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A collocation method on a Gartland-type mesh for a singularly perturbed reaction-diffusion problem

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Abstract. A singularly perturbed reaction-diffusion problem in one dimension is solved numerically by a collocation method with quadratic C^1-splines. Using an appropriately graded mesh of Gartland type, second order of convergence is obtained in the supremum norm uniformly, up to a logarithmic factor, in the singular perturbation parameter. The aim of this paper is to establish the advantage of using the Gartland-type mesh. The method presented here generates results that are superior to those obtained on the so-called "smoothed Shishkin mesh". Results of numerical experiments which illustrate our theoretical findings are presented. Furthermore, numerical results for a two-dimensional problem reveal the same order of the convergence as in the one-dimensional case, though efforts to establish its theoretical foundation are still ongoing.

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Key words: Reaction-diffusion problems, collocation, singular perturbation, Gartland-type mesh

1. Introduction

Singularly perturbed reaction-diffusion problems represent mathematical models of various phenomena in many areas, such as fluid mechanics, heat transfer, etc. These problems can be solved by various methods, such as difference schemes or finite element methods of different flavours, see [6, 9, 12] and references therein. In contrast, collocation methods are studied only in a small number of papers [10, 14].

The main goal in the construction of numerical methods for singularly perturbed problems is to obtain convergence that is uniform with respect to the perturbation parameter. Let u_ε be the solution of a singularly perturbed problem, and u_ε^N its numerical approximation obtained by a numerical method with N degrees of freedom. Then the numerical method is said to be uniformly convergent of order $p > 0$ in the norm $\|\cdot\|$ if there exist a constant C and an integer N_0, both independent of ε, such that

$$\|u_\varepsilon - u_\varepsilon^N\| \le CN^{-p}, \quad \text{for all} \quad N \ge N_0. \tag{1}$$

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In the present paper, we use a layer-adapted mesh constructed in a special way, in order to ensure the uniform convergence of a collocation method.

Consider the problem of finding u such that

$$\begin{gathered}\big(\mathcal{L}u\big)(x) := -\varepsilon^2 u''(x) + r(x)u(x) = f(x), \quad x \in (0,1),\\ u(0) = g_0, \quad u(1) = g_1,\end{gathered} \tag{2}$$

where $\varepsilon \in (0,1]$, $r, f \in C^4([0,1])$ and $0 < \varrho^2 \le r(x)$, $x \in (0,1)$, with some positive constant ϱ. Under these conditions, problem (2) possesses a unique solution. If ε is a small parameter, then our problem is singularly-perturbed and the solution exhibits sharp boundary layers of identical widths in the vicinity of $x = 0$ and $x = 1$.

A general theory for spline-collocation methods applied to classical problems, i.e. problems with $\varepsilon = 1$, has been developed in [1]. A first error bound for the collocation method applied to a reaction-diffusion problem was obtained in [14]. In that paper, the authors studied collocation with a quadratic spline on a Shishkin mesh. However, error bounds were given for the discrete maximum norm only.

In the present paper, we aim at improving the theoretical and numerical results obtained in [10]. We analyse the quadratic spline collocation method on a Gartland-type mesh, that we adapt to (2) from [4]. Our analysis requires some properties of the mesh that the smoothed Shishkin mesh and the Gartland-type mesh satisfy, but not the standard Shishkin and Bakhvalov meshes (see Remark 2).

The standard Shishkin mesh is characterised by a transition point

$$\tau := \min\left\{\frac{\sigma\varepsilon}{\varrho}\ln N, q\right\}, \tag{3}$$

where $\sigma, \varrho > 0$ and $q \in (0, 1/2)$. It is generated by $x_i = \varphi_S(i/N)$, $i = 0, 1, \dots, N$, with the mesh generating function

$$\varphi_S(t) := \begin{cases} \dfrac{\tau}{q}t, & t \in [0,q],\\ \tau + \dfrac{1-2\tau}{1-2q}(t-q), & t \in [q, 1/2],\\ 1 - \varphi_S(1-t), & t \in [1/2, 1].\end{cases}$$

Typically, $q = 1/4$ is chosen in the literature. The intervals $[0,\tau]$ and $[1-\tau, 1]$ are split into $N/4$ subintervals of equal length, and $[\tau, 1-\tau]$ into $N/2$ subintervals.

The smoothed Shishkin mesh is a modification of the Shishkin mesh presented in [15]. It is generated by nodes $x_i = \varphi_{sS}(i/N)$, $i = 0, 1, \dots, N$, with the mesh generating function

$$\varphi_{sS}(t) := \begin{cases} \frac{\lambda}{q}t, & t \in [0,q],\\ \kappa(t) := p(t-q)^3 + \frac{\lambda}{q}t, & t \in [q, 1/2],\\ 1 - \varphi_{sS}(1-t), & t \in [1/2, 1],\end{cases}$$

where p is chosen such that $\varphi_{sS}\left(\frac{1}{2}\right) = \frac{1}{2}$, i.e. $p = \frac{1}{2}\big(1 - \frac{\lambda}{q}\big)\big(\frac{1}{2} - q\big)^{-3}$. On this mesh, uniform convergence of almost second order was established for a collocation method with quadratic splines in [10].

The main contribution of the present paper is uniform convergence of order two up to a logarithmic factor in the supremum norm. The results obtained in numerical examples for the Gartland-type mesh are better then those for the smoothed Shishkin mesh. The orders of convergence are better and the errors are actually smaller on the graded mesh than on the smoothed Shishkin mesh, see Section 5.

For the reaction-diffusion problem posed on a square, we expect that the a priori analysis can be extended to biquadratic C^1-splines using tensor-product meshes of Gartland-type in both space directions. The numerical results obtained on these graded meshes are better than the results obtained on tensor-product smoothed Shishkin meshes.

The outline of the paper is as follows. In Section 2, we introduced the recursively defined graded mesh. The following section contains error analysis for the one-dimensional discretisation and the main convergence result. The collocation method for 2D reaction-diffusion problems is introduced in Section 4. Finally, in Section 5, numerical experiments are presented that confirm our theoretical findings.

Notation: *Throughout the paper C will denote a generic positive constant that is independent of the perturbation parameter ε and of the number of mesh points N. For any set $I \subset [0,1]$ and any function u defined on I we set $\|w\|_{\infty,I} := \sup_{x\in I} |w(x)|$.*

2. The Gartland-type mesh

A recursively defined graded mesh was proposed by Gartland, [7]. It is an interesting alternative to the Shishkin mesh. Moreover, this type of layer-adapted meshes was used in [5] for the reaction-diffusion problem and in [4, 13] for convection-diffusion problems in one and two dimension. Paper [7] is related to a finite difference method, while in [5], a finite element method was considered. We shall use the mesh defined in [4], but adjusted to the reaction-diffusion problem (2).

Given a parameter $0 < h < 1$, we follow [4] and define a graded mesh by

$$\begin{cases} x_0 = 0, \\ x_i = ih\varepsilon, & \text{for } 1 \le i < \dfrac{1}{h} + 1, \\ x_i = x_{i-1} + hx_{i-1}, & \text{for } \dfrac{1}{h} + 1 \le i < M, \\ x_M = \dfrac{1}{2}, \\ x_i = 1 - x_{2M-i}, & \text{for } M+1 \le i \le 2M, \end{cases} \tag{4}$$

where M is that uniquely defined integer with

$$x_{M-1} < \tfrac{1}{2} \quad \text{and} \quad x_{M-1} + hx_{M-1} \ge \tfrac{1}{2}.$$

We modify the construction by imposing equality in the last inequality, i.e.

$$\tfrac{1}{2} = x_{M-1} + hx_{M-1}. \tag{5}$$

Then we have

$$|h_i - h_{i-1}| \le Ch^2, \quad \text{for all } i = 1, 2, \dots, N, \tag{6}$$

which is essential to our later convergence analysis. In practice, we fix M, compute the corresponding h and then construct the mesh according to (4). Some examples of these Gartland-type meshes are depicted in Figure 1.

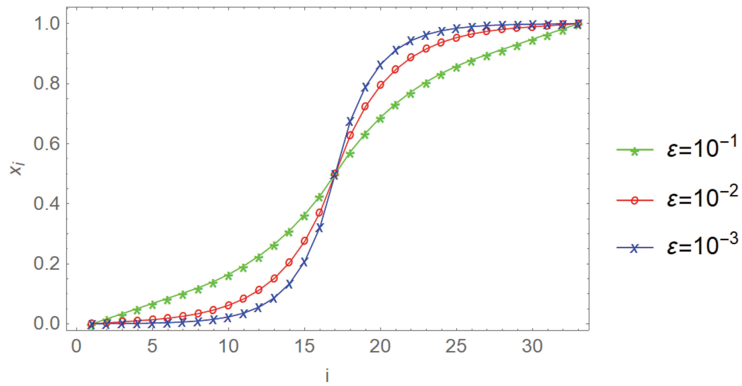

Figure 1: *The graded meshes with $N = 32$ mesh points and various ε*

We introduce a further notation. Let $N := 2M$ and $M_1 := \lceil \frac{1}{h} \rceil$. From (4) we have

$$x_i = x_{i-1} + \frac{h}{1+h}(1 - x_{i-1}), \quad \text{for} \;\; i = M+2, M+3, ..., N - M_1.$$

The mesh step sizes will be denoted by $h_i := x_i - x_{i-1}$, $i = 1, 2, \ldots, N$.

Remark 1. *The choice $x_1 = h\varepsilon$ is motivated by the necessity to have $h_1 = \mathcal{O}(\varepsilon)$ which is required to obtain uniform convergence, see [12, 13].*

For $i = M+1, M+2, ..., N - M_1 - 1$, we have

$$x_i = 1 - x_{N-i} = 1 - (x_{N-i-1} + hx_{N-i-1}) = x_{i+1} - h(1 - x_{i+1}),$$

because $M_1 + 1 \leq N - i \leq M - 1$. Then for $M + 1 \leq i \leq N - M_1 - 1$,

$$x_{i+1} = x_i + \frac{h}{1+h}(1 - x_i) \qquad \text{and} \qquad h_{i+1} = h(1 - x_{i+1}).$$

Now, we prove that for any integer $M \geq M_0 \in \mathbb{N}$, there exists an h such that (5) holds. To this end, let $M_1 \in \mathbb{N}\backslash\{1\}$. The number M_1 is chosen so that $M_1 h \geq 1$ and $(M_1 - 1)h < 1$, because of the definition of the function $\lceil \cdot \rceil$. Hence, for fixed M_1, we can choose any $h \in \left[\frac{1}{M_1}, \frac{1}{M_1 - 1}\right)$. From (4) we get

$$x_{M-1} = (1+h)^{M-M_1-1} x_{M_1} = (1+h)^{M-M_1-1} M_1 h\varepsilon,$$

and (5) yields

$$\tfrac{1}{2} = x_M = (1+h)^{M-M1} M_1 h\varepsilon.$$

Therefore

$$M = \frac{\ln \dfrac{1}{2M_1\varepsilon} + \ln \dfrac{1}{h}}{\ln(1+h)} + M_1.$$

In order to find h such that $M \in \mathbb{N}$, we consider the following function

$$\tilde{f}_{M_1}(h) = \frac{\ln \dfrac{1}{2M_1\varepsilon} + \ln \dfrac{1}{h}}{\ln(1+h)} + M_1, \qquad h \in \Phi := \left[\frac{1}{M_1}, \frac{1}{M_1 - 1}\right).$$

The first-order derivative of the function $\tilde{f}_{M_1}$ is

$$\tilde{f}'_{M_1}(h) = -\frac{\dfrac{\ln(1+h)}{h} + \dfrac{\ln(2hM_1\varepsilon)^{-1}}{1+h}}{\ln^2(1+h)} < -\frac{\dfrac{\ln(1+h)}{h} + \dfrac{\ln(4\varepsilon)^{-1}}{1+h}}{\ln^2(1+h)} < 0,$$

for $\varepsilon \le 1/4$. Obviously, $\tilde{f}_{M_1}$ is a continuous and decreasing function on Φ. Thus, $\tilde{f}_{M_1}$ takes all values in the interval $\left(\tilde{f}_{M_1}\left(\frac{1}{M_1-1}\right), \tilde{f}_{M_1}\left(\frac{1}{M_1}\right)\right]$.

Next, consider

$$\tilde{f}_{M_1+1}(h) = \frac{\ln \dfrac{1}{2(M_1+1)\varepsilon} + \ln \dfrac{1}{h}}{\ln(1+h)} + M_1 + 1, \qquad h \in \Phi.$$

Again, $\tilde{f}_{M_1+1}$ is continuous and decreasing on Φ. Hence, $\tilde{f}_{M_1+1}$ takes all values in the interval $\left(\tilde{f}_{M_1+1}\left(\frac{1}{M_1}\right), \tilde{f}_{M_1+1}\left(\frac{1}{M_1+1}\right)\right]$.

Furthermore,

$$\lim_{h \to 1/M_1 - 0} \tilde{f}_{M_1+1}(h) = \tilde{f}_{M_1}\left(\frac{1}{M_1}\right).$$

Hence, $\tilde{f}$ takes all values on the following set

$$S = \bigcup_{M_1=2}^{\infty} \left(\tilde{f}\left(\frac{1}{M_1 - 1}\right), \tilde{f}\left(\frac{1}{M_1}\right)\right] = (\tilde{f}(1), \infty),$$

where $\tilde{f}$ is defined by

$$\tilde{f} := \bigcup_{M_1=2}^{\infty} \tilde{f}_{M_1}.$$

Consequently, we can always choose $0 < h < 1$ such that condition (5) is satisfied, for all $M \ge \tilde{f}(1) = \tilde{f}_2(1) = \frac{-\ln(4\varepsilon)}{\ln 2} + 2$.

Table 1 presents different values of the perturbation parameter $\varepsilon \le 1/4$, those values of h for which condition (5) is satisfied.

M \ ε	10^{-2}	10^{-4}	10^{-6}	10^{-8}	10^{-10}
2^5	0.1631827	0.3404057	0.5443164	0.7791426	1.0535690
2^6	0.0791271	0.1589883	0.2448140	0.3370096	0.4357913
2^7	0.0389655	0.0768791	0.1161645	0.1568887	0.1991299
2^8	0.0193347	0.0378008	0.0566015	0.0757433	0.0952300
2^9	0.0096305	0.0187436	0.0279390	0.0372174	0.0465793
2^{10}	0.0048061	0.0093329	0.0138801	0.0184478	0.0230360
2^{11}	0.0024007	0.0046567	0.0069178	0.0091840	0.0114553
2^{12}	0.0011998	0.0023260	0.0034534	0.0045821	0.0057120
2^{13}	0.0005998	0.0011624	0.0017253	0.0022886	0.0028521
2^{14}	0.0002998	0.0005810	0.0008623	0.0011437	0.0014251

Table 1: *Values of h, in dependence of $M = N/2$ and ε, that generate mesh* (4)-(5).

Because of (5), the Gartland-type mesh (4) satisfies

$$h_i = \begin{cases} h\varepsilon & \text{for} \quad i \le M_1 \text{ and } i \ge N - M_1 + 1, \\ hx_{i-1} & \text{for} \quad M_1 < i \le M, \\ h(1-x_i) & \text{for} \quad M+1 \le i < N - M_1 + 1. \end{cases} \tag{7}$$

The following property is essential to the a priori analysis on the Gartland-type mesh.

Lemma 1. *Given mesh* (4)-(5)*, we have*

$$|h_i - h_{i-1}| = \begin{cases} 0 & \textit{for} \quad i \le M_1, \quad i = M+1 \textit{ and } \quad i > N - M_1 + 1, \\ hh_{i-1} & \textit{for} \quad M_1 + 1 < i \le M, \\ hh_i & \textit{for} \quad M+1 < i \le N - M_1. \end{cases}$$

Specifically, $|h_i - h_{i-1}| \le h^2\varepsilon$ *for* $i = M_1 + 1$ *and* $i = N - M_1 + 1$.

Proof. First, consider $i \le M + 1$. Obviously, the mesh sizes h_i $i \le M$, are nondecreasing. Consequently $h_i - h_{i-1} \ge 0$, for $i \le M$.

If $i \le M_1$, then $h_i - h_{i-1} = 0$.

If $i = M_1 + 1$, then

$$h_i - h_{i-1} = h\varepsilon\,(M_1 h - 1) < h\varepsilon\left(\left(\frac{1}{h} + 1\right)h - 1\right) = h^2\varepsilon.$$

For $M_1 + 1 < i \le M$ we obtain $h_i - h_{i-1} = hx_{i-1} - hx_{i-2} = hh_{i-1}$, and $i = M+1$ implies that $h_{M+1} - h_M = 0$.

The nodes in mesh (4)-(5) are symmetric with respect to the point $x_M = 1/2$ and therefore the proposition of the lemma holds for $i > M + 1$ as well. $\square$

The error bounds will be given in terms of the number N of mesh points. The following lemma establishes a relation between h and N.

Lemma 2. *Let $h < 1$. Then for the Gartland-type mesh*

$$h < 4N^{-1} \ln \frac{e^2}{2\varepsilon}.$$

Proof. Assuming $h < 1$, we have

$$\begin{aligned} M - M_1 - 2 &= \sum_{i=M_1+1}^{M-2} 1 = \sum_{i=M_1+1}^{M-2} \frac{1}{h_{i+1}} \int_{x_i}^{x_{i+1}} dx = \sum_{i=M_1+1}^{M-2} \frac{1}{hx_i} \int_{x_i}^{x_{i+1}} dx \\ &\leq \sum_{i=M_1+1}^{M-2} \frac{2}{hx_{i+1}} \int_{x_i}^{x_{i+1}} dx, \end{aligned}$$

because $x_{i+1} = x_i + hx_i \leq 2x_i$ for $i = M_1+1, M_1+2, \ldots, M-2$.
We know that $\dfrac{1}{x_{i+1}} \leq \dfrac{1}{x}$ for all $x \in [x_i, x_{i+1}]$ and then

$$M - M_1 - 2 \leq \sum_{i=M_1+1}^{M-2} \frac{2}{h} \int_{x_i}^{x_{i+1}} \frac{1}{x} dx \leq \frac{2}{h} \int_{\varepsilon}^{1/2} \frac{1}{x} dx \leq \frac{2}{h} \ln \frac{1}{2\varepsilon}.$$

Recalling $M = (M_1 + 1) + (M - M_1 - 2) + 1$, we obtain

$$\frac{N}{2} = M < \frac{1}{h} + 2 + \frac{2}{h} \ln \frac{1}{2\varepsilon} + 1.$$

Further, $1 < \dfrac{1}{h}$ implies

$$\frac{N}{2} < \frac{4}{h} + \frac{2}{h} \ln \frac{1}{2\varepsilon} = \frac{2}{h} \left(2 + \ln \frac{1}{2\varepsilon} \right),$$

i.e.

$$N < \frac{4}{h} \ln \frac{e^2}{2\varepsilon}.$$

Now, the proposition of the lemma follows. □

Remark 2. *Either the standard Shishkin or the Bakhvalov mesh do not satisfy* (6). *This inequality is violated where the meshes change from fine to coarse. However,* (6) *is essential to our a priori analysis (see the proof of Theorem 1) and we have been unable to prove convergence of order (almost) two for those meshes. This is the reason why the smoothed Shishkin mesh was used in [10]. The previous lemma implies that the Gartland-type mesh satisfies $|h_{i+1} - h_i| \leq CN^{-2} \ln^2 \varepsilon^{-1}$.*

3. Error analysis

Let Δ be an arbitrary partition of $[0, 1]$. For $m, l \in \mathbb{N}$ set

$$\mathcal{S}_m^l(\Delta) := \left\{ s \in C^m[0,1] : s|_{J_i} \in \Pi_l, \text{ for } i = 1, 2, \ldots, N \right\},$$

where Π_l is the space of polynomials of highest degree l. Then our discretization is:
Find $u_\Delta \in \mathcal{S}_2^1(\Delta)$ such that

$$u_{\Delta,0} = g_0, \quad \left(\mathcal{L}u_\Delta\right)_{i-1/2} = f_{i-1/2}, \quad i = 1, 2, \ldots, N, \quad u_{\Delta,N} = g_1. \tag{8}$$

3.1. Solution properties

The solution u of problem (2) has two boundary layers according to the following lemma, which we quote from [9].

Lemma 3. *Let $r, f \in C^4[0,1]$. Then*

$$\left|u^{(k)}(x)\right| \leq C\left\{1+\varepsilon^{-k}\mathrm{e}^{-\varrho x/\varepsilon}+\varepsilon^{-k}\mathrm{e}^{-\varrho(1-x)/\varepsilon}\right\}, \text{ for } \; x\in(0,1), \quad k=0,\dots,4.$$

Furthermore, the solution u can be decomposed as $u=v+w_0+w_1$. For $k=0,\dots,4$, the regular solution component v satisfies $\left\|v^{(k)}\right\|_\infty \leq C$, while for the layer parts w_0 and w_1 we have

$$\left|w_0^{(k)}(x)\right| \leq C\varepsilon^{-k}\mathrm{e}^{-\varrho x/\varepsilon}, \quad \left|w_1^{(k)}(x)\right| \leq C\varepsilon^{-k}\mathrm{e}^{-\varrho(1-x)/\varepsilon}, \quad x\in[0,1].$$

The following lemma gives additional properties for the layer components of the exact solution that shall be employed later.

Lemma 4. *The components w_0 and w_1 of the exact solution satisfy*

$$\varepsilon^l\left\|x^k\frac{d^{l+k}w_0}{dx^{l+k}}\right\|_{\infty,J_i} \leq C \quad \text{and} \quad \varepsilon^l\left\|(1-x)^k\frac{d^{l+k}w_1}{dx^{l+k}}\right\|_{\infty,J_i} \leq C,$$

for $l,k\in\mathbb{N}_0$ and $l+k\leq 4$, where $J_i=[x_{i-1},x_i]$.

Proof. Lemma 3 gives

$$\varepsilon^l\left\|x^k\frac{d^{l+k}w_0}{dx^{l+k}}\right\|_{\infty,J_i} \leq C\left\|\varepsilon^{-k}x^k e^{-\varrho x/\varepsilon}\right\|_{\infty,J_i}.$$

For $k=0$ the proposition of lemma follows easily.

Now, for $k>0$, set $s(x)=\varepsilon^{-k}x^k e^{-\varrho x/\varepsilon}$. We have to determine the maximum of this function. Differentiation yields

$$s'(x)=\varepsilon^{-k-1}x^{k-1}e^{-\varrho x/\varepsilon}(k\varepsilon-x\varrho),$$

and then the function s attains its maximum in the point $x=k\varepsilon/\varrho$. Thus,

$$\left\|\varepsilon^{-k}x^k e^{-\varrho x/\varepsilon}\right\|_{\infty,J_i} \leq s\left(\frac{k\varepsilon}{\varrho}\right) \leq C.$$

The first inequality in the proposition of the lemma follows. The bounds for w_1 and its derivatives follow by symmetry. $\square$

In the rest of the paper, we use the following bound

$$\tilde{s}_i(\varepsilon)=\varepsilon^{-i}e^{-\varrho/(2\varepsilon)} \leq \tilde{s}_i\left(\frac{\varrho}{2i}\right) \leq C \qquad \text{for } \; \varepsilon>0, \quad i=1,2,3,4. \tag{9}$$

3.2. $\mathcal{S}_2^0$-interpolation

For an arbitrary function $g \in C^0[0,1]$, we define the piecewise quadratic function $I_2^0 g \in \mathcal{S}_2^0(\Delta)$ with

$$\left(I_2^0 g\right)_i = g_i,\ i = 0,\dots,N, \quad \text{and} \quad \left(I_2^0 g\right)_{i-1/2} = g_{i-1/2},\ \ i = 1,\dots,N, \tag{10}$$

where $g_i = g(x_i)$ and $g_{i-1/2} = g(x_{i-1/2})$ for each i. Here $x_{i-1/2} = (x_{i-1} + x_i)/2$. We have the following interpolation error bounds, see [10].

Lemma 5. *Let $s \in \mathcal{S}_2^0(\Delta)$ with $s_{i-1/2} = 0,\ \ i = 1,2,\dots,N$. Then*

$$\|s\|_{\infty,J_i} \le \max\Big\{|s_{i-1}|\,,|s_i|\Big\}, \quad \|s''\|_{\infty,J_i} \le \frac{8}{h_i^2}\max\Big\{|s_{i-1}|\,,|s_i|\Big\},\ i = 1,\dots,N.$$

Theorem 1. *Assume $r, f \in C^4[0,1]$. Let u be the solution of (2). Then the interpolation error $I_2^0 u - u$ on the Gartland-type mesh (4)-(5) satisfies*

$$\left\|u - I_2^0 u\right\|_\infty \le Ch^3 \qquad \textit{and} \qquad \varepsilon^2 \max_{i=1,\dots,N}\left|\left(u - I_2^0 u\right)''_{i-1/2}\right| \le Ch^2.$$

Proof. The following two bounds are well-known from Lagrange interpolation and Taylor expansion:

$$\left\|u - I_2^0 u\right\|_{\infty,J_i} \le \frac{h_i^3}{72\sqrt{3}}\,\|u'''\|_{\infty,J_i} \quad \text{and} \quad \left|\left(u - I_2^0 u\right)''_{i-1/2}\right| \le \frac{h_i^2}{48}\left\|u^{(4)}\right\|_{\infty,J_i}. \tag{11}$$

Using the linearity of I_2^0, we split the interpolation error corresponding to the decomposition of the exact solution, cf. Lemma 3:

$$u - I_2^0 u = \left(v - I_2^0 v\right) + \left(w_0 - I_2^0 w_0\right) + \left(w_1 - I_2^0 w_1\right).$$

The regular component satisfies

$$\left\|v - I_2^0 v\right\|_{\infty,J_i} \le \frac{h_i^3}{72\sqrt{3}}\,\|v'''\|_{\infty,J_i} \le C\frac{h_i^3}{72\sqrt{3}} \le Ch^3,$$

while for the singular component w_0, we shall distinguish three cases.

(i) If $J_i \subset [0, x_{M_1}]$, then $h_i = h\varepsilon$ and

$$\left\|w_0 - I_2^0 w_0\right\|_{\infty,J_i} \le \frac{h_i^3}{72\sqrt{3}}\,\|w_0'''\|_{\infty,J_i} \le \frac{h^3\varepsilon^3}{72\sqrt{3}}C\varepsilon^{-3}\|e^{-\varrho x/\varepsilon}\|_{\infty,J_i} \le Ch^3.$$

(ii) Consider $J_i \subset [x_{M_1}, x_M]$. Then $h_i \le hx_{i-1} \le hx$ for $x \in [x_{i-1}, x_i]$ and

$$\left\|w_0 - I_2^0 w_0\right\|_{\infty,J_i} \le \frac{h^3 x_{i-1}^3}{72\sqrt{3}}\,\|w_0'''\|_{\infty,J_i} \le \frac{h^3}{72\sqrt{3}}\,\left\|x^3 w_0'''\right\|_{\infty,J_i} \le Ch^3, \quad \text{by Lemma 4.}$$

(iii) Let $J_i \subset [x_M, x_N]$. Then

$$\left\|w_0 - I_2^0 w_0\right\|_{\infty,J_i} \le \frac{h_i^3}{72\sqrt{3}}\varepsilon^{-3}\|e^{-\varrho x/\varepsilon}\|_{\infty,J_i} \le \frac{h_i^3}{72\sqrt{3}}\varepsilon^{-3}e^{-\varrho/(2\varepsilon)} \le Ch^3, \quad \text{by (9).}$$

Because of symmetry we have identical bounds for $\left\|w_1 - I_2^0 w_1\right\|_{\infty,J_i}$. Then, using a triangle inequality, we complete the proof for the first inequality of the theorem.

The same technique, combined with Lemma 4, gives the desired bound for $\varepsilon^2\left(u - I_2^0 u\right)''$. $\square$

3.3. $\mathcal{S}_2^1$-interpolation

For an arbitrary function, we define $g \in C^0[0,1]$, a piecewise quadratic function $I_2^1 g \in \mathcal{S}_2^1(\Delta)$ with

$$\left(I_2^1 g\right)_0 = g_0, \qquad \left(I_2^1 g\right)_{i-1/2} = g_{i-1/2}, \quad i = 1, \dots, N, \qquad \left(I_2^1 g\right)_N = g_N. \tag{12}$$

For any $s \in \mathcal{S}_2^1(\Delta)$ from [10], we have

$$[Fs]_i := a_i s_{i-1} + 3s_i + c_i s_{i+1} = 4a_i s_{i-1/2} + 4c_i s_{i+1/2}, \tag{13}$$

where $a_i := h_{i+1}/(h_i + h_{i+1})$ and $c_i := 1 - a_i = h_i/(h_i + h_{i+1})$, see [8]. For the operator F we have the following stability properties, which we quote from [10]:

Lemma 6. *For all vectors $s \in \mathbb{R}^{N+1}$ with $s_0 = s_N = 0$, there holds*

$$\max_{i=1,\dots,N-1} |s_i| \le \frac{1}{2} \max_{i=1,\dots,N-1} |[Fs]_i| .$$

Theorem 2. *Assume $r, f \in C^4[0,1]$. Let u be the solution of (2). Then the interpolation error $I_2^0 1 - u$ on the Gartland-type mesh (4)-(5) satisfies*

$$\max_{i=0,\dots,N} \left|\left(u - I_2^1 u\right)_i\right| \le Ch^4, \tag{14a}$$

$$\left\|u - I_2^1 u\right\|_\infty \le Ch^3, \tag{14b}$$

$$\varepsilon^2 \max_{i=1,\dots,N} \left|\left(u - I_2^1 u\right)''_{i-1/2}\right| \le Ch^2. \tag{14c}$$

Proof. **(a)** We start by deriving (14a). The interpolation error satisfies $\left(u - I_2^1 u\right)_0 = \left(u - I_2^1 u\right)_N = 0$ and

$$\left[F\left(u - I_2^1 u\right)\right]_i = a_i u_{i-1} - 4a_i u_{i-1/2} + 3u_i - 4c_i u_{i+1/2} + c_i u_{i+1} =: \tau_i,$$

for $i = 1, 2, \dots, N$. Taylor expansion gives

$$|\tau_i| \le \frac{1}{12} h_i h_{i+1} |h_{i+1} - h_i| \, |u_i'''| + \frac{5}{96} h_i^4 \left\|u^{(4)}\right\|_{\infty, J_i} + \frac{5}{96} h_{i+1}^4 \left\|u^{(4)}\right\|_{\infty, J_{i+1}}. \tag{15}$$

With the technique used in the proof of Theorem 1, we can bound the last two terms and obtain

$$\frac{5}{96} h_i^4 \left\|u^{(4)}\right\|_{\infty, J_i} \le Ch^4 \quad \text{and} \quad \frac{5}{96} h_{i+1}^4 \left\|u^{(4)}\right\|_{\infty, J_{i+1}} \le Ch^4, \tag{16}$$

for $i = 1, \dots, N-1$.

Bounding the first term in (15) is more tedious. Set

$$\psi_{g,i} := \frac{1}{12} h_i h_{i+1} |h_{i+1} - h_i| \, |g_i'''| \quad \text{for any} \;\; g \in C^3[0,1].$$

Clearly, we have

$$\psi_{u,i} \le \psi_{v,i} + \psi_{w_0,i} + \psi_{w_1,i} \, .$$

For the regular solution component v, Lemma 4 and (6) yield $\psi_{v,i} \le Ch^4$.

For the layer component w_0, the arguments split into six cases.

(i) Let $i < M_1$ or $i > N - M_1$. Then $\psi_{w_0,i} = 0$.

(ii) Let $i = M_1$. Then $h_{i+1} - h_i = h^2\varepsilon$ and $h_i \le h_{i+1} \le hx_{i+1}$. Thus

$$\psi_{w_0,i} \le Ch_i h_{i+1} h^2 \varepsilon |w_{0,i}'''| \le Ch^2\varepsilon \|x^2 w_0'''\|_{\infty,J_{i+1}} \le Ch^4,$$

by Lemma 4.

(iii) Consider $M_1 < i \le M-1$. Then $h_{i+1} = hx_i \le hx$ for $x \in J_{i+1}$ and consequently

$$\psi_{w_0,i} \le Ch_i h_{i+1} h h_{i+1} |w_{0,i}'''| \le Ch_{i+1}^3 h \|w_0'''\|_{\infty,J_{i+1}} \le Ch^4 \|x^3 w_0'''\|_{\infty,J_{i+1}} \le Ch^4,$$

by Lemma 4.

(iv) If $i = M$, then $\psi_{w_0,M} = 0$.

(v) In the case when $M + 1 \le i < N - M_1$, we have

$$\psi_{w_0,i} \le Ch_i h_{i+1} h h_{i+1} \varepsilon^{-3} e^{-\varrho/(2\varepsilon)} \le Ch^4 \varepsilon^{-3} e^{-\varrho/(2\varepsilon)} \le Ch^4, \quad \text{because of (9).}$$

(vi) Let $i = N - M_1$. Then $|h_{i+1} - h_i| \le h^2\varepsilon$ and $h_{i+1} = h\varepsilon$ and

$$\psi_{w_0,i} \le Ch_i h_{i+1} (h^2\varepsilon) \varepsilon^{-3} e^{-\varrho/(2\varepsilon)} \le Ch^4 \varepsilon^{-1} e^{-\varrho/(2\varepsilon)} \le Ch^4, \quad \text{by (9).}$$

Gathering all six cases, we have established that $\psi_{w_0,i} \le Ch^4$ for $i = 1, 2, \dots, N$. Clearly, we have an identical bound for $\psi_{w_1,i}$. Combining these with our earlier bound for $\psi_{v,i}$, we get

$$\psi_{u,i} \le Ch^4, \quad \text{for} \quad i = 1, 2, \dots, N.$$

This inequality together with (16) and (15) establishes

$$\left| \left[F\big(u - I_2^1 u\big) \right]_i \right| = |\tau_i| \le Ch^4, \quad \text{for} \quad i = 1, 2, \dots, N.$$

Finally, application of Lemma 6 completes the proof of (14a).

(b) The following short argument proves inequality (14b).

$$\begin{aligned} \left\| u - I_2^1 u \right\|_\infty &\le \left\| u - I_2^0 u \right\|_\infty + \left\| I_2^0 u - I_2^1 u \right\|_\infty \\ &= \left\| u - I_2^0 u \right\|_\infty + \max_{i=0,\dots,N} \left| \big(u - I_2^1 u\big)_i \right| \le Ch^3 + Ch^4 \le Ch^3, \end{aligned}$$

by Theorem 1 and (14a).

(c) Finally, we verify (14c). Again,

$$\varepsilon^2 \left| \big(u - I_2^1 u\big)_{i-1/2}'' \right| \le \varepsilon^2 \left| \big(u - I_2^0 u\big)_{i-1/2}'' \right| + \varepsilon^2 \left| \big(I_2^0 u - I_2^1 u\big)_{i-1/2}'' \right|, \tag{17}$$

and from Lemma 5, we obtain

$$\varepsilon^2 \left| \big(I_2^0 u - I_2^1 u\big)_{i-1/2}'' \right| \le \frac{8\varepsilon^2}{h_i^2} \max \left\{ \left| \big(u - I_2^1 u\big)_{i-1} \right|, \left| \big(u - I_2^1 u\big)_i \right| \right\}.$$

If $i \leq M_1$ or $i \geq N - M_1 + 1$, then $h_i = h\varepsilon$. Otherwise, $h_{N-i} = h_i = hx_i \geq h\varepsilon$ for $i = M_1 + 1, M_1 + 2, \ldots, M$. Consequently,

$$\varepsilon^2 \max_{i=1,2,\ldots,N} \left| \left(I_2^0 u - I_2^1 u\right)''_{i-1/2} \right| \leq \max_{i=1,2,\ldots,N} \frac{8\varepsilon^2}{h_i^2} \left| \left(u - I_2^1 u\right)_i \right| \leq Ch^2,$$

because of (14a). Combining this and (17), we obtain

$$\varepsilon^2 \max_{i=1,2,\ldots,N} \left| \left(u - I_2^0 u\right)''_{i-1/2} \right| \leq Ch^2,$$

and the proof is complete. $\square$

3.4. A priori error analysis

Let $\{B_{2,i}\}_{i=0}^{N+1}$ be the B-spline basis in $\mathcal{S}_2^1(\Delta)$. Then we may represent u_Δ as

$$u_\Delta(x) := \sum_{i=0}^{N+1} \alpha_i B_{2,i}(x),$$

where α_i are determined by the collocation, i.e. equation (8). That equation is equivalent to

$$\alpha_0 = \gamma_0, \quad [\boldsymbol{L\alpha}]_{i-1/2} = f_{i-1/2}, \quad i = 1, \ldots, N, \quad \alpha_{N+1} = \gamma_1, \tag{18}$$

with $\boldsymbol{\alpha} = (\alpha_0, \ldots, \alpha_{N+1})^T \in \mathbb{R}^{N+2}$ and

$$\begin{aligned} [\boldsymbol{L\alpha}]_{i-1/2} &:= -\varepsilon^2 \left[\frac{2(\alpha_{i+1} - \alpha_i)}{h_i(h_i + h_{i+1})} - \frac{2(\alpha_i - \alpha_{i-1})}{h_i(h_{i-1} + h_i)} \right] \\ &\quad + r_{i-1/2} \left[q_i^+ \alpha_{i+1} + \left(1 - q_i^+ - q_i^-\right) \alpha_i + q_i^- \alpha_{i-1} \right], \quad i = 1, \ldots, N, \\ q_i^+ &:= \frac{h_i}{4(h_i + h_{i+1})} \quad \text{and} \quad q_i^- := \frac{h_i}{4(h_i + h_{i-1})}, \end{aligned}$$

where we have formally set $h_0 = h_{N+1} = 0$, see [10].

The step sizes of the Gartland-type mesh satisfy $\max\{h_{i+1}, h_{i-1}\} \geq h_i$ for $i = 1, \ldots, N$. Therefore, from [10] we have the following stability result.

Theorem 3. *The operator $\boldsymbol{L}$ is maximum-norm stable on the mesh* (4)-(5) *with*

$$\|\boldsymbol{\gamma}\|_\infty \leq \frac{4}{\varrho^2} \|\boldsymbol{L\gamma}\|_\infty \quad \text{for all } \boldsymbol{\gamma} \in \mathbb{R}_0^{N+2},$$

where $\mathbb{R}_0^{N+2} = \left\{ v \in \mathbb{R}^{N+2} : v_0 = v_{N+1} = 0 \right\}$.

Theorem 4. *Let u be the solution of* (2), $0 < \varepsilon \leq \frac{1}{4}$ *and u_Δ its approximation by the collocation method* (8) *on the Gartland-type mesh* (4)-(5). *Then*

$$\|u - u_\Delta\|_\infty \leq Ch^2.$$

Proof. We start with the triangle inequality:

$$\|u - u_\Delta\|_\infty \le \left\|u - I_2^1 u\right\|_\infty + \left\|I_2^1 u - u_\Delta\right\|_\infty .$$

The interpolant $I_2^1 u$ can be represented by means of the quadratic B-spline basis $\{B_{2,k}\}$ as

$$I_2^1 u = \sum_{k=0}^{N+1} \beta_k B_{2,k} \quad \text{with some} \quad \boldsymbol{\beta} \in \mathbb{R}_0^{N+2}.$$

Using (10) and (18), we get

$$\left[\boldsymbol{L}(\boldsymbol{\alpha} - \boldsymbol{\beta})\right]_{i-1/2} = \mathcal{L}(u_\Delta - I_2^1 u)_{i-1/2} = \varepsilon^2 (I_2^1 u - u)''_{i-1/2}, \;\; i = 1, \dots, N.$$

Because $\boldsymbol{\alpha} - \boldsymbol{\beta} \in \mathbb{R}_0^{N+2}$, (14c) and Theorem 3 give $\|\boldsymbol{\alpha} - \boldsymbol{\beta}\|_\infty \le Ch^2$. Next, the stability of the B-spline basis implies

$$\left\|I_2^1 u - u_\Delta\right\|_\infty \le C \left\|\boldsymbol{\alpha} - \boldsymbol{\beta}\right\|_\infty \le Ch^2.$$

Using (14b), we complete the proof. □

A direct consequence of Lemma 2 and Theorem 4 is our main result.

Theorem 5. *Let u be the solution of* (2), $0 < \varepsilon \le \frac{1}{4}$ *and u_Δ its approximation by the collocation method* (8) *on the Gartland-type mesh* (4)-(5). *Then*

$$\|u - u_\Delta\|_\infty \le CN^{-2} \ln^2(1/\varepsilon).$$

Remark 3. *The error estimate in Theorem 5 appears to be quite sharp. Our numerical experiments in Section 5 (Tables 3 and 5) confirm a dependence on the perturbation parameter ε that is only marginally weaker than $\ln^2(1/\varepsilon)$.*

4. Collocation method for a 2D reaction-diffusion problem

In this section, we consider 2D reaction-diffusion problem of finding $u \in C(\bar{\Omega}) \cap C^2(\Omega)$ such that

$$\begin{gathered}\big(\mathcal{L}u\big)(x,y) := -\varepsilon^2 \triangle u(x,y) + r(x,y)u(x,y) = f(x,y) \;\text{ in }\; \Omega = (0,1)\times(0,1),\\ u(x,y) = g(x,y), \quad (x,y) \in \partial\Omega,\end{gathered} \tag{19}$$

where $\varepsilon \in (0,1]$, $\triangle = \partial^2/\partial x^2 + \partial^2/\partial y^2$ is the Laplace operator, $r, f \in C(\Omega)$ and $0 \le \beta < r$ on Ω with some positive constant β. Under these conditions, problem (19) has a unique solution, [11]. If ε is a small parameter, then our problem is singularly-perturbed and the solution exhibits sharp boundary layers of width $\mathcal{O}(\varepsilon \ln \varepsilon^{-1})$ along the boundary $\partial\Omega$, see [3].

We extend the collocation method (8) to 2D. Consider the rectangle $\bar{\Omega} = \Omega \cup \partial\Omega = [0,1]\times[0,1]$ and let

$$\begin{aligned}\Delta_x &: 0 = x_0 < x_1 < \ldots < x_N = 1,\\ \Delta_y &: 0 = y_0 < y_1 < \ldots < y_M = 1,\end{aligned}$$

be two partitions of the interval $[0,1]$. Then $\Delta = \Delta_x \times \Delta_y$ forms a rectangular tensor-product grid on $\bar{\Omega}$.

We look for a biquadratic C^1-spline u_Δ that satisfies differential equation (19) in the midpoints of the mesh rectangles, see [2] for some classical problems, i.e. problems without layers. A basis for the space of biquadratic C^1-splines on $\bar{\Omega}$ can be constructed by forming tensor products of the one-dimensional B-spline basis functions $\{B_{2,i}(x)\}_{i=0}^{N+1}$ and $\{\bar{B}_{2,j}(y)\}_{j=0}^{M+1}$.

Then, we seek an approximation u_Δ of the solution u of (19) as

$$u_\Delta(x,y) := \sum_{i=0}^{N+1} \sum_{j=0}^{M+1} \alpha_{i,j} B_{2,i}(x) \bar{B}_{2,j}(y),$$

where $\alpha_{i,j}$ are determined such that u_Δ satisfies differential equation (19) in the midpoints $(x_{i-1/2}, y_{j-1/2})$ of the partition, i.e.,

$$-\varepsilon^2 \triangle u_{\Delta,i-1/2,j-1/2} + r_{i-1/2,j-1/2} u_{\Delta,i-1/2,j-i/2} = f_{i-1/2,j-1/2}, \tag{20}$$

for $i = 1, \ldots, N$ and $j = 1, \ldots, M$, while the non-homogeneous boundary conditions are discretised by imposing

$$u_\Delta(x_i,0) = g(x_i,0), \quad u_\Delta(x_i,1) = g(x_i,1), \quad i = 0, \tfrac{1}{2}, 1, 1+\tfrac{1}{2}, \ldots, N,$$

and

$$u_\Delta(1,y_j) = g(1,y_j), \quad u_\Delta(1,y_j) = g(1,y_j), \quad j = 0, \tfrac{1}{2}, 1, 1+\tfrac{1}{2}, \ldots, M.$$

We expect that the a priori analysis from Section 3 can be extended to biquadratic C^1-splines on the tensor-product of the smoothed Shishkin meshes and on the tensor-product of our Gartland-type meshes. However, this is ongoing work and the subject of a forthcoming paper. Nonetheless, in Section 5, we presume numerical results for a 2D problem suggesting properties of the scheme extended directly to this case. Our tests confirm that our Gartland-type mesh outperforms the smoothed Shishkin mesh.

5. Numerical experiments

5.1. Numerical experiments for a one-dimensional problem

We verify the theoretical results of the preceding section by applying the collocation method to two different test problems.

The first problem is:

$$-\varepsilon^2 u''(x) + 4u(x) = \cos 12x, \quad x \in (0,1), \; u(0) = u(1) = 0. \tag{21}$$

Its exact solution can be found easily. We approximate the supremum-norm errors by

$$\|u - u_\Delta\|_\infty \approx \chi_{N,\varepsilon} := \max_{\substack{i=1,\ldots,N \\ m=0,\ldots,K}} \left| (u - u_\Delta)(x_{i-1} + mK^{-1}h_i) \right|.$$

In our experiments, we have chosen $K = 7$. The rates of convergence are computed using the following formula

$$p_{N,\varepsilon} = \frac{\ln \chi_{N,\varepsilon} - \ln \chi_{2N,\varepsilon}}{\ln 2}.$$

Also, we choose the parameters $\sigma = 4$ and $q = 1/4$ for the smoothed Shishkin mesh.

In Table 2, we compare numerical results obtained on the Gartland-type mesh and on the smoothed Shishkin mesh, for the different values of N, but for fixed ε. The second and third column contain the errors $\chi^{sS}_{N,\varepsilon}$ and convergence rates $p^{sS}_{N,\varepsilon}$ for the smoothed Shishkin mesh, while the fourth and fifth column contain the corresponding values $\chi^{G}_{N,\varepsilon}$ and $p^{G}_{N,\varepsilon}$ for the Gartland-type mesh. For increasing N the collocation method on the Gartland-type mesh gives much smaller errors than on the smoothed Shishkin mesh.

N	$\chi^{sS}_{N,\varepsilon}$	$p^{sS}_{N,\varepsilon}$	$\chi^{G}_{N,\varepsilon}$	$p^{G}_{N,\varepsilon}$
2^6	3.198e-03	1.93	2.827e-02	3.10
2^7	8.375e-04	1.69	3.307e-03	2.97
2^8	2.588e-04	1.73	4.234e-04	3.18
2^9	7.800e-05	1.74	4.659e-05	3.06
2^{10}	2.335e-05	1.75	5.570e-06	2.04
2^{11}	6.940e-06	1.76	1.351e-06	2.02
2^{12}	2.046e-06	1.78	3.326e-07	2.01
2^{13}	5.971e-07	1.79	8.252e-08	2.01
2^{14}	1.726e-07	1.80	2.055e-08	2.00
2^{15}	4.947e-08	1.81	5.128e-09	2.00
2^{16}	1.406e-08	—	1.285e-09	—

Table 2: *Supremum-norm errors of the collocation method on the smoothed Shishkin mesh and on the Gartland-type mesh for test problem* (21) *with* $\varepsilon = 10^{-6}$

ε	$\chi^{sS}_{N,\varepsilon}$	$\chi^{G}_{N,\varepsilon}$	$\tilde{q}_\varepsilon$	$\bar{q}_\varepsilon$
10^{-2}	4.929e-08	6.174e-10	2.166	2.250
10^{-3}	4.947e-08	1.337e-09	1.741	1.778
10^{-4}	4.947e-08	2.328e-09	1.543	1.563
10^{-5}	4.947e-08	3.592e-09	1.428	1.440
10^{-6}	4.947e-08	5.129e-09	1.353	1.361
10^{-7}	4.947e-08	6.939e-09	1.300	1.306
10^{-8}	4.947e-08	9.025e-09	1.261	1.266
10^{-9}	4.947e-08	1.138e-08	1.231	1.235
10^{-10}	4.947e-08	1.402e-08	—	—

Table 3: *Dependence of the supremum-norm errors on* ε *for test problem* (21) *with* $N = 2^{15}$. *The errors of the collocation method on the smoothed Shishkin mesh and on the Gartland-type mesh are shown in the second and third column, respectively.*

Table 3 shows the errors for the two meshes when the number N of mesh points is fixed and ε attains various values. We compute two quotients:

$$\tilde{q}_\varepsilon = \frac{\chi_{N,\varepsilon/10}}{\chi_{N,\varepsilon}} \quad \text{and} \quad \bar{q}_\varepsilon = \frac{\ln^2 10/\varepsilon}{\ln^2 1/\varepsilon},$$

for the Gartland-type mesh to check the correlation between the actual error and the logarithmic factor present in the error bound of Theorem 5. We see that the theoretical and numerical results do strongly correlate. We conclude that the error bound in Theorem 5 cannot be improved. Also, note that $\lim_{\varepsilon\to 0} \bar{q}_\varepsilon = 1$, which justifies the use of the Gartland-type mesh.

Theoretically, for sufficiently small values of ε, the smoothed Shishkin mesh will be a better choice than the Gartland-type graded mesh. However, in practice, this does not happen for reasonable values of ε.

The second test problem is:

$$-\varepsilon^2 u''(x) + (1+x^2)u(x) = e^x, \quad x \in (0,1), \; u(0) = u(1) = 0. \tag{22}$$

Its exact solution is not available. Therefore, we approximate the errors by comparison with the numerical solution u^* on a mesh obtained by uniformly bisecting the original mesh (”the double mesh principle”), i.e.

$$\|u - u_\Delta\|_\infty \approx \|u_\Delta - u^*\|_\infty \approx \chi_{N,\varepsilon} := \max_{\substack{i=1,\dots,N\\ m=0,\dots,K}} \left|(u_\Delta - u^*)\left(x_{i-1} + mK^{-1}h_i\right)\right|.$$

Again, we choose $K = 7$. Our numerical tests for this example reveal similar behaviour as for (21). The results are documented in Tables 4 and 5.

N	$\chi^{sS}_{N,\varepsilon}$	$p^{sS}_{N,\varepsilon}$	$\chi^{G}_{N,\varepsilon}$	$p^{G}_{N,\varepsilon}$
2^5	7.323e-02	1.52	4.315e-02	2.81
2^6	2.552e-02	1.66	6.157e-03	2.80
2^7	8.050e-03	1.81	8.821e-04	2.43
2^8	2.290e-03	1.78	1.635e-04	2.24
2^9	6.647e-04	1.76	3.472e-05	2.13
2^{10}	1.964e-04	1.77	7.917e-06	2.04
2^{11}	5.769e-05	1.77	1.920e-06	2.00
2^{12}	1.686e-05	1.78	4.795e-07	2.00
2^{13}	4.899e-06	1.79	1.198e-07	2.00
2^{14}	1.412e-06	1.81	2.995e-08	2.00
2^{15}	4.041e-07	—	7.487e-09	—

Table 4: *Supremum-norm errors of the collocation method for test problem* (22) *with* $\varepsilon = 10^{-6}$.

ε	$\chi_{N,\varepsilon}^{sS}$	$\chi_{N,\varepsilon}^{G}$	$\tilde{q}_\varepsilon$	$\bar{q}_\varepsilon$
10^{-2}	1.471e-07	9.054e-10	2.158	2.250
10^{-3}	4.045e-07	1.954e-09	1.740	1.778
10^{-4}	4.042e-07	3.400e-09	1.542	1.563
10^{-5}	4.041e-07	5.245e-09	1.428	1.440
10^{-6}	4.041e-07	7.487e-09	1.353	1.361
10^{-7}	4.041e-07	1.013e-08	1.300	1.306
10^{-8}	4.041e-07	1.317e-08	1.261	1.266
10^{-9}	4.041e-07	1.660e-08	1.231	1.235
10^{-10}	4.041e-07	2.044e-08	—	—

Table 5: *Dependence of the supremum-norm errors on ε for test problem* (22) *with $N = 2^{15}$. The errors of the collocation method on the smoothed Shishkin mesh and on the Gartland-type mesh are shown in the second and third column, respectively.*

5.2. Numerical experiments for two-dimensional problem

Finally, we test the collocation method when applied to the following 2D problem:

$$-\varepsilon^2 \triangle u(x,y) + (1 + x^2 y^2 e^{xy}) u(x,y) = f(x,y) \quad \text{in} \quad \Omega = (0,1) \times (0,1), \tag{23}$$

where the source term f and the boundary conditions are such that

$$u(x,y) = \sin{(x+y)}\frac{\pi}{2} + (x+y)\left(e^{-x/\varepsilon} + e^{-(1-x)/\varepsilon} + e^{-2y/\varepsilon} + e^{-2(1-y)/\varepsilon}\right)$$

is the solution of (23). We approximate the supremum-norm errors by

$$\|u - u_\Delta\|_\infty \approx \chi_N = \max_{\substack{i,j=1,\dots,N \\ m,n=0,\dots,k}} \left|(u - u_\Delta)\left(x_{i-1} + mk^{-1}h_i, y_{j-1} + nk^{-1}k_j\right)\right|.$$

In our experiments, we have chosen $k = 7$. Table 6 contains the errors of the collocation method on the smoothed Shishkin mesh and on our Gartland-type meshes for a fixed value of ε. In addition, Table 7 illustrates the dependence of the errors on the perturbation parameter ε when N is fixed. We observe that our Gartland-type mesh outperforms the smoothed Shishkin mesh for two-dimensional reaction-diffusion problems, too.

N	$\chi_{N,\varepsilon}^{sS}$	$p_{N,\varepsilon}^{sS}$	$\chi_{N,\varepsilon}^{G}$	$p_{N,\varepsilon}^{G}$
2^5	2.374e-01	1.21	1.551e-01	2.41
2^6	1.029e-01	1.56	2.920e-02	2.69
2^7	3.498e-02	1.86	4.528e-03	2.32
2^8	9.640e-03	1.70	9.027e-04	2.14
2^9	2.971e-03	1.74	2.048e-04	2.08
2^{10}	8.908e-04	—	4.854e-05	—

Table 6: *Supremum-norm errors of the collocation method on the smoothed Shishkin mesh and on the Gartland-type mesh for test problem* (23) *with $\varepsilon = 10^{-6}$*

ε	10^{-2}	10^{-4}	10^{-6}	10^{-8}
smoothed Shishkin mesh	3.006e-03	2.971e-03	2.971e-03	2.971e-03
Gartland-type mesh	2.338e-05	9.005e-05	2.048e-04	3.708e-04

Table 7: *Dependence of the supremum-norm errors on ε for 2D test problem* (23) *with* $N = 2^9$

6. Conclusion

Using the Gartland-type mesh (4)-(5) we have obtained optimal error bounds in the supremum norm for the collocation method (8) using quadratic C^1-splines. The established rate of convergence is two. The factor $\ln^2(1/\varepsilon)$ is not significant in practice. For reasonably small values of ε, the results in the present paper are better than those obtained for the smoothed Shishkin mesh in [10],

Special meshes (smoothed Shishkin and Gartland-type) have been constructed for the collocation method, because we were unable to prove uniform convergence of this method for standard Shishkin and Bakhvalov meshes.

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