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Higher integrabilities and boundednesses for minimizers of weighted anisotropic integral functionals*

Tingfu Feng[1] and Yan Dong[†2]

[1] *Department of Applied Mathematics, Northwestern Polytechnical University, Xi'an, Shaanxi, 710 072, China*
[2] *Department of Applied Mathematics, Hubei University of Economics, Wuhan, Hubei, 430 205, China*

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Abstract. We consider the weighted anisotropic integral functionals

$$I(u)=\int_\Omega f(x,Du(x))dx,$$

where $\Omega\subset\mathbb{R}^n$ $(n\geqslant 2)$ is a bounded open set, $u:\Omega\subset\mathbb{R}^n\to\mathbb{R}$, $f:\Omega\times\mathbb{R}^n\to[0,+\infty)$ is a Carathéodory function which satisfies the nonstandard growth condition

$$\sum_{i=1}^n \nu_i|z_i|^{p_i}\le f(x,z)\le c\left(1+\sum_{i=1}^n \nu_i|z_i|^{q_i}\right),$$

where $c>0$ is a constant, $1<p_i<q_i<n$, $i=1,2,\cdots,n$, ν_i is the positive weighted function on Ω and

$$\nu_i\in L^1_{loc}(\Omega),\quad \left(\frac{1}{\nu_i}\right)^{m_i}\in L^1(\Omega),\quad m_i\ge\frac{1}{p_i-1}.$$

By using the weighted anisotropic Sobolev inequality and the iteration Lemma, we prove the higher integrability for the minimizer u of $I(u)$ when the boundary datum has the higher integrability. We also obtain the global boundednesses of exponential form and $L^\infty(\Omega)$ for the minimizer, respectively. Furthermore, similar results for the minimizer of the obstacle problem to $I(u)$ are given.

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Key words: weighted anisotropic integral functional, minimizer, higher integrability, boundedness

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†Corresponding author. *Email addresses:* `ftfml@mail.nwpu.edu.cn` (T. Feng), `dongyan@mail.nwpu.edu.cn` (Y. Dong)

1. Introduction

Anisotropic integral functionals are a very important and useful framework in dealing with anisotropic physical properties of some reinforced materials ([26, 30]). A typical form is

$$\int_{\Omega}(|D_1u|^{p_1}+|D_2u|^{p_2}+\cdots+|D_nu|^{p_n})dx,$$

where $\Omega\subset\mathbb{R}^n$ $(n\geqslant 2)$ is a bounded open set, $D_iu=\frac{\partial u}{\partial x_i}\in L^{p_i}(\Omega)$, $1<p_i<+\infty$, $i=1,2,\cdots,n$. For anisotropic integral functionals

$$I(u)=\int_{\Omega}f(x,Du(x))dx,$$

where $f\,(x,z)$ satisfies

$$\sum_{i=1}^{n}|z_i|^{p_i}\leq f(x,z)\leq c\left(1+\sum_{i=1}^{n}|z_i|^{p_i}\right), \tag{1}$$

where $c>0$ is a constant. Note that the Euler-Lagrange equation of anisotropic integral functionals $I(u)$ with (1) is

$$\sum_{i=1}^{n}D_i\left(a_i\left(x,Du(x)\right)\right)=0,\qquad x\in\Omega,$$

where $a_i\,(x,z)=\frac{\partial f(x,z)}{\partial z_i}$ and satisfies

$$|a_i\,(x,z)|\leqslant c(1+|z_i|)^{p_i-1}. \tag{2}$$

Fusco-Sbordone [9] obtained the local boundedness of minimizers of anisotropic integral functionals $I(u)$ with (1) in a limit case. Leonetti-Siepe [22] considered $I(u)$ with (1). Assume that the boundary datum $u_*\in W^{1,(q_i)}(\Omega)$, namely

$$u_*\in W^{1,1}(\Omega)\quad\text{ with }\quad D_iu_*\in L^{q_i}(\Omega)\quad\text{ and }\quad q_i\in(p_i,+\infty)$$

for every $i=1,\cdots,n$. If $u\in u_*+W_0^{1,(p_i)}(\Omega)$ is a minimizer of $I(u)$ with (1), that is,

$$\int_{\Omega}f(x,Du)dx\leqslant\int_{\Omega}f(x,Dw)dx,\quad\forall w\in u_*+W_0^{1,(p_i)}(\Omega),$$

then

$$u\in u_*+L^{t}_{weak}(\Omega),$$

where $t=\frac{\overline{p}\overline{p}^*}{\overline{p}-b\overline{p}^*}>\overline{p}^*$, $\overline{p}=\left(\frac{1}{n}\sum\limits_{i=1}^{n}\frac{1}{p_i}\right)^{-1}$ is the harmonic average of $\{p_i\}_{i=1}^n$, $\overline{p}^*=\frac{n\overline{p}}{n-\overline{p}}$ is the anisotropic Sobolev exponent for $\overline{p}<n$; moreover, b is any number such that

$$0<b\leqslant\min_{i=1,\cdots,n}\left\{1-\frac{p_i}{q_i}\right\}\quad\text{ and }\quad b<\frac{\overline{p}}{\overline{p}^*}.$$

It is worth noting that Cupini-Marcellini-Mascolo [6] pointed out an integral of the calculus of variations satisfying anisotropic growth conditions that may have unbounded minimizers if the growth exponents are too far apart. Under sharp assumptions on the exponents, they proved the local boundedness of minimizers of functionals with anisotropic p, q-growth via the De Giorgi method. As a by-product, regularity of minimizers of some non-coercive functionals was also obtained by reduction to coercive ones in [6], and the results of [22] were generalized .

Gao-Cui-Liang [10] extended the results of Leonetti-Siepe [22] to the obstacle problem. Namely, for

$$K_{\Psi,\vartheta}^{(p_i)}(\Omega) = \left\{ v \in W^{1,(p_i)}(\Omega), v \geqslant \Psi,\ a.e.,\ v - \vartheta \in W_0^{1,(p_i)}(\Omega) \right\},$$

where the function Ψ is an obstacle and ϑ denotes the boundary value, if $u \in K_{\Psi,\vartheta}^{(p_i)}(\Omega)$ is a minimizer of the obstacle problem of $I(u)$ with (1), that is,

$$\int_\Omega f(x, Du)dx \leqslant \int_\Omega f(x, Dw)dx, \quad \forall w \in K_{\Psi,\vartheta}^{(p_i)}(\Omega).$$

Then they show that the higher integrability of the boundary datum

$$\theta_* = \max\{\vartheta, \Psi\} \in \vartheta + W_0^{1,(q_i)}(\Omega)$$

forces minimizer u to have a higher integrability as well.

Cupini-Marcellini-Mascolo [5] obtained some regularity results for local minimizers $u : \Omega \subset \mathbb{R}^3 \to \mathbb{R}^3$ of a special class of polyconvex functionals, for example

$$\int_\Omega \left(\sum_{\alpha=1}^{3} |Du^\alpha|^p + |adj_2 Du|^q + |\det Du|^r \right) dx, \quad p, q, r > 1,$$

where $u = (u^1, u^2, u^3)$, adj_i denotes the adjugate matrix of order i, and det denotes the determinant of matrix. Under some structure assumptions on the energy density, they proved that local minimizers u are locally bounded. Furthermore, the regularity of minimizers of integrals of the calculus of variations with nonstandard growth conditions can be found in Marcellini [27] (see also [3, 4, 24]).

For isotropic integral functionals, Leonetti-Petricca [25] considered $I(u)$ with

$$c|z|^p - g(x) \leqslant f(x, z), \tag{3}$$

where the constants $p \in (1, n)$ and $c \in (0, +\infty)$, the function $g : \Omega \to [0, +\infty)$ for almost every $x \in \Omega$ and for all $z \in \mathbb{R}^n$ and $g \in L^\sigma(\Omega)$, the function $x \to f(x, Du_*) \in L^\sigma(\Omega)$, where $\sigma \in (1, +\infty)$. If $u \in u_* + W_0^{1,p}(\Omega)$ is a minimizer of $I(u)$ with (3), that is,

$$\int_\Omega f(x, Du)dx \leqslant \int_\Omega f(x, Dw)dx, \quad \forall w \in u_* + W_0^{1,p}(\Omega),$$

then

$$\begin{array}{lcc} \sigma < \frac{n}{p} & \Rightarrow & u - u_* \in L_{weak}^{\frac{np\sigma}{n-p\sigma}}(\Omega), \\ \sigma = \frac{n}{p} & \Rightarrow & \exists \alpha > 0 : e^{\alpha|u-u_*|} \in L^1(\Omega), \\ \sigma > \frac{n}{p} & \Rightarrow & u - u_* \in L^\infty(\Omega), \end{array}$$

where $\frac{np\sigma}{n-p\sigma} > \frac{np}{n-p}$.

Leonetti-Siepe [23] (see also [15]) considered the following homogeneous boundary value problem

$$\begin{cases} \sum\limits_{i=1}^{n} D_i\left(a_i\left(x, Du(x)\right)\right) = 0, & x \in \Omega, \\ u(x) = u_*(x), & x \in \partial\Omega, \end{cases}$$

where $a_i(x,z)$ satisfies (2). They showed that the higher integrability of the boundary datum u_* forces solutions u to have a higher integrability as well. Gao-Huang [11] (see also [12]) extended the results of [23] to the obstacle problem.

Kovalevsky [16] considered the following nonhomogeneous boundary value problem

$$\begin{cases} -\sum\limits_{i=1}^{n} D_i\left(a_i\left(x, Du(x)\right)\right) = -\sum\limits_{i=1}^{n} D_i f_i + f, & x \in \Omega, \\ u(x) = u_*(x), & x \in \partial\Omega, \end{cases}$$

where $f \in L^{\frac{\overline{p}^*}{\overline{p}^*-1}}(\Omega)$, $f_i \in L^{\frac{p_i}{p_i-1}}(\Omega)$, $a_i(x,z)$ satisfies

$$\sum_{i=1}^{n} |a_i(x,z)|^{\frac{p_i}{p_i-1}} \leqslant c\sum_{i=1}^{n} |z_i|^{p_i} + g_1(x), \quad 0 \leqslant g_1(x) \in L^1(\Omega),$$
$$\sum_{i=1}^{n} a_i(x,z)z_i \geqslant c\sum_{i=1}^{n} |z_i|^{p_i} - g_2(x), \quad 0 \leqslant g_2(x) \in L^1(\Omega)$$

and

$$\sum_{i=1}^{n} (a_i(x,z) - a_i(x,\tilde{z}))(z_i - \tilde{z}_i) \geqslant 0,$$

they obtained some similar integrability results to Leonetti-Siepe [23] as well.

Tersenov-Tersenov [31] considered the Dirichlet problem of anisotropic degenerate elliptic equation

$$\begin{cases} -\sum\limits_{i=1}^{n} \mu_i D_i\left(|D_i u|^{p_i-2} D_i u\right) = b(x)g(u) + f(x), & x \in \Omega, \\ u = 0, & x \in \partial\Omega, \end{cases}$$

where $\Omega \subset \mathbb{R}^n$ $(n \geqslant 2)$ with $\Omega \subset \{x : -l_i \leqslant x_i \leqslant l_i, i = 1, \ldots, n\}$, the constants $\mu_i > 0$ and $p_i > 1$, the functions $b(x), f(x)$ are bounded in $\overline{\Omega}$, the function g is Hölder continuous and satisfies

$$g(0) = 0, g(z) > 0 \quad \text{if} \quad z > 0,$$

and

$$|g(z)| > |g(c)| \quad \text{for} \quad |z| < c,$$

where c is an arbitrary positive constant. New priori estimates for solutions and for the gradient of solutions are established. Based on these estimates, sufficient conditions guaranteeing solvability of the problem are formulated. The results are new even in the semilinear case when the principal part is the Laplace operator.

Some existence and regularity results of solutions to the nonlinear problems related to [31], namely for

$$\begin{cases} -\sum\limits_{i=1}^{n} D_i\left(|D_i u|^{p_i-2} D_i u\right) = f(x), & x \in \Omega, \\ u = 0, & x \in \partial\Omega \end{cases}$$

and

$$\begin{cases} -\sum\limits_{i=1}^{n} D_i\left(|D_i u|^{p_i-2} D_i u\right) = \lambda u^p, & x \in \Omega, \\ u = 0, & x \in \partial\Omega, \end{cases}$$

can be found in Di Castro [7] and Fragala-Gazzola-Kawohl [8], respectively.

Kovalevsky-Nicolosi [19] considered the Dirichlet problem for a degenerate nonlinear elliptic equation

$$-\sum_{i=1}^{n} D_i\left(\mu |Du|^{p-2} D_i u\right) = f, \quad \text{in} \quad \Omega,$$

where $p \in (1, n)$, $f \in L^1(\Omega)$, and the weighted function $\mu = |x|^{\alpha}, x \in \Omega, \alpha \in (0, 1]$. They proved that if $p > 2 - \frac{1-\alpha}{n}$, then the Dirichlet problem has weak solutions for every L^1-right-hand side. On the other hand, they found out that if $p \leqslant 2 - \frac{1-\alpha}{n}$, then there exists an L^1-datum such that the corresponding Dirichlet problem does not have weak solutions. Similar results to the weighted anisotropic case are obtained by Kovalevsky-Gorban [18].

Inspired by the ideas in Tersenov-Tersenov [31] and Kovalevsky-Nicolosi [19], a question is natural whether tthere exist weighted anisotropic integral functionals. In this paper, we investigate weighted anisotropic integral functionals $I(u) = \int_{\Omega} f(x, Du(x))dx$, where $u : \Omega \subset \mathbb{R}^n \to \mathbb{R}$, $f : \Omega \times \mathbb{R}^n \to [0, +\infty)$ is a Carathéodory function (i.e., $x \to f(x, z)$ measurable for $z \in \mathbb{R}^n$, $z \to f(x, z)$ is continuous for a.e. $x \in \Omega$) and satisfies the nonstandard growth condition

$$\sum_{i=1}^{n} \nu_i |z_i|^{p_i} \le f(x, z) \le c\left(1 + \sum_{i=1}^{n} \nu_i |z_i|^{q_i}\right), \tag{4}$$

where $c > 0$ is a constant, $1 < p_i < q_i < n$, $i = 1, 2, \ldots, n$, ν_i is the positive weighted function on Ω and

$$\nu_i \in L^1_{loc}(\Omega), \quad \left(\frac{1}{\nu_i}\right)^{m_i} \in L^1(\Omega), \quad m_i \ge \frac{1}{p_i - 1}. \tag{5}$$

For the boundary datum $u_* : \Omega \to \mathbb{R}$, we suppose $u_* \in W^{1,(q_i)}(\Omega, \nu_i)$. Taking advantage of the weighted anisotropic Sobolev inequality (see Theorem 3 below) and the iteration lemma (see Lemma 1 below), due to Stampacchia [28, 29], we get that the minimizer of $I(u)$ with (4) has a higher integrability if u_* has a higher integrability. Furthermore, the global boundednesses of exponential form and $L^{\infty}(\Omega)$ for the minimizer are proved. For the minimizer of the obstacle problem to $I(u)$

with (4), similar results are obtained when the boundary datum $\theta_* = \max\{\theta, \Psi\} \in \vartheta + W_0^{1,(q_i)}(\Omega, \nu_i)$. Hence we generalize the results of [10, 22] to weighted anisotropic integral functionals.

This paper is organized as follows. In Section 2, we state the main results (Theorem 1 and Theorem 2) and describe preliminary knowledge. The proofs of Theorem 1 and Theorem 2 are given in Section 3 and Section 4, respectively.

2. Main results and preliminary knowledge

Anisotropic Sobolev spaces are introduced and studied by Adams [1], Kruzhkow-Kolodii [21] and Troisi [32].

Let $\Omega \subset \mathbb{R}^n$ $(n \geqslant 2)$ be a bounded open set and denote

$$\overline{p}^* = \frac{n\overline{p}}{n-\overline{p}}, \quad \text{for} \quad \overline{p} < n, \quad \overline{p} = \left(\frac{1}{n}\sum_{i=1}^{n}\frac{1}{p_i}\right)^{-1}, \quad 1 < p_i < +\infty \quad (i = 1, 2, \ldots, n).$$

Two anisotropic Sobolev spaces are defined by

$$W^{1,(p_i)}(\Omega) = \left\{u \in W^{1,1}(\Omega) : D_i u \in L^{p_i}(\Omega), i = 1, \ldots, n\right\}$$

and

$$W_0^{1,(p_i)}(\Omega) = \left\{u \in W_0^{1,1}(\Omega) : D_i u \in L^{p_i}(\Omega), i = 1, \ldots, n\right\}$$

the corresponding norms

$$\|u\|_{W^{1,(p_i)}(\Omega)} = \int_\Omega |u|\, dx + \sum_{i=1}^{n}\left(\int_\Omega |D_i u|^{p_i}\, dx\right)^{\frac{1}{p_i}}$$

and

$$\|u\|_{W_0^{1,(p_i)}(\Omega)} = \sum_{i=1}^{n}\left(\int_\Omega |D_i u|^{p_i}\, dx\right)^{\frac{1}{p_i}},$$

respectively. Note that $W^{1,(p_i)}(\Omega)$ and $W_0^{1,(p_i)}(\Omega)$ are reflexive Banach spaces. In Troisi [32], it is proved that if $\overline{p} < n$. Then

$$W_0^{1,(p_i)}(\Omega) \hookrightarrow L^r, \forall r \in [1, \overline{p}^*].$$

This embedding is continuous and also compact if $r \in [1, \overline{p}^*)$. The following anisotropic Sobolev inequality is also proved: If $\overline{p} < n$, then there exists a positive constant c such that for every $u \in W_0^{1,(p_i)}(\Omega)$,

$$\left(\int_\Omega |u|^{\overline{p}^*} dx\right)^{\frac{1}{\overline{p}^*}} \leqslant c\left(\prod_{i=1}^{n}\left(\int_\Omega |D_i u|^{p_i} dx\right)^{\frac{1}{p_i}}\right)^{\frac{1}{n}} = c\prod_{i=1}^{n}\|D_i u\|_{L^{p_i}(\Omega)}^{\frac{1}{n}}.$$

If $\overline{p} \geqslant n$, then $W_0^{1,(p_i)}(\Omega) \hookrightarrow L^r(\Omega)$ for any $r \in [1, +\infty)$.

Based on the method of Troisi [32], Kovalevsky-Gorban [17] (see also [14, 18]) extended anisotropic Sobolev spaces to weighted anisotropic Sobolev spaces

$$W^{1,(p_i)}(\Omega,\nu_i)=\left\{u\in W^{1,1}(\Omega):\nu_i|D_iu|^{p_i}\in L^1(\Omega),i=1,\dots,n\right\}$$

and

$$W_0^{1,(p_i)}(\Omega,\nu_i)=\left\{u\in W_0^{1,1}(\Omega):\nu_i|D_iu|^{p_i}\in L^1(\Omega),i=1,\dots,n\right\}.$$

The norms are

$$\|u\|_{W^{1,(p_i)}(\Omega,\nu_i)}=\int_\Omega |u|\,dx+\sum_{i=1}^n\left(\int_\Omega \nu_i|D_iu|^{p_i}dx\right)^{\frac{1}{p_i}}$$

and

$$\|u\|_{W_0^{1,(p_i)}(\Omega,\nu_i)}=\sum_{i=1}^n\left(\int_\Omega \nu_i|D_iu|^{p_i}dx\right)^{\frac{1}{p_i}},$$

respectively, where the weighted function ν_i satisfies (5) and characterizes degeneration or singularity to the spatial variable. If the weighted function $\nu_i=\nu$, the positive exponents $p_i=p$ for every $i=1,\cdots,n$, then weighted anisotropic Sobolev space $W^{1,(p_i)}(\Omega,\nu_i)$ is weighted isotropic Sobolev space $W^{1,p}(\Omega,\nu)$ with the norm (see Kovalevsky-Nicolosi [19], see also Kovalevsky-Rudakova [20])

$$\|u\|_{W^{1,p}(\Omega,\nu)}=\int_\Omega |u|\,dx+\sum_{i=1}^n\left(\int_\Omega \nu|D_iu|^{p}dx\right)^{\frac{1}{p}}.$$

It is worth noting that we can construct a weighted function ν_i, namely

$$\nu_i=|x_i|\,,\quad x_i\neq 0,\quad i=1,\dots,n.$$

Note that $W^{1,(p_i)}(\Omega,\nu_i)$ and $W_0^{1,(p_i)}(\Omega,\nu_i)$ are reflexive Banach spaces. For any $m=(m_1,m_2,\dots,m_n)\in\mathbb{R}^n$, $m_i\geq\frac{1}{p_i-1}$, we set

$$p_m=\frac{n}{\sum\limits_{i=1}^n\frac{1+m_i}{m_ip_i}-1}. \tag{6}$$

Note that (6) and

$$\sum_{i=1}^n\frac{1+m_i}{m_ip_i}=\sum_{i=1}^n\left(\frac{1}{m_i}+1\right)\frac{1}{p_i}\leq\sum_{i=1}^n(p_i-1+1)\frac{1}{p_i}=n;$$

then $p_m\geq\frac{n}{n-1}>1$ and $W_0^{1,(p_i)}(\Omega,\nu_i)\hookrightarrow L^{p_m}(\Omega)$ (see Theorem 3 below).

The weak L^p space on Ω, also known as the Marcinkiewicz space (see [1, 2, 13]), denoted by $L^p_{weak}(\Omega)$, is the set of all measurable functions $f(x)$ satisfying

$$meas\,\{x\in\Omega:|f(x)|>t\}\leq\frac{c}{t^p} \tag{7}$$

for $t > 0$ and some positive constants $c = c(f)$, $p \geq 1$, where $meas\,E$ is the n-dimensional Lebesgue measure of $E \subset \mathbb{R}^n$. We recall the facts that if $f \in L^p_{weak}(\Omega)$, then $f \in L^q(\Omega)$ for every $1 \leq q < p$; furthermore, $L^\infty_{weak}(\Omega) = L^\infty(\Omega)$.

The minimizer u of $I(u)$ with (4) can be rewritten as $u = u_* + (u - u_*)$. Our first aim is to prove when the boundary datum u_* has a higher integrability, $u - u_*$ also has a higher integrability.

Theorem 1. *Let $p_m > \overline{p}$, $\left(1 + \sum\limits_{i=1}^n \nu_i |D_i u_*|^{q_i}\right) \in L^r(\Omega)$, $r > 1$, and $u \in u_* + W_0^{1,(p_i)}(\Omega, \nu_i)$ be a minimizer of $I(u)$ with (4), that is,*

$$\int_\Omega f(x, Du)dx \leq \int_\Omega f(x, Dw)dx \tag{8}$$

for any $w \in u_* + W_0^{1,(p_i)}(\Omega, \nu_i)$. Hence,

(i) (global integrability) if $1 < r < \frac{p_m}{p_m - \overline{p}}$, then

$$u - u_* \in L_{weak}^{\frac{r\overline{p}p_m}{r\overline{p} - rp_m + p_m}}(\Omega),$$

where $\frac{r\overline{p}p_m}{r\overline{p} - rp_m + p_m} > p_m$;

(ii) (global boundedness of exponential form) if $r = \frac{p_m}{p_m - \overline{p}}$, then there is a positive constant $\theta < \tau$ such that

$$\int_\Omega \left(e^{\theta|u - u_*|} - 1\right)dx \leq \frac{\theta e}{\tau - \theta} meas\,\Omega,$$

where $\tau = (ec)^{-\frac{1}{p_m}} > 0$;

(iii) (global boundedness of $L^\infty(\Omega)$) if $r > \frac{p_m}{p_m - \overline{p}}$, then

$$u - u_* \in L^\infty(\Omega).$$

Remark 1. *In Theorem 1, if $f : \Omega \times \mathbb{R}^n \to [0, +\infty)$ is a Carathéodory function and satisfies the standard growth condition*

$$\sum_{i=1}^n \nu_i |z_i|^{p_i} \leq f(x, z) \leq c\left(1 + \sum_{i=1}^n \nu_i |z_i|^{p_i}\right),$$

where $c > 0$ is a constant, $1 < p_i = q_i < n$, $i = 1, 2, \cdots, n$, ν_i is the positive weighted function as in (5), then the condition "$\left(1 + \sum\limits_{i=1}^n \nu_i |D_i u_*|^{q_i}\right) \in L^r(\Omega)$, $r > 1$"is replaced by "$\left(1 + \sum\limits_{i=1}^n \nu_i |D_i u_*|^{p_i}\right) \in L^r(\Omega)$, $r > 1$".

Next, we consider the obstacle problem to $I(u)$ with (4). Let Ψ be a function on Ω with the value in $\mathbb{R} \cup \{\pm\infty\}$, $\vartheta \in W^{1,(p_i)}(\Omega, \nu_i)$. Set

$$K_{\Psi,\vartheta}^{(p_i)}(\Omega, \nu_i) = \left\{v \in W^{1,(p_i)}(\Omega, \nu_i), v \geq \Psi,\ a.e.,\ v - \vartheta \in W_0^{1,(p_i)}(\Omega, \nu_i)\right\},$$

where Ψ is an obstacle and ϑ denotes the boundary value.

We call that $u \in K^{(p_i)}_{\Psi,\vartheta}(\Omega,\nu_i)$ is a minimizer of the obstacle problem to $I(u)$ with (4) if

$$\int_\Omega f(x,Du)dx \le \int_\Omega f(x,D\phi)dx \tag{9}$$

for any $\phi \in K^{(p_i)}_{\Psi,\vartheta}(\Omega,\nu_i)$.

Theorem 2. *Let $p_m > \overline{p}$, $\left(1+\sum_{i=1}^n \nu_i|D_i\theta_*|^{q_i}\right) \in L^s(\Omega), s>1$, $\theta_* = \max\{\vartheta,\Psi\} \in \vartheta + W_0^{1,(q_i)}(\Omega,\nu_i)$, and let $u \in K^{(p_i)}_{\Psi,\vartheta}(\Omega,\nu_i)$ be a minimizer of the obstacle problem to $I(u)$ with (4). Hence,*

(iv) (global integrability) if $1<s<\frac{p_m}{p_m-\overline{p}}$, then

$$u-\theta_* \in L_{weak}^{\frac{s\overline{p}p_m}{s\overline{p}-sp_m+p_m}}(\Omega),$$

where $\frac{s\overline{p}p_m}{s\overline{p}-sp_m+p_m} > p_m$;

(v) (global boundedness of exponential form) if $s=\frac{p_m}{p_m-\overline{p}}$, then there is a positive constant $\lambda<\tau$ such that

$$\int_\Omega \left(e^{\lambda|u-\theta_*|}-1\right)dx \le \frac{\lambda e}{\tau-\lambda}\, meas\,\Omega,$$

where $\tau = (ec)^{-\frac{1}{p_m}} > 0$;

(vi) (global boundedness of $L^\infty(\Omega)$) if $s>\frac{p_m}{p_m-\overline{p}}$, then

$$u-\theta_* \in L^\infty(\Omega).$$

The following Theorem 3 and Lemma 1 will be useful in the proofs of Theorem 1 and Theorem 2.

Theorem 3 (Weighted anisotropic Sobolev inequality, see [14, 17, 18])**.** *Let $\Omega \subset \mathbb{R}^n$ be a bounded open set, $m=(m_1,m_2,\dots,m_n) \in \mathbb{R}^n$, $m_i \ge \frac{1}{p_i-1}$ and $\frac{1}{\nu_i} \in L^{m_i}(\Omega)$ for $i \in \{1,2,\dots,n\}$. Then $W_0^{1,(p_i)}(\Omega,\nu_i) \hookrightarrow L^{p_m}(\Omega)$ and there is a positive constant c such that for any $u \in W_0^{1,(p_i)}(\Omega,\nu_i)$,*

$$\left(\int_\Omega |u|^{p_m}dx\right)^{\frac{1}{p_m}} \le c\left(\prod_{i=1}^n\left(\int_\Omega \nu_i|D_iu|^{p_i}dx\right)^{\frac{1}{p_i}}\right)^{\frac{1}{n}} \tag{10}$$

where p_m satisfies (6).

Lemma 1 (see [28, 29])**.** *Let $\varphi(t)$ be a nonnegative and nonincreasing function on $[k_0,+\infty)$ satisfying*

$$\varphi(h) \le \frac{c}{(h-k)^\alpha}[\varphi(k)]^\beta, \quad h>k\ge k_0,$$

where c, α and β are positive constants. If $\beta < 1$, $k_0 > 0$, then

$$\varphi(h) \leq \left[c^{\frac{1}{1-\beta}} + (2k_0)^{\frac{\alpha}{1-\beta}} \varphi(k_0)\right] 2^{\frac{\alpha}{(1-\beta)^2}} \left(\frac{1}{h}\right)^{\frac{\alpha}{1-\beta}}; \tag{11}$$

if β=1, then

$$\varphi(h) \leq e^{1-\tau(h-k_0)} \varphi(k_0), \tag{12}$$

where $\tau = (ec)^{-\frac{1}{\alpha}} > 0$; if $\beta > 1$, then

$$\varphi(k_0 + d) = 0, \tag{13}$$

where $d = c(\varphi(k_0))^{\frac{\beta-1}{\alpha}} 2^{\frac{\beta}{\beta-1}}$.

3. Proof of Theorem 1

For any $k \in (0, +\infty)$, let $T_k : \mathbb{R} \to \mathbb{R}$ be a function such that

$$T_k(u - u_*) = \begin{cases} u - u_*, & \text{if} \quad |u - u_*| \leq k, \\ k \frac{u-u_*}{|u-u_*|}, & \text{if} \quad |u - u_*| > k. \end{cases} \tag{14}$$

Denoting $\psi = u - u_* - T_k(u - u_*)$, it follows from (14) that

$$\psi = \begin{cases} u - u_* + k, & \text{if} \quad u - u_* < -k, \\ 0, & \text{if} \quad -k \leq u - u_* \leq k, \\ u - u_* - k, & \text{if} \quad u - u_* > k. \end{cases} \tag{15}$$

By $u \in u_* + W_0^{1,(p_i)}(\Omega, \nu_i)$ and (15), it follows

$$\psi \in W_0^{1,(p_i)}(\Omega, \nu_i), \tag{16}$$

$$D\psi = (Du - Du_*)1_{\{|u-u_*|>k\}}, \tag{17}$$

$$|\psi| = (|u - u_*| - k)\, 1_{\{|u-u_*|>k\}}, \tag{18}$$

where $1_A(x) = 1$ if $x \in A$, $1_A(x) = 0$ if $x \notin A$. Setting

$$w = u - \psi, \tag{19}$$

we have $w \in u_* + W_0^{1,(p_i)}(\Omega, \nu_i)$ and from (17) and (19) it follows that

$$\begin{aligned} Dw &= Du - D\psi \\ &= Du - (Du - Du_*)1_{\{|u-u_*|>k\}} \\ &= Du - (Du)1_{\{|u-u_*|>k\}} + (Du_*)1_{\{|u-u_*|>k\}} \\ &= (Du)1_{\{|u-u_*|\leq k\}} + (Du)1_{\{|u-u_*|>k\}} - (Du)1_{\{|u-u_*|>k\}} \\ &\quad + (Du_*)1_{\{|u-u_*|>k\}} \\ &= (Du)1_{\{|u-u_*|\leq k\}} + (Du_*)1_{\{|u-u_*|>k\}}. \end{aligned} \tag{20}$$

By $\Omega = \{|u-u_*| \leq k\} \cup \{|u-u_*| > k\}$, (8) and (20), we obtain

$$\begin{aligned}
&\int_{\{|u-u_*|\leq k\}} f(x,Du)dx + \int_{\{|u-u_*|>k\}} f(x,Du)dx \\
&\quad \leq \int_{\{|u-u_*|\leq k\}} f(x,Dw)dx + \int_{\{|u-u_*|>k\}} f(x,Dw)dx \\
&\quad = \int_{\{|u-u_*|\leq k\}} f(x,Du)dx + \int_{\{|u-u_*|>k\}} f(x,Du_*)dx, \qquad (21)
\end{aligned}$$

and then by (21),

$$\int_{\{|u-u_*|>k\}} f(x,Du)dx \leq \int_{\{|u-u_*|>k\}} f(x,Du_*)dx. \qquad (22)$$

Combining (16), (17) and (10) in Theorem 3, we conclude

$$\begin{aligned}
\left(\int_\Omega |\psi|^{p_m} dx\right)^{\frac{1}{p_m}} &\leqslant c\left(\prod_{i=1}^n \left(\int_\Omega \nu_i |D_i\psi|^{p_i} dx\right)^{\frac{1}{p_i}}\right)^{\frac{1}{n}} \\
&= c\left(\prod_{i=1}^n \left(\int_{\{|u-u_*|>k\}} \nu_i |D_iu - D_iu_*|^{p_i} dx\right)^{\frac{1}{p_i}}\right)^{\frac{1}{n}}. \qquad (23)
\end{aligned}$$

By (4) and (22),

$$\begin{aligned}
&\int_{\{|u-u_*|>k\}} \sum_{i=1}^n \nu_i |D_iu - D_iu_*|^{p_i} dx \\
&\quad \leq \int_{\{|u-u_*|>k\}} \sum_{i=1}^n \nu_i \left[2^{p_{\max}} \left(|D_iu|^{p_i} + |D_iu_*|^{p_i}\right)\right] dx \\
&\quad = 2^{p_{\max}} \int_{\{|u-u_*|>k\}} \sum_{i=1}^n \nu_i |D_iu|^{p_i} dx + 2^{p_{\max}} \int_{\{|u-u_*|>k\}} \sum_{i=1}^n \nu_i |D_iu_*|^{p_i} dx \\
&\quad \leq 2^{p_{\max}} \int_{\{|u-u_*|>k\}} f(x,Du)dx + 2^{p_{\max}} \int_{\{|u-u_*|>k\}} f(x,Du_*)dx \\
&\quad \leq 2^{p_{\max}} \int_{\{|u-u_*|>k\}} f(x,Du_*)dx + 2^{p_{\max}} \int_{\{|u-u_*|>k\}} f(x,Du_*)dx \\
&\quad \leq 2^{p_{\max}} 2c \int_{\{|u-u_*|>k\}} \left(1 + \sum_{i=1}^n \nu_i |D_iu_*|^{q_i} dx\right), \qquad (24)
\end{aligned}$$

where $p_{\max} = \max\limits_{i=1,\cdots,n}\{p_i\}$. Denote $\Phi = 1 + \sum\limits_{i=1}^{n} \nu_i |D_i u_*|^{q_i}$; therefore $\Phi \in L^r(\Omega)$ and $\|\Phi\|_{L^r(\Omega)} \le c$, $c > 0$. Using Hölder's inequality, we get

$$\begin{aligned}
\int_{\{|u-u_*|>k\}} \Phi dx &\le \left(\int_{\{|u-u_*|>k\}} |\Phi|^r dx \right)^{\frac{1}{r}} [meas\,\{|u-u_*|>k\}]^{\frac{r-1}{r}} \\
&\le \|\Phi\|_{L^r(\Omega)} [meas\,\{|u-u_*|>k\}]^{\frac{r-1}{r}} \\
&\le c[meas\,\{|u-u_*|>L\}]^{\frac{r-1}{r}}. \qquad (25)
\end{aligned}$$

By (24) and (25),

$$\begin{aligned}
\sum_{i=1}^{n} \int_{\{|u-u_*|>k\}} \nu_i |D_i u - D_i u_*|^{p_i} dx &= \int_{\{|u-u_*|>k\}} \sum_{i=1}^{n} \nu_i |D_i u - D_i u_*|^{p_i} dx \\
&\leqslant c[meas\,\{|u-u_*|>k\}]^{\frac{r-1}{r}}. \qquad (26)
\end{aligned}$$

Now we lower the left-hand side of (26) by considering just one integral. Then we raise both sides to the power $\frac{1}{p_i}$ and take the product with respect to i obtaining that

$$\begin{aligned}
\prod_{i=1}^{n} \left(\int_{\{|u-u_*|>k\}} \nu_i |D_i u - D_i u_*|^{p_i} dx \right)^{\frac{1}{p_i}} &\leqslant c[meas\,\{|u-u_*|>k\}]^{\frac{r-1}{r}\left(\sum\limits_{i=1}^{n} \frac{1}{p_i}\right)} \\
&= c[meas\,\{|u-u_*|>k\}]^{\frac{n(r-1)}{r\overline{p}}}. \qquad (27)
\end{aligned}$$

Combining (23) and (27),

$$\begin{aligned}
\left(\int_{\Omega} |\psi|^{p_m} dx \right)^{\frac{1}{p_m}} &\leqslant c \left(\prod_{i=1}^{n} \left(\int_{\{|u-u_*|>k\}} \nu_i |D_i u - D_i u_*|^{p_i} dx \right)^{\frac{1}{p_i}} \right)^{\frac{1}{n}} \\
&\leqslant c[meas\,\{|u-u_*|>k\}]^{\frac{(r-1)}{r\overline{p}}}. \qquad (28)
\end{aligned}$$

For any $h > k \ge k_0$, by using (18) we have that

$$\begin{aligned}
(h-k)^{p_m} [meas\,\{|u-u_*|>h\}] &= \int_{\{|u-u_*|>h\}} (h-k)^{p_m} dx \\
&\le \int_{\{|u-u_*|>h\}} (|u-u_*|-k)^{p_m} dx \\
&\le \int_{\{|u-u_*|>k\}} (|u-u_*|-k)^{p_m} dx \\
&= \int_{\Omega} |\psi|^{p_m} dx. \qquad (29)
\end{aligned}$$

Finally, by (28) and (29) we easily obtain

$$meas\,\{|u-u_*|>h\} \le \frac{c}{(h-k)^{p_m}} [meas\,\{|u-u_*|>k\}]^{\frac{(r-1)p_m}{r\overline{p}}}. \qquad (30)$$

Hence, let

$$\varphi(h) = meas\left\{|u-u_*|>h\right\}, \varphi(k) = meas\left\{|u-u_*|>k\right\},$$
$$\alpha = p_m, \beta = \frac{(r-1)\,p_m}{r\overline{p}}$$

in (30), from Lemma 1, we prove, respectively.

(i) If $1<r<\frac{p_m}{p_m-\overline{p}}$, then $\beta<1$. For $k_0>0$, one has by (11) that

$$\begin{aligned} meas\left\{|u-u_*|>h\right\} &\leq \left[c^{\frac{1}{1-\beta}} + (2k_0)^{\frac{\alpha}{1-\beta}} meas\left\{|u-u_*|>k_0\right\}\right] 2^{\frac{\alpha}{(1-\beta)^2}} \left(\frac{1}{h}\right)^{\frac{\alpha}{1-\beta}} \\ &\leq \left[c^{\frac{1}{1-\beta}} + (2k_0)^{\frac{\alpha}{1-\beta}} meas\ \Omega\right] 2^{\frac{\alpha}{(1-\beta)^2}} \left(\frac{1}{h}\right)^{\frac{\alpha}{1-\beta}}; \end{aligned} \tag{31}$$

hence from (31) and (7),

$$u-u_* \in L^{\frac{\alpha}{1-\beta}}_{weak}(\Omega),$$

where

$$\frac{\alpha}{1-\beta} = \frac{p_m}{1-\frac{(r-1)p_m}{r\overline{p}}} = \frac{r\overline{p}p_m}{r\overline{p}-rp_m+p_m} > p_m.$$

(ii) If $r=\frac{p_m}{p_m-\overline{p}}$, then $\beta=1$, by (12) we have that

$$meas\left\{|u-u_*|>h\right\} \leq\ e^{1-\tau(h-k_0)} meas\left\{|u-u_*|>k_0\right\}, \tag{32}$$

where $\tau=(ec)^{-\frac{1}{p_m}}>0$. If $k_0\leq 0$, then

$$e^{1-\tau(h-k_0)} = ee^{-\tau(h-k_0)} \leq ee^{-\tau h}, \tag{33}$$

and

$$meas\left\{|u-u_*|>k_0\right\} = meas\,\Omega. \tag{34}$$

Substituting (33) and (34) into (32), yields

$$meas\left\{|u-u_*|>h\right\} \leq ee^{-\tau h} meas\ \Omega. \tag{35}$$

Note that there is a positive constant $\theta<\tau$ such that

$$\begin{aligned} e^{\theta|u-u_*|}-1 &= \int_0^{|u-u_*|} \theta e^{\theta h}dh \\ &= \int_0^\infty \theta e^{\theta h} 1_{\{|u-u_*|>h\}}dh. \end{aligned} \tag{36}$$

By (36) it follows that

$$\begin{aligned}\int_\Omega \left(e^{\theta|u-u_*|}-1\right)dx &= \int_\Omega\int_0^\infty \theta e^{\theta h}1_{\{|u-u_*|>h\}}dhdx\\ &= \int_\Omega 1_{\{|u-u_*|>h\}}dx\int_0^\infty \theta e^{\theta h}dh\\ &= \int_{\{|u-u_*|>h\}} 1dx\int_0^\infty \theta e^{\theta h}dh\\ &= meas\left\{|u-u_*|>h\right\}\int_0^\infty \theta e^{\theta h}dh\\ &= \int_0^\infty \theta e^{\theta h} meas\left\{|u-u_*|>h\right\}dh. \qquad (37)\end{aligned}$$

Taking (35) into (37),

$$\begin{aligned}\int_\Omega \left(e^{\theta|u-u_*|}-1\right)dx &\le \int_0^\infty \theta e^{\theta h} e e^{-\tau h} meas\,\Omega\, dh\\ &= \frac{\theta e}{\tau-\theta}\, meas\,\Omega.\end{aligned}$$

(iii) If $r>\frac{p_m}{p_m-\overline{p}}$, then $\beta>1$ and by (13),

$$\varphi\left(k_0+d\right)=meas\left\{|u-u_*|>k_0+d\right\}=0, \qquad (38)$$

where

$$d=c(meas\left\{|u-u_*|>k_0\right\})^{\frac{rp_m-p_m-r\overline{p}}{r\overline{p}p_m}}2^{\frac{rp_m-p_m}{rp_m-p_m-r\overline{p}}}.$$

Hence from (38) and (7), we have $u-u_*\in L^\infty(\Omega)$.

4. Proof of Theorem 2

Without loss of generality, we assume $\vartheta\ge\Psi$ a.e on $\partial\Omega$, otherwise $K^{(p_i)}_{\Psi,\vartheta}(\Omega,\nu_i)$ will be empty. Let $u\in K^{(p_i)}_{\Psi,\vartheta}(\Omega,\nu_i)$ be a minimizer of the obstacle problem to $I(u)$ with (4).

For any $L\in(0,+\infty)$, we introduce a function $T_L:\mathbb{R}\to\mathbb{R}$ by

$$T_L(u-\theta_*)=\begin{cases} u-\theta_*, & \text{if} \quad |u-\theta_*|\le L,\\ L\frac{u-\theta_*}{|u-\theta_*|}, & \text{if} \quad |u-\theta_*|>L.\end{cases} \qquad (39)$$

Denoting $\phi=\theta_*+T_L(u-\theta_*)$, from (39) it follows that

$$\phi=\begin{cases}\theta_*-L, & \text{if} \quad u-\theta_*<-L,\\ u, & \text{if} \quad -L\le u-\theta_*\le L,\\ \theta_*+L, & \text{if} \quad u-\theta_*>L\end{cases} \qquad (40)$$

and

$$D\phi=(Du)1_{\{|u-\theta_*|\le L\}}+(D\theta_*)1_{\{|u-\theta_*|>L\}}. \qquad (41)$$

We claim $\phi \in K^{(p_i)}_{\Psi,\vartheta}(\Omega,\nu_i)$. Actually, by $u \in K^{(p_i)}_{\Psi,\vartheta}(\Omega,\nu_i)$, it follows $\phi \in W^{1,(p_i)}(\Omega,\nu_i)$. From (40),

$$\phi = \begin{cases} \theta_* - L > u \geq \Psi, & \text{if} \quad u - \theta_* < -L, \\ u \geq \Psi, & \text{if} \quad -L \leq u - \theta_* \leq L, \\ \theta_* + L \geq \theta_* \geq \Psi, & \text{if} \quad u - \theta_* > L, \end{cases}$$

then $\phi \geq \Psi$ a.e on Ω. Finally, we notice that $\theta_* = \vartheta$ on $\partial\Omega$, it follows $T_L(u-\theta_*) = 0$ on $\partial\Omega$, so $\phi = \theta_* = \vartheta$ on $\partial\Omega$, that is, $\phi - \vartheta \in W^{1,(p_i)}_0(\Omega,\nu_i)$.

By $\Omega = \{|u-\theta_*| \leq L\} \cup \{|u-\theta_*| > L\}$, $\phi \in K^{(p_i)}_{\Psi,\vartheta}(\Omega,\nu_i)$, (9) and (41), we have

$$\begin{aligned} &\int_{\{|u-\theta_*|\leq L\}} f(x,Du)dx + \int_{\{|u-\theta_*|>L\}} f(x,Du)dx \\ &\quad \leq \int_{\{|u-\theta_*|\leq L\}} f(x,D\phi)dx + \int_{\{|u-\theta_*|>L\}} f(x,D\phi)dx \\ &\quad = \int_{\{|u-\theta_*|\leq L\}} f(x,Du)dx + \int_{\{|u-\theta_*|>L\}} f(x,D\theta_*)dx, \end{aligned} \tag{42}$$

and then by (42),

$$\int_{\{|u-\theta_*|>L\}} f(x,Du)dx \leq \int_{\{|u-\theta_*|>L\}} f(x,D\theta_*)dx. \tag{43}$$

From (40), (41) and $\phi \in K^{(p_i)}_{\Psi,\vartheta}(\Omega,\nu_i)$, we deduce that $\phi - u \in W^{1,(p_i)}_0(\Omega,\nu_i)$ and

$$|\phi - u| = |u-\phi| = (|u-\theta_*| - L)\, 1_{\{|u-\theta_*|>L\}}, \tag{44}$$

$$\begin{aligned} D\phi - Du &= (Du)1_{\{|u-\theta_*|\leq L\}} + (D\theta_*)1_{\{|u-\theta_*|>L\}} - (Du)1_{\{|u-\theta_*|\leq L\}} \\ &\quad - (Du)1_{\{|u-\theta_*|>L\}} \\ &= (D\theta_*)1_{\{|u-\theta_*|>L\}} - (Du)1_{\{|u-\theta_*|>L\}}. \end{aligned} \tag{45}$$

By $\phi - u \in W^{1,(p_i)}_0(\Omega,\nu_i)$, (45) and (10) in Theorem 3, it derives

$$\begin{aligned} \left(\int_\Omega |\phi-u|^{p_m} dx\right)^{\frac{1}{p_m}} &\leqslant c\left(\prod_{i=1}^n \left(\int_\Omega \nu_i |D_i\phi - D_i u|^{p_i} dx\right)^{\frac{1}{p_i}}\right)^{\frac{1}{n}} \\ &= c\left(\prod_{i=1}^n \left(\int_{\{|u-\theta_*|>L\}} \nu_i |D_i u - D_i\theta_*|^{p_i} dx\right)^{\frac{1}{p_i}}\right)^{\frac{1}{n}}. \end{aligned} \tag{46}$$

By using (4) and (43),

$$\begin{aligned}
&\int_{\{|u-\theta_*|>L\}} \sum_{i=1}^{n} \nu_i |D_i u - D_i\theta_*|^{p_i} dx \\
&\leq 2^{p_{\max}} \int_{\{|u-\theta_*|>L\}} \sum_{i=1}^{n} \nu_i |D_i u|^{p_i} dx + 2^{p_{\max}} \int_{\{|u-\theta_*|>L\}} \sum_{i=1}^{n} \nu_i |D_i\theta_*|^{p_i} dx \\
&\leq 2^{p_{\max}} \int_{\{|u-\theta_*|>L\}} f(x, Du) dx + 2^{p_{\max}} \int_{\{|u-\theta_*|>L\}} f(x, D\theta_*) dx \\
&\leq 2^{p_{\max}} \int_{\{|u-\theta_*|>L\}} f(x, D\theta_*) dx + 2^{p_{\max}} \int_{\{|u-\theta_*|>L\}} f(x, D\theta_*) dx \\
&\leq 2^{p_{\max}} 2c \int_{\{|u-\theta_*|>L\}} \left(1 + \sum_{i=1}^{n} \nu_i |D_i\theta_*|^{q_i}\right) dx, \qquad (47)
\end{aligned}$$

where $p_{\max} = \max_{i=1,\cdots,n} \{p_i\}$. Denote $H = 1 + \sum_{i=1}^{n} \nu_i |D_i\theta_*|^{q_i}$; then $H \in L^s(\Omega)$ and $\|H\|_{L^s(\Omega)} \leq c$, $c > 0$. By Hölder's inequality, we obtain

$$\begin{aligned}
\int_{\{|u-\theta_*|>L\}} H dx &\leq \left(\int_{\{|u-\theta_*|>L\}} |H|^s dx\right)^{\frac{1}{s}} [meas\,\{|u-\theta_*| > L\}]^{\frac{s-1}{s}} \\
&\leq \left(\int_{\Omega} |H|^s dx\right)^{\frac{1}{s}} [meas\,\{|u-\theta_*| > L\}]^{\frac{s-1}{s}} \\
&\leq c[meas\,\{|u-\theta_*| > L\}]^{\frac{s-1}{s}}. \qquad (48)
\end{aligned}$$

By (47) and (48),

$$\begin{aligned}
\sum_{i=1}^{n} \int_{\{|u-\theta_*|>L\}} \nu_i |D_i u - D_i\theta_*|^{p_i} dx &= \int_{\{|u-\theta_*|>L\}} \sum_{i=1}^{n} \nu_i |D_i u - D_i\theta_*|^{p_i} dx \\
&\leq c[meas\,\{|u-\theta_*| > L\}]^{\frac{s-1}{s}}. \qquad (49)
\end{aligned}$$

Now we lower the left-hand side of (49) by considering just one integral. Then we raise both sides to the power $\frac{1}{p_i}$ and take the product with respect to i obtaining that

$$\begin{aligned}
\prod_{i=1}^{n} \left(\int_{\{|u-\theta_*|>L\}} \nu_i |D_i u - D_i\theta_*|^{p_i} dx\right)^{\frac{1}{p_i}} &\leqslant c[meas\,\{|u-\theta_*| > L\}]^{\frac{s-1}{s}\left(\sum_{i=1}^{n} \frac{1}{p_i}\right)} \\
&= c[meas\,\{|u-\theta_*| > L\}]^{\frac{n(s-1)}{s\overline{p}}}. \qquad (50)
\end{aligned}$$

Combining (46) and (50),

$$\begin{aligned}
\left(\int_{\Omega} |\phi - u|^{p_m} dx\right)^{\frac{1}{p_m}} &\leqslant c\left(\prod_{i=1}^{n} \left(\int_{\{|u-\theta_*|>L\}} \nu_i |D_i u - D_i\theta_*|^{p_i} dx\right)^{\frac{1}{p_i}}\right)^{\frac{1}{n}} \\
&\leqslant c[meas\,\{|u-\theta_*| > L\}]^{\frac{(s-1)}{s\overline{p}}}. \qquad (51)
\end{aligned}$$

For any $\widetilde{L} > L \geq L_0$, by (44) we have that

$$\begin{aligned}\left(\widetilde{L}-L\right)^{p_m}\left[meas\left\{|u-\theta_*|>\widetilde{L}\right\}\right] &= \int_{\left\{|u-\theta_*|>\widetilde{L}\right\}}\left(\widetilde{L}-L\right)^{p_m}dx\\ &\leq \int_{\left\{|u-\theta_*|>\widetilde{L}\right\}}(|u-\theta_*|-L)^{p_m}dx\\ &\leq \int_{\{|u-\theta_*|>L\}}(|u-\theta_*|-L)^{p_m}dx\\ &= \int_\Omega |\phi-u|^{p_m}dx. \end{aligned} \tag{52}$$

From (51) and (52), one has

$$meas\left\{|u-\theta_*|>\widetilde{L}\right\} \leq \frac{c}{\left(\widetilde{L}-L\right)^{p_m}}[meas\{|u-\theta_*|>L\}]^{\frac{(s-1)p_m}{s\overline{p}}}. \tag{53}$$

Let

$$\varphi(\widetilde{L}) = meas\left\{|u-\theta_*|>\widetilde{L}\right\}, \varphi(L) = meas\{|u-\theta_*|>L\},$$
$$\alpha = p_m, \beta = \tfrac{(s-1)p_m}{s\overline{p}}$$

in (53). Similarly to the remaining process in the proof of Theorem 1, we can prove (iv), (v) and (vi) of Theorem 2 by Lemma 1.

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