b]$ as usual.

1.) By (4)

$$I = [-\lambda ab, 0]$$

and by (5)

$$X = [\lambda ab, \lambda(a + b)].$$

Therefore the midpoint of the side \overline{XI} is

$$\left[0, \frac{\lambda(a + b)}{2} \right]$$

which lies on $(1 : 0 : 0)$, the tangent at the vertex to the parabola.

2.) The line through X parallel to the axis of the parabola intersects the parabola at

$$N \equiv \left[\frac{\lambda(a+b)^2}{4}, \lambda(a+b) \right].$$

By (1) the tangent at N is

$$(4 : -2(a+b) : \lambda(a+b)^2)$$

which is parallel to line AB from (4). □

Theorem 15 For any point C there is a line $l = l(C)$ with the property that if a chord AB of the parabola p passes through C , then the external point of the chord lies on l .

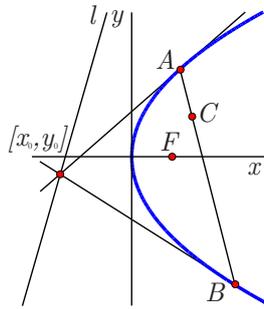


Figure 8.

Proof. Let the external point of the chord be $[x_0, y_0]$ and $C \equiv [h, k]$. By assumption the chord passes through $[h, k]$, so by (3)

$$2\lambda h - y_0 k + 2\lambda x_0 = 0.$$

Hence $[x_0, y_0]$ lies on the line

$$l \equiv (2\lambda : -k : 2\lambda h). \quad \square$$

C is called the **pole** of l , while l is called the **polar** of C .

Theorem 16 Let $A_1 \equiv [x_1, y_1]$ and $A_2 \equiv [x_2, y_2]$ be two external points to a parabola. If the line containing the chord of contact from A_1 passes through A_2 , then the line containing the chord of contact from A_2 passes through A_1 .

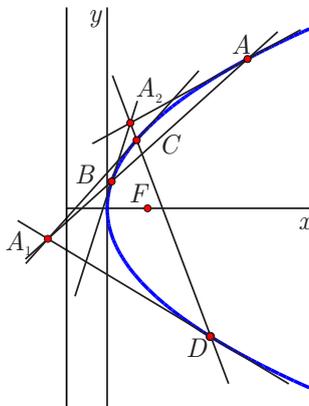


Figure 9.

Proof. Since the chord of contact from the point A_1 , shown as CD in the Figure, passes through A_2 by (3), we have

$$2\lambda x_2 - y_1 y_2 + 2\lambda x_1 = 0.$$

Rearranging the equation shows that $A_1 \equiv [x_1, y_1]$ lies on

$$(2\lambda : -y_2 : 2\lambda x_2)$$

which is the chord of contact from the point A_2 , shown as AB in the Figure. □

7 Quadrance properties of a parabola

Theorem 17 Let A lie on a parabola with focus F . Let T be the point where the tangent at A meets the axis of the parabola. Then $Q(A, F) = Q(F, T)$.

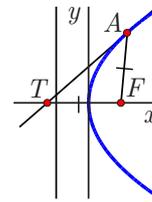


Figure 10.

Proof. Let $A \equiv [\lambda a^2, 2\lambda a]$. By (2) the tangent at A meets the axis at the point

$$T = [-\lambda a^2, 0].$$

Now

$$Q(A, F) = (\lambda a^2 - \lambda)^2 + 4\lambda^2 a^2 = \lambda^2 (a^2 + 1)^2$$

and

$$Q(F, T) = (\lambda + \lambda a^2)^2 = \lambda^2 (a^2 + 1)^2.$$

Hence $Q(A, F) = Q(F, T)$. □

Theorem 18 Let AB be a chord of the parabola p from the external point X . Let F be the focus of p . Then

$$Q(X, F)^2 = Q(A, F) Q(F, B).$$

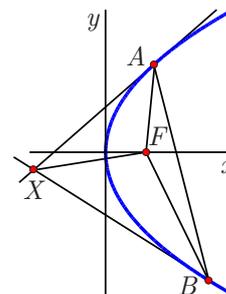


Figure 11.

Proof. Let $A \equiv [\lambda a^2, 2\lambda a]$ and $B \equiv [\lambda b^2, 2\lambda b]$. By (5) the external point of the chord AB is $X = [\lambda ab, \lambda(a+b)]$. Now

$$\begin{aligned} Q(X, F)^2 &= (\lambda^2(ab - 1)^2 + \lambda^2(a+b)^2)^2 \\ &= \lambda^4(a^2 + b^2 + a^2b^2 + 1)^2 \\ &= \lambda^4((a^2 + 1)(b^2 + 1))^2. \end{aligned}$$

Moreover

$$Q(A, F) = (\lambda a^2 - \lambda)^2 + 4\lambda^2 a^2 = \lambda^2(a^2 + 1)^2.$$

Similarly

$$Q(F, B) = \lambda^2(b^2 + 1)^2.$$

Hence $Q(X, F)^2 = Q(A, F) Q(F, B)$. □

Theorem 19 *If a focal chord AB of a parabola with focus F meets the directrix at D , then*

$$Q(A, F) : Q(F, B) = Q(A, D) : Q(D, B).$$

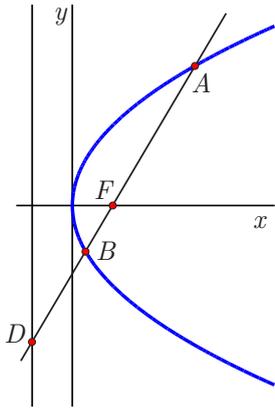


Figure 12.

Proof. Let $A \equiv [\lambda a^2, 2\lambda a]$ and $B \equiv [\lambda b^2, 2\lambda b]$. Then

$$\begin{aligned} Q(A, F) &= (\lambda a^2 - \lambda)^2 + 4\lambda^2 a^2 = \lambda^2(a^2 + 1)^2 \\ Q(F, B) &= (\lambda b^2 - \lambda)^2 + 4\lambda^2 b^2 = \lambda^2(b^2 + 1)^2. \end{aligned}$$

Therefore

$$Q(A, F) : Q(F, B) = (1 + a^2)^2 : (1 + b^2)^2$$

By (4) the chord AB intersects the directrix at

$$D = \left[-\lambda, \frac{2\lambda(ab - 1)}{a + b} \right]$$

where $a \neq -b$. Therefore

$$\begin{aligned} Q(A, D) &= (\lambda a^2 + \lambda)^2 + \left(2\lambda a - \frac{2\lambda(ab - 1)}{a + b} \right)^2 \\ &= \lambda^2(a^2 + 1)^2 \left(1 + \frac{4}{(a + b)^2} \right). \end{aligned}$$

Similarly

$$Q(D, B) = \lambda^2(b^2 + 1)^2 \left(1 + \frac{4}{(a + b)^2} \right).$$

Hence

$$Q(A, F) : Q(F, B) = Q(A, D) : Q(D, B). \quad \square$$

Theorem 20 *Let A and B be two non-null points on a parabola with focus F . If AB is a focal chord and the tangent at A meets the latus rectum at C . Then*

$$Q(C, F)^2 = Q(A, F) Q(F, B).$$

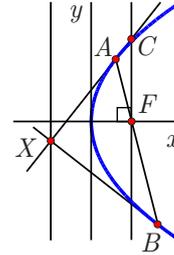


Figure 13.

Proof. Let $A \equiv [\lambda a^2, 2\lambda a]$. The latus rectum is

$$(1 : 0 : -\lambda)$$

which intersects the tangent at A

$$(1 : -a : \lambda a^2)$$

at

$$C \equiv \left[\lambda, \frac{\lambda(1 + a^2)}{a} \right].$$

Now

$$Q(C, F)^2 = \left(\left(\frac{\lambda(1 + a^2)}{a} \right)^2 \right)^2 = \frac{\lambda^4(1 + a^2)^4}{a^4}$$

Moreover

$$Q(A, F) = (\lambda a^2 - \lambda)^2 + 4\lambda^2 a^2 = \lambda^2(a^2 + 1)^2.$$

Similarly with $B \equiv [\lambda b^2, 2\lambda b]$,

$$Q(F, B) = \lambda^2(b^2 + 1)^2.$$

Since AB is a focal chord, it passes through F . By (4) $ab = -1$. Therefore

$$\begin{aligned} Q(A, F) Q(F, B) &= \lambda^2(a^2 + 1)^2 \lambda^2(b^2 + 1)^2 \\ &= \lambda^4(a^2 + 1)^2 \left(\frac{1}{a^2} + 1 \right)^2 \\ &= \lambda^4(a^2 + 1)^2 \left(\frac{1 + a^2}{a^2} \right)^2 \\ &= \frac{\lambda^4(a^2 + 1)^4}{a^4}. \end{aligned}$$

Hence

$$Q(C, F)^2 = Q(A, F)Q(F, B). \quad \square$$

8 Spread properties of a parabola

Theorem 21 *Let A lie on a parabola with focus F. Suppose the tangent at A meets the directrix at D. Then AF is perpendicular to DF.*

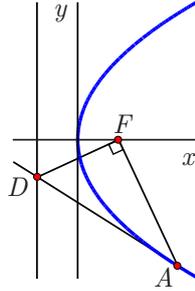


Figure 14.

Proof. Let $A \equiv [\lambda a^2, 2\lambda a]$. From (2) the tangent at A meets the directrix at

$$D = \left[-\lambda, \frac{\lambda(a^2 - 1)}{a} \right]$$

where $a \neq 0$. Now the line AF is

$$(2a : 1 - a^2 : -2\lambda a)$$

and the line DF

$$(a^2 - 1 : 2a : -\lambda(a^2 - 1)).$$

So AF is perpendicular to DF. □

Theorem 22 *Let A be a non-null point on a parabola with focus F. Then the spread between the line AF and the axis of the parabola is a square number.*

Proof. Let $A \equiv [\lambda a^2, 2\lambda a]$. Then the line AF has equation

$$(2a : 1 - a^2 : -2\lambda a).$$

The axis of the parabola is $(0 : 1 : 0)$. The spread between these two lines is

$$\frac{(2a)^2}{(2a)^2 + (1 - a^2)^2} = \left(\frac{2a}{1 + a^2} \right)^2$$

which is a square. □

Theorem 23 *Let A and B be two non-null points on a parabola with focus F. Then \overline{AB} subtends a square spread at the focus.*

Proof. Let $A \equiv [\lambda a^2, 2\lambda a], B \equiv [\lambda b^2, 2\lambda b]$ Then the lines AF and BF are

$$(2a : 1 - a^2 : -2\lambda a) \quad \text{and} \quad (2b : 1 - b^2 : -2\lambda b)$$

respectively. Now the spread between AF and BF is

$$\begin{aligned} s(AF, BF) &= \frac{(2a(1 - b^2) - 2b(1 - a^2))^2}{((2a)^2 + (1 - a^2)^2)((2b)^2 + (1 - b^2)^2)} \\ &= \frac{(2(a - b)(1 + ab))^2}{(1 + a^2)^2(1 + b^2)^2} \end{aligned}$$

which is a square. □

Theorem 24 *Let A and B be two points on a parabola with focus F with at least one of the points non-null. Let X be the external point of the chord AB, and suppose the line AB meets the directrix of the parabola at D. Then DF is perpendicular to FX.*

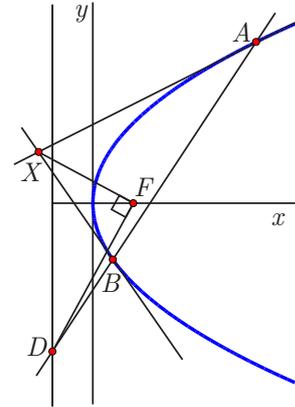


Figure 15.

Proof. With A and B the usual points, by (4) the line AB intersects the directrix at

$$D = \left[-\lambda, \frac{2\lambda(ab - 1)}{a + b} \right]$$

where $a \neq -b$. By (5)

$$X = [\lambda ab, \lambda(a + b)].$$

Now the line DF is

$$(ab - 1 : a + b : -\lambda(ab - 1))$$

and the line FX is

$$(a + b : 1 - ab : -\lambda(a + b)).$$

Thus DF is perpendicular to FX. □

Theorem 25 *Let A and B be two non-null points lying on a parabola with focus F. Let the tangents at A and B intersect the directrix at C and D respectively. Then \overline{AB} and \overline{CD} subtend equal spreads at the focus.*

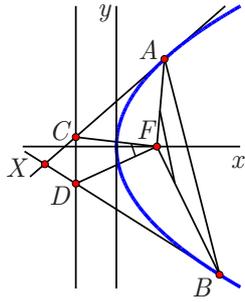


Figure 16.

Proof. By (2) the tangents at the usual points A and B are

$$(1 : -a : \lambda a^2) \quad \text{and} \quad (1 : -b : \lambda b^2)$$

respectively. They intersect the directrix at

$$C \equiv \left[-\lambda, \frac{\lambda(a^2 - 1)}{a} \right] \quad \text{and} \quad D \equiv \left[-\lambda, \frac{\lambda(b^2 - 1)}{b} \right]$$

respectively. The line AF is

$$(2a : 1 - a^2 : -2\lambda a).$$

Similarly the line BF is $(2b : 1 - b^2 : -2\lambda b)$. Thus

$$\begin{aligned} s(AF, BF) &= \frac{(2a(1 - b^2) - 2b(1 - a^2))^2}{(4a^2 + (1 - a^2)^2)(4b^2 + (1 - b^2)^2)} \\ &= \frac{4((a - b)(ab + 1))^2}{(1 + a^2)^2(1 + b^2)^2}. \end{aligned}$$

The line CF is $(a^2 - 1 : 2a : -\lambda(a^2 - 1))$ and similarly the line DF is $(b^2 - 1 : 2b : -\lambda(b^2 - 1))$. Thus

$$\begin{aligned} s(CF, DF) &= \frac{(2b(a^2 - 1) - 2a(b^2 - 1))^2}{((a^2 - 1)^2 + 4a^2)((b^2 - 1)^2 + 4b^2)} \\ &= \frac{4((a - b)(1 + ab))^2}{(1 + a^2)^2(1 + b^2)^2}. \end{aligned}$$

Hence $s(AF, BF) = s(CF, DF)$. \square

Theorem 26 Suppose A and B are two non-null points on a parabola. If \overline{AB} is a side of the parabola with the external point X , then the sides \overline{AX} and \overline{BX} subtend equal spreads at the focus.

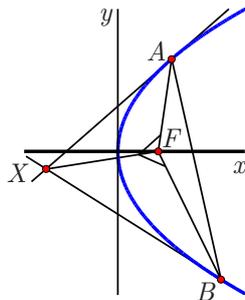


Figure 17.

Proof. Let $A \equiv [\lambda a^2, 2\lambda a]$ and $B \equiv [\lambda b^2, 2\lambda b]$. By (5) the tangents intersect at

$$X = [\lambda ab, \lambda(a + b)].$$

Then

$$AF = (2a : 1 - a^2 : -2\lambda a)$$

$$BF = (2b : 1 - b^2 : -2\lambda b)$$

$$XF = (a + b : 1 - ab : -\lambda(a + b)).$$

Therefore

$$\begin{aligned} s(AF, FX) &= \frac{(2a(1 - ab) - (1 - a^2)(a + b))^2}{(4a^2 + (1 - a^2)^2)((a + b)^2 + (1 - ab)^2)} \\ &= \frac{(a - a^2b + a^3 - b)^2}{(1 + a^2)^2(a^2 + b^2 + a^2b^2 + 1)} \\ &= \frac{((a - b)(1 + a^2))^2}{(1 + a^2)^2(1 + a^2)(1 + b^2)} \\ &= \frac{(a - b)^2}{(1 + a^2)(1 + b^2)}. \end{aligned}$$

Since this expression is symmetrical in a and b ,

$$s(AF, FX) = s(XF, FB). \quad \square$$

Theorem 27 Suppose the non-null tangent at A to a parabola with focus F passes through an external point X to a parabola. Then the spread between the tangent to the parabola at A and its axis is equal to the spread between the other non-null tangent and line XF .

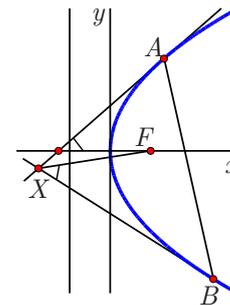


Figure 18.

Proof. Let $A \equiv [\lambda a^2, 2\lambda a]$ and $B \equiv [\lambda b^2, 2\lambda b]$ lie on the parabola. By (5) the tangents intersect at

$$X = [\lambda ab, \lambda(a + b)].$$

The axis of the parabola is

$$n \equiv (0 : 1 : 0).$$

Then by (2)

$$AX = (1 : -a : \lambda a^2) \quad \text{and} \quad BX = (1 : -b : \lambda b^2).$$

The equation of the line through X and the focus F is

$$XF = (a + b : 1 - ab : -\lambda(a + b)).$$

Therefore

$$s(AX, n) = \frac{1}{1 + a^2}.$$

Moreover

$$\begin{aligned} s(BX, XF) &= \frac{((1 - ab) + b(a + b))^2}{(1 + b^2)((a + b)^2 + (1 - ab)^2)} \\ &= \frac{(1 + b^2)^2}{(1 + b^2)(a^2 + b^2 + a^2b^2 + 1)} \\ &= \frac{(1 + b^2)^2}{(1 + b^2)(1 + a^2)(1 + b^2)} \\ &= \frac{1}{1 + a^2}. \end{aligned}$$

Hence $s(AX, n) = s(BX, XF)$. □

Theorem 28 *Let A and B be two non-null points on a parabola with focus F . Let the tangents at A and B intersect at X . Let the spread subtended by the side \overline{AB} at X be s . Then the spread subtended by \overline{AB} at the focus F is equal to $S_2(s) = 4s(1 - s)$.*

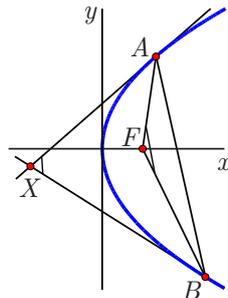


Figure 19.

Proof. Suppose $A \equiv [\lambda a^2, 2\lambda a]$ and $B \equiv [\lambda b^2, 2\lambda b]$ are points on the parabola. By (5) the tangents intersect at

$$X = [\lambda ab, \lambda(a + b)].$$

The spread between the tangents is

$$s \equiv \frac{(a - b)^2}{(1 + a^2)(1 + b^2)}.$$

Therefore

$$\begin{aligned} S_2(s) &= 4s(1 - s) \\ &= 4 \frac{(a - b)^2}{(1 + a^2)(1 + b^2)} \left(1 - \frac{(a - b)^2}{(1 + a^2)(1 + b^2)} \right) \\ &= \frac{4(a - b)^2}{(1 + a^2)(1 + b^2)} \left(\frac{1 + a^2b^2 + 2ab}{(1 + a^2)(1 + b^2)} \right) \\ &= \frac{4(a - b)^2(1 + ab)^2}{(1 + a^2)^2(1 + b^2)^2}. \end{aligned}$$

The lines AF and FB are

$$(2a : 1 - a^2 : -2\lambda a) \quad (2b : 1 - b^2 : -2\lambda b)$$

respectively. Therefore

$$\begin{aligned} s(AF, FB) &= \frac{(2a(1 - b^2) - 2b(1 - a^2))^2}{(4a^2 + (1 - a^2)^2)(4b^2 + (1 - b^2)^2)} \\ &= \frac{4((a - b)(1 + ab))^2}{(1 + a^2)^2(1 + b^2)^2}. \end{aligned}$$

Hence $s(AF, FB) = S_2(s) = 4s(1 - s)$. □

Theorem 29 *Suppose that two congruent parabolas have the same vertex, and that their axes are perpendicular to each other. If the characteristic of the field is not 5, then the parabolas intersect with a spread of $9/25$. Otherwise the two parabolas intersect at one of the null points on the two parabolas.*

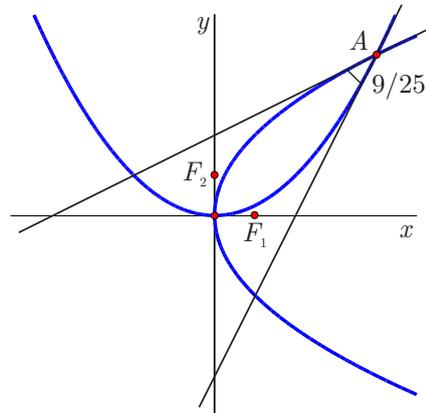


Figure 20.

Proof. Let the two congruent parabolas be

$$y^2 = 4\lambda x \quad \text{and} \quad x^2 = 4\lambda y \tag{7}$$

with foci F_1 and F_2 respectively.

Suppose first that the two parabolas are defined over a field of characteristic other than 5. Then they intersect at

$$A \equiv [4\lambda, 4\lambda].$$

The tangents at A in (7) are

$$(1 : -2 : 4\lambda) \quad \text{and} \quad (2 : -1 : -4\lambda)$$

respectively. Hence their spread is

$$\frac{((1)(-1) - (2)(-2))^2}{(1^2 + (-2)^2)(2^2 + (-1)^2)} = \frac{3^2}{5^2} = \frac{9}{25}.$$

On the other hand, if the characteristic of the field is 5, then

$$Q(A, F_1) = (4\lambda - \lambda)^2 + (4\lambda)^2 = 25\lambda^2 = 0$$

$$Q(A, F_2) = (4\lambda)^2 + (4\lambda - \lambda)^2 = 25\lambda^2 = 0$$

which shows that the point A is a null point on the parabolas. \square

Theorem 30 *Suppose that two congruent parabolas have a common focus but different vertices. If their axes are the same, then the parabolas intersect perpendicularly.*

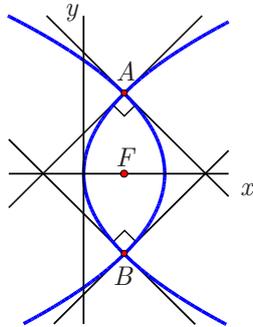


Figure 21.

Proof. Let the parabolas be

$$y^2 = 4\lambda x \quad \text{and} \quad y^2 = -4\lambda(x - 2\lambda). \tag{8}$$

They intersect at

$$A = [\lambda, 2\lambda] \quad \text{and} \quad B = [\lambda, -2\lambda].$$

The tangents at A for (8) are

$$(1 : -1 : \lambda) \quad \text{and} \quad (1 : 1 : -3\lambda)$$

respectively, which are perpendicular. Similarly the tangents at B in (8) are

$$(1 : 1 : \lambda) \quad \text{and} \quad (1 : -1 : -3\lambda)$$

respectively, which are also perpendicular. \square

9 Signed areas and anti-symmetric polynomials

In this section we review some definitions, notation and results from [16] which we quickly review. If $A_1 \equiv [x_1, y_1]$,

$A_2 \equiv [x_2, y_2]$ and $A_3 \equiv [x_3, y_3]$, then the **signed area** of the oriented triangle $\overrightarrow{A_1A_2A_3}$ is the number

$$a(\overrightarrow{A_1A_2A_3}) \equiv \frac{x_1y_2 - x_2y_1 + x_2y_3 - x_3y_2 + x_3y_1 - x_1y_3}{2} \\ = \frac{1}{2} \begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}.$$

This concept is extended to more general n -gons, by for example defining the signed area of an oriented quadrilateral $\overrightarrow{A_1A_2A_3A_4}$ to be

$$a(\overrightarrow{A_1A_2A_3A_4}) = a(\overrightarrow{A_1A_2A_3}) + a(\overrightarrow{A_1A_3A_4}).$$

To deal with a wide variety of anti-symmetric expressions in the variable x_i and y_j for $i, j = 1, 2$ and 3 , introduce the following notation. For any polynomial r in these variables, define $[r]_3^-$ to be the alternating sum of the six terms obtained from r by applying the following changes to the indices:

$$2 \leftrightarrow 3 \quad 1 \leftrightarrow 2 \quad 2 \leftrightarrow 3 \quad 1 \leftrightarrow 2 \quad 2 \leftrightarrow 3.$$

in this order. The expression in the numerator of $a(\overrightarrow{A_1A_2A_3})$ is $[x_1y_2]_3^-$. Here is another example:

$$[x_1x_2^3y_2]_3^- = x_1x_2^3y_2 - x_1x_3^3y_3 + x_2x_3^3y_3 - x_3x_2^3y_2 + x_3x_1^3y_1 - x_2x_1^3y_1.$$

Theorem 31 (Circumcenter formula) *If $A_1 \equiv [x_1, y_1]$, $A_2 \equiv [x_2, y_2]$ and $A_3 \equiv [x_3, y_3]$, then the circumcenter C of the triangle $A_1A_2A_3$ is*

$$C = \left[\frac{[x_1^2y_2]_3^- + [y_1^2y_2]_3^-}{2[x_1y_2]_3^-}, \frac{[x_1x_2^2]_3^- + [x_1y_2^2]_3^-}{2[x_1y_2]_3^-} \right].$$

Theorem 32 (Orthocentre formula) *If $A_1 \equiv [x_1, y_1]$, $A_2 \equiv [x_2, y_2]$ and $A_3 \equiv [x_3, y_3]$, then the orthocentre O of the triangle $A_1A_2A_3$ is*

$$O = \left[\frac{[x_1x_2y_2]_3^- + [y_1y_2^2]_3^-}{[x_1y_2]_3^-}, \frac{[x_1y_1y_2]_3^- + [x_1^2x_2]_3^-}{[x_1y_2]_3^-} \right].$$

We generally use determinants to evaluate anti-symmetric expressions.

10 Parabolic triangles

If all three points lie on the parabola, then we have a **parabolic triangle**.

Theorem 33 *Let A, B and C be three distinct points on a parabola. Let the external points of the sides \overline{AB} , \overline{BC} and \overline{CA} be X, Y and Z respectively. Then*

$$a(\overrightarrow{ABC}) : a(\overrightarrow{XYZ}) = -2 : 1$$

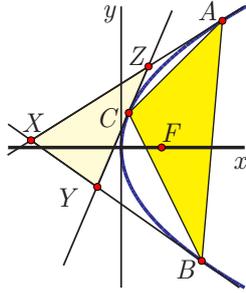


Figure 22.

Proof. Let the points on the parabola be

$$A \equiv [\lambda a^2, 2\lambda a] \quad B \equiv [\lambda b^2, 2\lambda b] \quad C \equiv [\lambda c^2, 2\lambda c].$$

Then by (5) the corresponding external points to the parabola are

$$X \equiv [\lambda ab, \lambda(a+b)] \quad Y \equiv [\lambda bc, \lambda(b+c)] \quad Z \equiv [\lambda ca, \lambda(c+a)].$$

Now by the signed area formula

$$\begin{aligned} a(\overrightarrow{ABC}) &= \frac{1}{2} \begin{vmatrix} \lambda a^2 & 2\lambda a & 1 \\ \lambda b^2 & 2\lambda b & 1 \\ \lambda c^2 & 2\lambda c & 1 \end{vmatrix} \\ &= \lambda^2 \begin{vmatrix} a^2 & a & 1 \\ b^2 & b & 1 \\ c^2 & c & 1 \end{vmatrix} \\ &= \lambda^2 \begin{vmatrix} (a-b)(a+b) & (a-b) & 0 \\ (b-c)(b+c) & (b-c) & 0 \\ c^2 & c & 1 \end{vmatrix} \\ &= \lambda^2 \begin{vmatrix} (a-b)(a+b) & (a-b) \\ (b-c)(b+c) & (b-c) \end{vmatrix} \\ &= \lambda^2(a-b)(b-c) \begin{vmatrix} a+b & 1 \\ b+c & 1 \end{vmatrix} \\ &= \lambda^2(a-b)(b-c)(a-c). \end{aligned}$$

Moreover

$$\begin{aligned} a(\overrightarrow{XYZ}) &= \frac{1}{2} \begin{vmatrix} \lambda ab & \lambda(a+b) & 1 \\ \lambda bc & \lambda(b+c) & 1 \\ \lambda ca & \lambda(c+a) & 1 \end{vmatrix} \\ &= \frac{\lambda^2}{2} \begin{vmatrix} ab & a+b & 1 \\ bc & b+c & 1 \\ ca & c+a & 1 \end{vmatrix} \\ &= \frac{\lambda^2}{2} \begin{vmatrix} b(a-c) & a-c & 0 \\ c(b-a) & b-a & 0 \\ ca & c+a & 1 \end{vmatrix} \\ &= \frac{\lambda^2}{2} \begin{vmatrix} b(a-c) & a-c \\ c(b-a) & b-a \end{vmatrix} \\ &= \frac{\lambda^2(a-c)(b-a)}{2} \begin{vmatrix} b & 1 \\ c & 1 \end{vmatrix} \\ &= \frac{\lambda^2(a-c)(b-a)(b-c)}{2}. \end{aligned}$$

Hence

$$a(\overrightarrow{ABC}) : a(\overrightarrow{XYZ}) = -2 : 1. \quad \square$$

Corollary 1 Let A_1, A_2, \dots, A_n be n distinct points on a parabola. Let the external points of the sides $\overline{A_1A_2}, \overline{A_2A_3}, \dots, \overline{A_nA_1}$ be X_1, X_2, \dots, X_n respectively. Then from the previous theorem

$$a(\overrightarrow{A_1A_2 \cdots A_n}) : a(\overrightarrow{X_1X_2 \cdots X_n}) = -2 : 1.$$

Proof. Let the external points of the chords $\overline{A_1A_3}, \overline{A_1A_4}, \dots, \overline{A_1A_{n-1}}$ be Y_3, Y_4, \dots, Y_{n-1} respectively. Then

$$\begin{aligned} a(\overrightarrow{A_1A_2 \cdots A_n}) &= a(\overrightarrow{A_1A_2A_3}) + a(\overrightarrow{A_1A_3A_4}) + \cdots + a(\overrightarrow{A_1A_kA_{k+1}}) + \cdots \\ &\quad + a(\overrightarrow{A_1A_{n-1}A_n}) \\ &= -2(a(\overrightarrow{X_1X_2Y_3}) + a(\overrightarrow{Y_3X_3Y_4}) + \cdots + a(\overrightarrow{Y_kX_kY_{k+1}}) + \cdots \\ &\quad + a(\overrightarrow{Y_{n-1}X_{n-1}X_n})) \\ &= -2(a(\overrightarrow{X_1X_2Y_3}) + a(\overrightarrow{X_3Y_4Y_3}) + \cdots + a(\overrightarrow{X_kY_{k+1}Y_k}) + \cdots \\ &\quad + a(\overrightarrow{X_{n-1}X_nY_{n-1}})) \\ &= -2(a(\overrightarrow{X_1X_2X_3Y_4}) + \cdots + a(\overrightarrow{X_kY_{k+1}Y_k}) + \cdots \\ &\quad + a(\overrightarrow{X_{n-1}X_nY_{n-1}})) \\ &= -2(a(\overrightarrow{X_1X_2 \cdots X_kY_{k+1}}) + \cdots + a(\overrightarrow{X_{n-1}X_nY_{n-1}})) \\ &= -2a(\overrightarrow{X_1X_2 \cdots X_{n-1}Y_n}). \end{aligned}$$

$$\text{Hence } a(\overrightarrow{A_1A_2 \cdots A_n}) : a(\overrightarrow{X_1X_2 \cdots X_n}) = -2 : 1. \quad \square$$

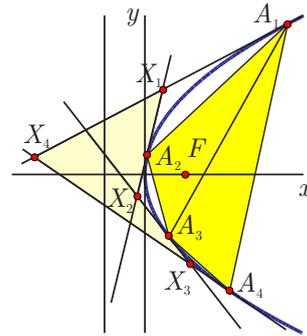


Figure 23.

The following is in [15].

Theorem 34 Let A, B and C be three distinct points on a parabola. Let the external points of the sides $\overline{AB}, \overline{BC}$ and \overline{CA} be X, Y and Z respectively. Then

$$\frac{Q(X,A)}{Q(A,Z)} = \frac{Q(B,X)}{Q(X,Y)} = \frac{Q(Y,Z)}{Q(Z,C)}.$$

Proof. Suppose the points on the parabola are

$$A \equiv [\lambda a^2, 2\lambda a] \quad B \equiv [\lambda b^2, 2\lambda b] \quad C \equiv [\lambda c^2, 2\lambda c].$$

Then by (5) the corresponding external points to the parabola are

$$X \equiv [\lambda ab, \lambda(a+b)] \quad Y \equiv [\lambda bc, \lambda(b+c)] \quad Z \equiv [\lambda ca, \lambda(c+a)].$$

Now

$$\begin{aligned} Q(X,A) &= (\lambda ab - \lambda a^2)^2 + (\lambda(a+b) - 2\lambda a)^2 \\ &= \lambda^2 a^2 (b-a)^2 + \lambda^2 (b-a)^2 \\ &= \lambda^2 (b-a)^2 (a^2 + 1) \end{aligned}$$

and

$$\begin{aligned} Q(A,Z) &= (\lambda a^2 - \lambda ca)^2 + (2\lambda a - \lambda(c+a))^2 \\ &= \lambda^2 a^2 (a-c)^2 + \lambda^2 (a-c)^2 \\ &= \lambda^2 (a-c)^2 (a^2 + 1). \end{aligned}$$

Therefore

$$\frac{Q(X,A)}{Q(A,Z)} = \frac{(b-a)^2}{(a-c)^2}.$$

Similarly

$$\frac{Q(B,X)}{Q(X,Y)} = \frac{(b-a)^2}{(a-c)^2} \quad \frac{Q(Y,Z)}{Q(Z,C)} = \frac{(b-a)^2}{(a-c)^2}.$$

Hence

$$\frac{Q(X,A)}{Q(A,Z)} = \frac{Q(B,X)}{Q(X,Y)} = \frac{Q(Y,Z)}{Q(Z,C)}.$$

□

Theorem 35 *Let A, B and C be three distinct points on a parabola. Let the external points of the sides AB, BC and CA be X, Y and Z respectively. If the parabola is defined over a field with -3 not a square, then the lines AY, BZ and CX are concurrent. Otherwise if the lines AY, BZ and CX are not concurrent, then they are mutually parallel to each other.*

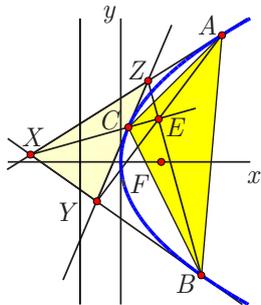


Figure 24.

Proof. Suppose the points on the parabola are

$$A \equiv [\lambda a^2, 2\lambda a] \quad B \equiv [\lambda b^2, 2\lambda b] \quad \text{and} \quad C \equiv [\lambda c^2, 2\lambda c]$$

so that

$$X \equiv [\lambda ab, \lambda(a+b)] \quad Y \equiv [\lambda bc, \lambda(b+c)] \quad Z \equiv [\lambda ca, \lambda(c+a)].$$

Then

$$AY = (2a - (b+c) : bc - a^2 : \lambda(a^2(b+c) - 2abc))$$

$$BZ = (2b - (c+a) : ca - b^2 : \lambda(b^2(c+a) - 2abc))$$

$$CX = (2c - (a+b) : ab - c^2 : \lambda(c^2(a+b) - 2abc)).$$

Suppose AY, BZ and CX are mutually parallel to each other. Then

$$\begin{aligned} (2a - (b+c)) \times (ca - b^2) - (2b - (c+a)) \times (bc - a^2) \\ = (b-a)(a^2 - ca - ba + b^2 + c^2 - bc) = 0 \end{aligned}$$

$$\begin{aligned} (2a - (b+c)) \times (ab - c^2) - (2c - (a+b)) \times (bc - a^2) \\ = (c-a)(a^2 - ca - ba + b^2 + c^2 - bc) = 0. \end{aligned}$$

Therefore a sufficient condition for the three lines to be mutually parallel is

$$a^2 + b^2 + c^2 - ab - bc - ca = 0. \tag{9}$$

If we solve for c in (9), then we obtain a quadratic in c . Its discriminant is

$$-3a^2 + 6ba - 3b^2 = -3(a-b)^2.$$

Hence if -3 is a square, then without loss of generality the point C on the parabola can be chosen such that AY, BZ and CX are mutually parallel.

Now suppose that -3 is not a square; then the three lines are not mutually parallel. Therefore

$$\begin{aligned} & \begin{vmatrix} 2a - (b+c) & bc - a^2 & \lambda(a^2(b+c) - 2abc) \\ 2b - (c+a) & ca - b^2 & \lambda(b^2(c+a) - 2abc) \\ 2c - (a+b) & ab - c^2 & \lambda(c^2(a+b) - 2abc) \end{vmatrix} \\ &= \begin{vmatrix} 3(a-b) & -(a-b)(a+b+c) & \lambda(a-b)(ab+bc+ca) \\ 2b - (c+a) & b^2 - ca & \lambda(b^2(c+a) - 2abc) \\ 2c - (a+b) & c^2 - ab & \lambda(c^2(a+b) - 2abc) \end{vmatrix} \\ &= \begin{vmatrix} 3(a-b) & -(a-b)(a+b+c) & \lambda(a-b)(ab+bc+ca) \\ 3(b-c) & -(b-c)(a+b+c) & \lambda(b-c)(ab+bc+ca) \\ 2c - (a+b) & c^2 - ab & \lambda(c^2(a+b) - 2abc) \end{vmatrix} \\ &= (a-b)(b-c) \begin{vmatrix} 3 & -(a+b+c) & \lambda(ab+bc+ca) \\ 3 & -(a+b+c) & \lambda(ab+bc+ca) \\ 2c - (a+b) & c^2 - ab & \lambda(c^2(a+b) - 2abc) \end{vmatrix} \\ &= 0. \end{aligned}$$

Therefore by the Concurrent lines theorem the lines AY, BZ and CX are concurrent. Hence if -3 is not a square, the lines AY, BZ and CX are always concurrent. Otherwise if the lines are not concurrent then the three lines must be mutually parallel. □

Call the common point of intersection, which is E in Figure 24, the **central point** of the parabolic triangle \overline{ABC} .

Theorem 36 Let A, B and C be three distinct points on a parabola. Let the external points of the sides \overline{AB} , \overline{BC} and \overline{CA} be X, Y and Z respectively. Then the centroids of the triangles \overline{XAB} , \overline{YBC} and \overline{ZCA} are collinear. Moreover the central point of \overline{ABC} lies on that line.

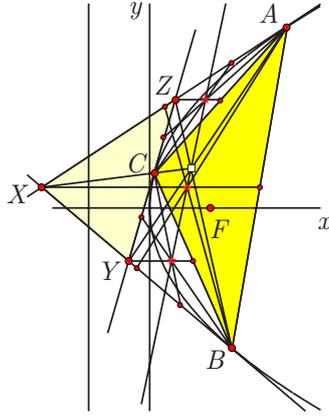


Figure 25.

Proof. Let the points on the parabola be

$$A \equiv [\lambda a^2, 2\lambda a] \quad B \equiv [\lambda b^2, 2\lambda b] \quad \text{and} \quad C \equiv [\lambda c^2, 2\lambda c].$$

Then by (5) the external points to the parabola are

$$X \equiv [\lambda ab, \lambda(a+b)] \quad Y \equiv [\lambda bc, \lambda(b+c)]$$

$$\text{and} \quad Z \equiv [\lambda ca, \lambda(c+a)].$$

The centroid of triangle \overline{XAB} is

$$\left[\frac{\lambda(a^2 + ab + b^2)}{3}, \lambda(a+b) \right].$$

Similarly the centroids of triangle \overline{YBC} and \overline{ZCA} are

$$\left[\frac{\lambda(b^2 + bc + c^2)}{3}, \lambda(b+c) \right] \quad \text{and}$$

$$\left[\frac{\lambda(c^2 + ca + a^2)}{3}, \lambda(c+a) \right]$$

respectively. Therefore

$$\begin{aligned} & \begin{vmatrix} \lambda(a^2 + ab + b^2)/3 & \lambda(a+b) & 1 \\ \lambda(b^2 + bc + c^2)/3 & \lambda(b+c) & 1 \\ \lambda(c^2 + ca + a^2)/3 & \lambda(c+a) & 1 \end{vmatrix} \\ &= \frac{\lambda^2}{3} \begin{vmatrix} a^2 + ab + b^2 & a+b & 1 \\ b^2 + bc + c^2 & b+c & 1 \\ c^2 + ca + a^2 & c+a & 1 \end{vmatrix} \\ &= \frac{\lambda^2}{3} \begin{vmatrix} (a+b+c)(a-c) & a-c & 0 \\ b^2 + bc + c^2 & b+c & 1 \\ c^2 + ca + a^2 & c+a & 1 \end{vmatrix} \\ &= \frac{\lambda^2}{3} \begin{vmatrix} (a+b+c)(a-c) & a-c & 0 \\ (a+b+c)(b-a) & b-a & 0 \\ c^2 + ca + a^2 & c+a & 1 \end{vmatrix} = 0. \end{aligned}$$

The line containing the three centroids is

$$(3 : -(a+b+c) : \lambda(ab+bc+ca)).$$

The coordinate of the central point is

$$\left[\begin{array}{c} \frac{\lambda(a^2(b-c)^2 + b^2(c-a)^2 + c^2(a-b)^2)}{(a-b)^2 + (b-c)^2 + (c-a)^2}, \\ \frac{2\lambda(a(b-c)^2 + b(c-a)^2 + c(a-b)^2)}{(a-b)^2 + (b-c)^2 + (c-a)^2} \end{array} \right].$$

Now

$$\begin{aligned} & 3 \left(\frac{\lambda(a^2(b-c)^2 + b^2(c-a)^2 + c^2(a-b)^2)}{(a-b)^2 + (b-c)^2 + (c-a)^2} \right) \\ & - (a+b+c) \left(\frac{2\lambda(a(b-c)^2 + b(c-a)^2 + c(a-b)^2)}{(a-b)^2 + (b-c)^2 + (c-a)^2} \right) \\ & + \lambda(ab+bc+ca) \\ &= \frac{\lambda \left(\begin{array}{c} 3(a^2(b-c)^2 + b^2(c-a)^2 + c^2(a-b)^2) \\ -2(a(b-c)^2 + b(c-a)^2 + c(a-b)^2)(a+b+c) \\ + (ab+bc+ca)((a-b)^2 + (b-c)^2 + (c-a)^2) \end{array} \right)}{(a-b)^2 + (b-c)^2 + (c-a)^2} \\ &= \frac{\lambda \left(\begin{array}{c} a^2(b-c)^2 + b^2(c-a)^2 + c^2(a-b)^2 \\ - (ab+ac)(b-c)^2 - (ab+bc)(c-a)^2 \\ - (ac+bc)(a-b)^2 \\ + ab(a-b)^2 + bc(b-c)^2 + ca(c-a)^2 \end{array} \right)}{(a-b)^2 + (b-c)^2 + (c-a)^2} \\ &= \frac{\lambda \left(\begin{array}{c} (c-a)(c-b)(a-b)^2 \\ + (a-b)(a-c)(b-c)^2 \\ + (b-c)(b-a)(c-a)^2 \end{array} \right)}{(a-b)^2 + (b-c)^2 + (c-a)^2} \\ &= \frac{\lambda(a-b)(b-c)(c-a)(-(a-b) - (b-c) - (c-a))}{(a-b)^2 + (b-c)^2 + (c-a)^2} \\ &= 0. \end{aligned}$$

Hence the central point lies on the line. \square

Theorem 37 Let A, B and C be three distinct points on a parabola. Let the external points of the sides \overline{AB} , \overline{BC} and \overline{CA} be X, Y and Z respectively. Then the orthocentre of triangle \overline{XYZ} lies on the directrix.

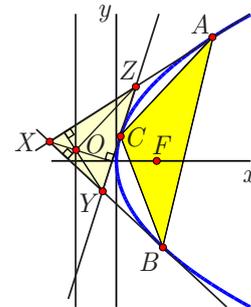


Figure 26.

Proof. Using our standard notation,

$$X \equiv [\lambda ab, \lambda(a+b)] \quad Y \equiv [\lambda bc, \lambda(b+c)]$$

and $Z \equiv [\lambda ca, \lambda(c+a)]$.

Calculating at the abscissa of the orthocentre

$$\begin{aligned} [x_1x_2y_2]_3^- &= \lambda^3 abc(b-a)(a+b+c) + \lambda^3 abc^2(a-b) \\ &\quad + \lambda^3 abc(a+b)(a-b) \\ &= \lambda^3 abc(a-b)(-(a+b+c) + c + a + b) \\ &= 0 \\ [y_1y_2^2]_3^- &= \begin{vmatrix} \lambda(a+b) & \lambda^2(a+b)^2 & 1 \\ \lambda(b+c) & \lambda^2(b+c)^2 & 1 \\ \lambda(c+a) & \lambda^2(c+a)^2 & 1 \end{vmatrix} \\ &= \lambda^3 \begin{vmatrix} a-c & (a+2b+c)(a-c) & 0 \\ b-a & (a+b+2c)(b-a) & 0 \\ c+a & (c+a)^2 & 1 \end{vmatrix} \\ &= \lambda^3(a-c)(b-a)(c-b) \\ [x_1y_2]_3^- &= \lambda^2(a-b)(b-c)(c-a). \end{aligned}$$

Therefore

$$\frac{[x_1x_2y_2]_3^- + [y_1y_2^2]_3^-}{[x_1y_2]_3^-} = \frac{0 + \lambda^3(a-c)(b-a)(c-b)}{\lambda^2(a-b)(b-c)(c-a)} = -\lambda.$$

Hence the orthocentre lies on the directrix. □

Theorem 38 *Let A, B and C be three distinct points on a parabola. Let the external points of the sides AB, BC and CA be X, Y and Z respectively. Let the orthocentres of the triangles XAB, YBC and ZCA be O_X, O_Y and O_Z respectively. Then*

$$a(\overrightarrow{O_X O_Y O_Z}) = -a(\overrightarrow{ABC}).$$

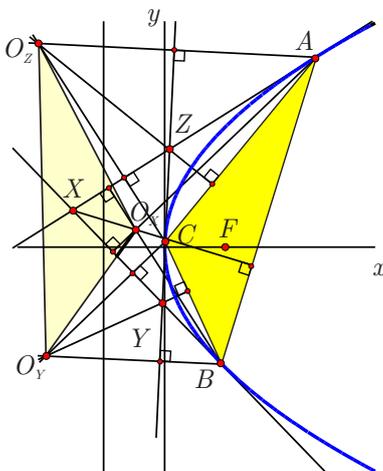


Figure 27.

Proof. Using the standard notation and the orthocentre formula

$$O_X = [-\lambda(2+ab), \lambda(2+ab)(a+b)]$$

$$O_Y = [-\lambda(2+bc), \lambda(2+bc)(b+c)]$$

$$O_Z = [-\lambda(2+ca), \lambda(2+ca)(c+a)].$$

Therefore by the signed area formula

$$\begin{aligned} a(\overrightarrow{O_X O_Y O_Z}) &= \frac{1}{2} \begin{vmatrix} -\lambda(2+ab) & \lambda(2+ab)(a+b) & 1 \\ -\lambda(2+bc) & \lambda(2+bc)(b+c) & 1 \\ -\lambda(2+ca) & \lambda(2+ca)(c+a) & 1 \end{vmatrix} \\ &= -\frac{\lambda^2}{2} \begin{vmatrix} 2+ab & (2+ab)(a+b) & 1 \\ 2+bc & (2+bc)(b+c) & 1 \\ 2+ca & (2+ca)(c+a) & 1 \end{vmatrix} \\ &= -\frac{\lambda^2}{2} \begin{vmatrix} b(a-c) & (a-c)(b^2+b(a+c)+2) & 0 \\ c(b-a) & (b-a)(c^2+c(b+a)+2) & 0 \\ 2+ca & (2+ca)(c+a) & 1 \end{vmatrix} \\ &= -\frac{\lambda^2(a-c)(b-a)}{2} \begin{vmatrix} b & b^2+b(a+c)+2 \\ c & c^2+c(b+a)+2 \end{vmatrix} \\ &= \lambda^2(a-b)(b-c)(c-a) \end{aligned}$$

But from the proof of Theorem 34,

$$a(\overrightarrow{ABC}) = \lambda^2(a-b)(b-c)(a-c).$$

Hence $a(\overrightarrow{O_X O_Y O_Z}) = -a(\overrightarrow{ABC})$. □

In the remaining theorem, we continue with our established notation. Recall that triangles are **similar** precisely when corresponding spreads are equal.

Theorem 39 *Let the circumcentres of the triangles XAB, YBC and ZCA be C_X, C_Y and C_Z respectively. Then triangle C_XC_YC_Z is similar to triangle XYZ.*

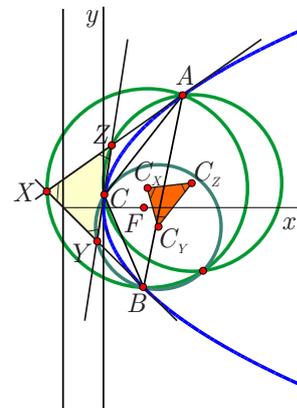


Figure 28.

Proof. (Using a computer) By the circumcenter formula, and using a computer,

$$C_X = \left[\frac{\lambda (a^2 + 2ab + b^2 + 2)}{2}, -\frac{\lambda (a+b)(ab-1)}{2} \right]$$

$$C_Y = \left[\frac{\lambda (b^2 + 2bc + c^2 + 2)}{2}, -\frac{\lambda (b+c)(bc-1)}{2} \right]$$

$$C_Z = \left[\frac{\lambda (a^2 + 2ca + c^2 + 2)}{2}, -\frac{\lambda (c+a)(ca-1)}{2} \right].$$

We may now calculate the spreads

$$s(C_Y C_X, C_X C_Z) = \frac{(a-b)^2}{(a^2+1)(b^2+1)}$$

$$s(C_Z C_Y, C_Y C_X) = \frac{(b-c)^2}{(b^2+1)(c^2+1)}$$

$$s(C_X C_Z, C_Z C_Y) = \frac{(a-c)^2}{(c^2+1)(a^2+1)}.$$

Similarly in triangle \overline{ABC} we find that

$$s(YX, XZ) = \frac{(a-b)^2}{(a^2+1)(b^2+1)}$$

$$s(ZY, YZ) = \frac{(b-c)^2}{(b^2+1)(c^2+1)}$$

$$s(XZ, ZY) = \frac{(a-c)^2}{(c^2+1)(a^2+1)}.$$

Therefore

$$s(C_Y C_X, C_X C_Z) = s(YX, XZ)$$

$$s(C_Z C_Y, C_Y C_X) = s(ZY, YZ)$$

$$s(C_X C_Z, C_Z C_Y) = s(XZ, ZY).$$

Thus triangles $\overline{C_X C_Y C_Z}$ and \overline{ABC} have identical spreads. Hence triangles $\overline{C_X C_Y C_Z}$ and \overline{ABC} are similar. \square

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