A priori estimates for finite-energy sequences of Müller's functional with non-coercive two-well potential with symmetrically placed wells

Andrija Raguž*

Department of Mathematics and Statistics, Zagreb School of Economics and Management, Jordanovac 110, HR-10 000 Zagreb

Abstract. In this paper we obtain a priori estimates for finite-energy sequences of Müller's functional

$$I_a^\varepsilon(v) = \int_0^1 \left(\varepsilon^2 v''^2(s) + W(v'(s)) + a(s)v^2(s) \right) ds,$$

where $v \in \mathrm{H}^2(0,1)$ and W is non-coercive two-well potential with symmetrically placed zero-points. We also prove Γ-convergence of corresponding relaxed functionals according to the approach of G. Alberti and S. Müller as $\varepsilon \longrightarrow 0$ for W, which satisfies $\int_{-\infty}^0 \sqrt{W} = \int_0^{+\infty} \sqrt{W} = +\infty$.

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1. Introduction

We study asymptotic behavior of the functional $I_a^\varepsilon : \mathrm{H}^2(0,1) \longrightarrow \mathbf{R}$,

$$I_a^\varepsilon(v) = \int_0^1 \left(\varepsilon^2 v''^2(s) + W(v'(s)) + a(s)v^2(s) \right) ds, \tag{1}$$

as a small parameter ε tends to zero, where $a \in \mathrm{L}^1(0,1)$ satisfies $a(s) \geq \alpha_0 > 0$ (a.e. $s \in (0,1)$). W is a non-negative continuous function with a suitable behavior at infinity such that $W(\zeta) = 0$ if and only if $\zeta \in \{-1,1\}$ holds true (in short, the two-well potential with symmetrically placed wells). In this paper we present some asymptotic properties of finite-energy sequences for the rescaled functional (1) (cf. [2], Section 3), derived from the corresponding properties of W (we recall that we say that (v_ε) is a finite-energy sequence (or an FE sequence) for $(\varepsilon^{-\frac{2}{3}} I_a^\varepsilon)$ if it holds that $\limsup_{\varepsilon \longrightarrow 0} \varepsilon^{-\frac{2}{3}} I_a^\varepsilon(v_\varepsilon) < +\infty$). A particular emphasis is placed on the optimality of the assumptions on W. Our results should be primarily viewed as a further development of considerations in [2]. The proofs are obtained by an application of results in Leoni's paper [15], which considers the well-known Modica-Mortola functional (i.e., the case when it holds that $a = 0$). Hence, on the one

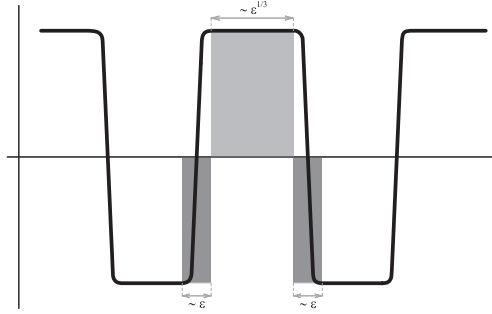*Corresponding author. *Email address:* `araguz@zsem.hr` (A. Raguž)

A. Raguž

Figure 1: *Two-scale structure of $s \mapsto v'_\varepsilon(s)$ for $\varepsilon \approx 0$*

hand, this paper is a follow-up of Leoni's paper [15], and on the other, it provides technical refinements of results sketched in Section 6 of [2]. In [2], G. Alberti and S. Müller calculated the (rescaled) asymptotic energy associated with I_a^ε as $\varepsilon \longrightarrow 0$ by means of Γ-convergence on the space of Young measures on micro-patterns (cf. [2], Section 3). Such a class of functionals appears in studying coherent solid-solid phase transformations and can be understood as a simplified one-dimensional model for a phase transition at a martensite-austenite interface (cf. [2], [20] and references therein). From the mathematical viewpoint, the minimizers of (1) exhibit rather interesting behavior as $\varepsilon \longrightarrow 0$: the derivatives of minimizers v_ε develop a two-scale structure (cf. Figure 1) due to an internally generated oscillatory scale of order $\varepsilon^{\frac{1}{3}}$, while the minimizers (v_ε) converge to zero strongly in $\mathrm{L}^2(0,1)$. It is reasonable to expect that FE sequences (v_ε) for $(\varepsilon^{-\frac{2}{3}} I_a^\varepsilon)$ are in a sense close to equi-Lipschitz FE sequences, since any deviation from the slope ± 1 comes at the cost of increasing the value of $I_a^\varepsilon(v_\varepsilon)$. Such behavior is a mathematical counterpart of the formation of a *microstructure* in complex physical systems such as diblock copolymer melts (cf. [9]). The approach of Alberti and Müller introduced in [2] uses the theorem of L. Modica and S. Mortola (cf. [18]) as a background Γ-convergence result. The main benefit of their approach is the fact that we are able to deal with the problem of calculation of rescaled asymptotic energies $\mathcal{E}_{a,per} := \lim_\varepsilon \min\{\varepsilon^{-\frac{2}{3}} I_a^\varepsilon(v) : v \in \mathrm{H}^2_{per}(0,1)\}$ and $\mathcal{E}_a := \lim_\varepsilon \min\{\varepsilon^{-\frac{2}{3}} I_a^\varepsilon(v) : v \in \mathrm{H}^2(0,1)\}$, as well as to describe the geometric behavior of minimizers for I_a^ε as $\varepsilon \longrightarrow 0$. For further progress in this respect, see [25]-[37]. Extensive literature is available on a wider subject, and our list of references is by no means complete, nor does it attempt to cite the most important contributions (a more complete list is available in, for instance, [2], [6], [21]). Most authors impose a growth condition on W at infinity which, in our notation (cf. (2)), corresponds to the case $q < 0$ (cf. [2, 3, 10, 14, 24, 38]). In particular, the classical Fonseca-Tartar assumption (cf. [13]), which corresponds to the case $q = -1$ (in such a case we say that W is a coercive two-well potential), should not play a decisive role in minimizing I_a^ε for small ε (coercive two-well potentials are usually introduced to simplify mathematical analysis of the model considered). In this paper, we show that this is indeed the case. We always assume that W satisfies: (i) $W(\zeta) = 0$ iff $\zeta \in \{\pm 1\}$, $W \geq 0$; (ii) Leoni's non-integrability condition $\int_0^{+\infty} \min\{\sqrt{W(\zeta)}, \sqrt{W(-\zeta)}\}d\zeta = +\infty$ (cf. [15]), whereby we allow the case $\liminf_{\zeta \longrightarrow \pm\infty} W(\zeta) = 0$. This already gives a substantially large class of non-coercive two-well potentials. We mostly have

in mind the following decay condition:

$$W(\zeta) \geq c_0 |\zeta|^{-q} \text{ for every } \zeta \text{ such that } |\zeta| \geq R_0, \tag{2}$$

where $R_0 > 1$ is large enough, $c_0 > 0$, $q \in [0, 2]$, and where the specific choice of q depends on the particular section. Our objective is to show that, in accordance with the procedure in [2], it is possible to recover the Γ-limit of the rescaled functionals $(\varepsilon^{-\frac{2}{3}} I_a^{\varepsilon})$ for such, quite general, two-well potentials W, and to extract the underlying geometric properties of FE sequences. The key contribution of the paper is the proof of the fact that Ball's condition in the statement of the fundamental theorem of Young measures (cf. [21], Theorem 3.1, or [4]) relating to first derivative of an FE sequence is still preserved under fairly weak assumption (3), which enables a sawtooth pattern of FE sequences to emerge. Furthermore, we prove that the expression for the rescaled asymptotic energy is independent of boundary conditions. Since we recover the results on the level of FE sequences (which are not necessarily actual or approximate minimizers), in contrast to the analysis in [20], we use purely variational arguments.

2. Notation

In what follows we use the notation ε to denote strictly positive small real numbers. To simplify the notation, we omit relabeling subsequences, and we set "a sequence (x_ε)" to mean a sequence in some metric space X defined only for (arbitrarily chosen) countably many $\varepsilon = \varepsilon_n$ such that $\varepsilon_n \longrightarrow 0$ as $n \longrightarrow +\infty$. A subsequence of (x_ε) is any sequence $(x_{\varepsilon_{n_k}})$, where ε_{n_k} is a subsequence of (ε_n) (so that $\varepsilon_{n_k} \longrightarrow 0$ as $k \longrightarrow +\infty$), and we say that (x_ε) is pre-compact in X if every subsequence of (x_ε) admits a further subsequence that converges in X. However, our results (with the exception of the proof of (29)) hold if the parameter $\varepsilon \in (0, 1)$ is allowed to take uncountably many values. If X is given metric space, we say that functionals $f^\varepsilon : X \longrightarrow \mathbf{R}$ are equi-coercive on X if $\limsup_{\varepsilon \longrightarrow 0} f^\varepsilon(x_\varepsilon) < +\infty$ implies that there exists a subsequence of (x_ε) which is convergent in X. We also mention that throughout the paper, without further comment, we use well-known argument of "the unique feature of the cluster point" pertaining to such sequences (x_ε), which, in effect, establishes the following: if an arbitrary subsequence allows further subsequence which has a certain property defined beforehand, then the whole sequence has such property. In this paper, measurability always means Borel measurability. We consider a compact metric space (K, d) (the space of patterns), which is a set of all measurable mappings $x : \mathbf{R} \longrightarrow [-\infty, +\infty]$ (modulo equivalence λ-almost everywhere, where λ is a one-dimensional Lebesgue measure), endowed with the metric d defined by

$$d(x_1, x_2) := \sum_{k=1}^{\infty} \frac{1}{2^k \alpha_k} | \int_{\mathbf{R}} y_k \left(\frac{2}{\pi} \arctan x_1 - \frac{2}{\pi} \arctan x_2 \right) d\lambda |,$$

where (y_k) is a sequence of bounded functions which are dense in $\mathrm{L}^1(\mathbf{R})$, such that the support of y_k is a subset of $(-k, k)$, with $\alpha_k := \|y_k\|_{\mathrm{L}^1} + \|y_k\|_{\mathrm{L}^\infty}$. The Banach space $\mathrm{C}(K)$ ($\mathrm{C}_0(\mathbf{R})$, resp.) is the space of all continuous real functions on K (the

space of all continuous real functions on $\mathbf{R}$ which vanish at infinity, resp.), endowed with the uniform norm. The dual of $\mathrm{C}(K)$ ($\mathrm{C}_0(\mathbf{R})$, resp.) is identified with the space of all real finite Radon measures on K (on $\mathbf{R}$, resp.), denoted by $\mathcal{M}(K)$ ($\mathcal{M}_b(\mathbf{R})$, resp.), endowed with the corresponding weak-star topology. The weak-star topology on $\mathcal{M}(K)$ is induced by the norm ϕ defined in [2], p. 799. By $\mathcal{P}(K)$ we denote the set of all probability measures in $\mathcal{M}(K)$. If $\mu \in \mathcal{M}(K)$, by $\|\mu\|$ we denote total variation of μ. If $\Omega \subseteq \mathbf{R}$ is a bounded measurable set, by $\mathrm{L}_{w*}^\infty(\Omega; \mathcal{M}(K))$ we denote the dual of $\mathrm{L}^1(\Omega; \mathrm{C}(K))$. The set of all K-valued Young measures (or *Young measures on micro-patterns*) denoted by $YM(\Omega; K)$ is the set of all $\boldsymbol{\nu} \in \mathrm{L}_{w*}^\infty(\Omega; \mathcal{M}(K))$ such that $\nu_s \in \mathcal{P}(K)$ for almost every $s \in \Omega$, where $\boldsymbol{\nu}(s) := \nu_s$, $s \in \Omega$. We always endow it with the weak-star topology of $\mathrm{L}_{w*}^\infty(\Omega; \mathcal{M}(K))$. The weak-star topology on bounded sets in $\mathrm{L}_{w*}^\infty(\Omega; \mathcal{M}(K))$ is induced by the norm Φ defined in [2], p. 769, and therefore $YM(\Omega; K)$ is metrized by Φ. The *elementary Young measure* associated with a measurable map $u : \Omega \longrightarrow K$ ($u : \Omega \longrightarrow \mathbf{R}$, resp.) is the map $\underline{\delta}_u : \Omega \longrightarrow \mathcal{M}(K)$ ($\underline{\delta}_u : \Omega \longrightarrow \mathcal{M}_b(\mathbf{R})$, resp.) given by $\underline{\delta}_u(s) := \delta_{u(s)}$, $s \in \Omega$. We say that a sequence of measurable maps $u^k : \Omega \longrightarrow K$ *generates the Young measure* $\boldsymbol{\nu}$, if the corresponding elementary Young measures $\underline{\delta}_{u^k}$ converge to $\boldsymbol{\nu}$ in the topology of $\mathrm{L}_{w*}^\infty(\Omega; \mathcal{M}(K))$. The basic result about $\mathbf{R}$-valued Young measures (and its proof) can be found in [4] (in accordance with the notation therein, for a given closed set $A \subseteq \mathbf{R}$ and a sequence of measurable functions $f_n : (0,1) \longrightarrow \mathbf{R}$, we write $f_n \xrightarrow{\lambda} A$ if for every open neighbourhood U of A it holds that $\lim_{n \longrightarrow +\infty} \lambda\{s \in (0,1) : f_n(s) \notin U\} = 0$). The version which is instrumental in setting up the machinery of Alberti and Müller can be found in [2], p. 770, and [21]. We say that $\mu \in \mathcal{M}(K)$ is *invariant with respect to translations* if for every $g \in \mathrm{C}(K)$ and every $\tau \in \mathbf{R}$ it holds that $\langle \mu, g \rangle = \langle \mu, g \circ T_\tau \rangle$, where $T_\tau : K \longrightarrow K$ is defined by $T_\tau x(t) := x(t - \tau)$, $x \in K$, $t \in \mathbf{R}$. $\mathcal{I}(K)$ denotes the class of all invariant measures in $\mathcal{P}(K)$. If $x \in K$ is periodic, the notation ϵ_x stands for the unique invariant probability measure supported on the orbit of x. If a Young measure $\boldsymbol{\nu} \in YM(\Omega; K)$ at almost every point s is equal to an elementary invariant measure (which is allowed to depend on s), we say that $\boldsymbol{\nu}$ is an elementary invariant Young measure (in such a case we write $\boldsymbol{\nu} = \mathcal{E}_x$, where $\mathcal{E}_x(s) := \epsilon_{x^s}$, $x^s \in K$ is periodic (a.e. $s \in \Omega$)). By $\mathcal{S}(\omega)$ we denote the set of all continuous piecewise affine functions on a bounded open interval $\omega \subseteq \mathbf{R}$ with a slope equal to 1 or -1 at almost every point in ω. By $\mathcal{S}_{per}(\omega)$ we denote the set of all real functions on a bounded open interval $\omega \subseteq \mathbf{R}$ extended to $\mathbf{R}$ by periodicity, which belong to $\mathcal{S}(J)$ for every bounded open interval $J \subseteq \mathbf{R}$. If $x \in \mathcal{S}(\omega)$, by $Sx' \cap \omega$ we denote the set of all jump discontinuities of x' on ω. $\mathrm{card}A$ stands for cardinality of the set A. Finally, if $x \in \mathcal{S}_{per}(a, b)$ satisfies $x(a) = x(b) = 0$, we write $x \in \mathcal{S}_{per,0}(a, b)$.

3. Outline

Our goal is to develop estimates which show how far an arbitrary FE sequence is from an equi-Lipschitz FE sequence in the case of W which satisfies a suitable non-integrability condition

$$\int_0^{+\infty} \sqrt{V(\xi)}d\xi = +\infty, \tag{3}$$

where $V : [0, +\infty) \longrightarrow [0, +\infty)$ is defined by $V(\xi) := \min\{W(z) : |z| = \xi\}$. All of our a priori estimates provide information regarding both the periodicity properties and the emergence of the transition layers (depicted in Figure 1) of arbitrary FE sequences. If a two-well potential W is coercive, these estimates are much easier to prove (and can be further improved) since in such a case we immediately deduce boundedness of (v'_ε) in $L^1(0, 1)$ (compare Corollary 8). We expect that results presented in this paper (with the exception of Section 4) do not hold provided $\int_{\mathbf{R}} \sqrt{W(\zeta)}d\zeta < +\infty$. As a typical example of W which satisfies (3) $(\int_{\mathbf{R}} \sqrt{W(\zeta)}d\zeta < +\infty$, resp.), we consider W such that for $0 \leq q \leq 2$ ($q > 2$, resp.) and $R_0 > 1$ it holds that

$$W(\zeta) \geq \frac{c_0}{|\zeta|^q} \quad \text{for every} \quad |\zeta| > R_0, \tag{4}$$

$$W(\zeta) \leq \frac{C_0}{|\zeta|^q} \quad \text{for every} \quad |\zeta| > R_0, \tag{5}$$

where $0 < c_0 \leq C_0 < +\infty$. If W satisfies both (4) and (5) with $q \in (0, 2]$ ($q > 2$, resp.), we say that W exhibits slow (fast, resp.) decay at infinity. If W satisfies both (4) and (5) with $q = 0$, we say that W exhibits steady behavior at infinity. In that context, the standard approach to asymptotic problem associated with (1) uses the so-called Zhang's Lemma (cf. [22]) which provides direct approximation of an arbitrary FE sequence by equi-Lipschitz FE sequence. However, if $q > 0$, such an approach is not easily applicable (or it is not applicable at all), and in this paper we adopt a different strategy, which is based on the area formula (compare [15]). Roughly speaking, by a convexity argument, we expect that FE sequences for the functional (1) with $a > 0$ for sufficiently small $\varepsilon > 0$ behave as FE sequences for the functional with $a = 0$ (the so-called Cahn-Hilliard functional) on "most" subintervals I_j^ε of order $O(\varepsilon^{\frac{1}{3}})$, where $j = 1, \ldots, O(\varepsilon^{-\frac{1}{3}})$. That is to say, we use the truncation argument in the domain rather than in the co-domain. We make this rigorous in Lemma 7. We also recover L^∞-estimates for $(|v'_\varepsilon|)$ in the case $a > 0$, which are the counterpart of Leoni's L^∞-estimates in the case $a = 0$ (cf. [15]). For a specific $q \geq 0$ we recover growth rates of v'_ε as $\varepsilon \longrightarrow 0$ explicitly in terms of q. The issue which complicates the analysis in the case of W which satisfies (3) is the fact that (v'_ε) is not necessarily bounded in $L^1(0, 1)$ (compare [5]). The paper is organized as follows. First, we develop some a priori estimates for FE sequences. In Section 4, we present some results relating to the choice of an arbitrary W, without specifying any asymptotic behavior of W at infinity (we do not even assume (3)). In Section 5 (Section 6, resp.), we consider the case of W which satisfies (2), where $0 < q \leq 2$ ($q = 0$, resp.) is specified (we are not aware of any corresponding results in the case $q > 2$). Second, in Section 7, we present some applications of the a priori estimates obtained beforehand: more precisely, we prove Γ-convergence of relaxed functionals associated with (1) on the space of micro-patterns $YM((0, 1); K)$ as $\varepsilon \longrightarrow 0$ under assumption (3). In particular, results in this paper show that the distinct sawtooth pattern of FE sequences (v_ε) emerges more explicitly as we progressively introduce ever stronger assumptions on W. We point out that not all of a priori estimates included in Section 4, Section 5 and Section 6 are used in the proof of our main result (Theorem 1), but we believe that all estimates

are of interest themselves. For instance, estimate (iv) ((iii), resp.) of Proposition 2 (Proposition 3, resp.) is complementary to estimate (10) of Proposition 1 in the case $0 < q < 1$ ($q = 0$, resp.). Throughout the paper we use the following notation. We set $N_\varepsilon := \varepsilon_\star^{-\frac{1}{3}} \in \mathbf{N}$, $\varepsilon_\star := \lceil \varepsilon^{-\frac{1}{3}} \rceil^{-3}$, $I_j^\varepsilon := (\frac{j-1}{N_\varepsilon}, \frac{j}{N_\varepsilon})$, $j = 1, \ldots, N_\varepsilon$, $\rho_{\varepsilon,\star} := (\varepsilon_\star \varepsilon^{-1})^{\frac{2}{3}}$, so that $0 < \rho_{\varepsilon,\star} \leq 1$, $\varepsilon \geq \varepsilon_\star > 0$ and $\rho_{\varepsilon,\star} \nearrow 1$ as $\varepsilon \longrightarrow 0$. Also, by (v_ε) we always denote an arbitrary FE sequence for $(\varepsilon^{-\frac{2}{3}} I_a^\varepsilon)$ in $\mathrm{H}^2(0,1)$, while by $M > 0$ we denote a chosen upper bound for $\varepsilon^{-\frac{2}{3}} I_a^\varepsilon(v_\varepsilon)$ provided that $\varepsilon > 0$ is sufficiently small.

4. The case of arbitrary W

We begin with some observations for an arbitrary two-well potential W.

Lemma 1. *For an arbitrary W we have:*

$$\limsup_{\varepsilon \longrightarrow 0} \sum_{j=1}^{N_\varepsilon} \frac{1}{N_\varepsilon} \inf_{I_j^\varepsilon} |v_\varepsilon'| < +\infty.$$

Proof. We set $a_\varepsilon^j := (j-1)\varepsilon_\star^{\frac{1}{3}}$, $b_\varepsilon^j := j\varepsilon_\star^{\frac{1}{3}}$, $j = 1, \ldots, N_\varepsilon$. By the integral mean value theorem there exists $\theta_{\varepsilon,j}^{(1)} \in (a_\varepsilon^j, a_\varepsilon^j + \frac{1}{4}\varepsilon_\star^{\frac{1}{3}})$ ($\theta_{\varepsilon,j}^{(2)} \in (a_\varepsilon^j + \frac{3}{4}\varepsilon_\star^{\frac{1}{3}}, b_\varepsilon^j)$, resp.) such that the following holds: $\int_{a_\varepsilon^j}^{a_\varepsilon^j + \frac{1}{4}\varepsilon_\star^{\frac{1}{3}}} \varepsilon^{-\frac{1}{3}} |v_\varepsilon(s)| ds = \varepsilon^{-\frac{1}{3}} |v_\varepsilon(\theta_{\varepsilon,j}^{(1)})| \frac{1}{4}\varepsilon_\star^{\frac{1}{3}} = \frac{1}{4}\rho_{\varepsilon,\star}^{\frac{1}{2}} |v_\varepsilon(\theta_{\varepsilon,j}^{(1)})|$ ($\int_{a_\varepsilon^j + \frac{3}{4}\varepsilon_\star^{\frac{1}{3}}}^{b_\varepsilon^j} \varepsilon^{-\frac{1}{3}} |v_\varepsilon(s)| ds = \varepsilon^{-\frac{1}{3}} |v_\varepsilon(\theta_{\varepsilon,j}^{(2)})| \frac{1}{4}\varepsilon_\star^{\frac{1}{3}} = \frac{1}{4}\rho_{\varepsilon,\star}^{\frac{1}{2}} |v_\varepsilon(\theta_{\varepsilon,j}^{(2)})|$, resp.). Since (v_ε) is an FE sequence, by the Hölder inequality we have $\|\varepsilon^{-\frac{1}{3}} v_\varepsilon\|_{\mathrm{L}^1(0,1)} \leq \alpha_0^{-\frac{1}{2}} M^{\frac{1}{2}}$, and it results in $\alpha_0^{-\frac{1}{2}} M^{\frac{1}{2}} \geq \sum_{j=1}^{N_\varepsilon} \int_{a_\varepsilon^j}^{b_\varepsilon^j} \varepsilon^{-\frac{1}{3}} |v_\varepsilon(s)| ds \geq \sum_{j=1}^{N_\varepsilon} \left(\frac{1}{8}|v_\varepsilon(\theta_{\varepsilon,j}^{(2)})| + \frac{1}{8}|v_\varepsilon(\theta_{\varepsilon,j}^{(1)})| \right)$. On the other hand, the Lagrange mean value theorem provides the existence of $\theta_{\varepsilon,j} \in (\theta_{\varepsilon,j}^{(1)}, \theta_{\varepsilon,j}^{(2)})$ with the property $|v_\varepsilon(\theta_{\varepsilon,j}^{(2)}) - v_\varepsilon(\theta_{\varepsilon,j}^{(1)})| = |v_\varepsilon'(\theta_{\varepsilon,j})| \cdot |\theta_{\varepsilon,j}^{(2)} - \theta_{\varepsilon,j}^{(1)}| \geq |v_\varepsilon'(\theta_{\varepsilon,j})| \frac{1}{2}\varepsilon_\star^{\frac{1}{3}}$. Thus, we get $8\alpha_0^{-\frac{1}{2}} M^{\frac{1}{2}} \geq \sum_{j=1}^{N_\varepsilon} \left(|v_\varepsilon(\theta_{\varepsilon,j}^{(2)})| + |v_\varepsilon(\theta_{\varepsilon,j}^{(1)})| \right) \geq \sum_{j=1}^{N_\varepsilon} \frac{1}{N_\varepsilon} \frac{1}{2} |v_\varepsilon'(\theta_{\varepsilon,j})|$, giving $\sum_{j=1}^{N_\varepsilon} \frac{1}{N_\varepsilon} \inf_{I_j^\varepsilon} |v_\varepsilon'| \leq 16\alpha_0^{-\frac{1}{2}} M^{\frac{1}{2}}$ (provided that $\varepsilon > 0$ is sufficiently small). $\square$

In the next lemma we show that for sufficiently wide intervals ω_ε, $\inf_{\omega_\varepsilon} |v_\varepsilon'|$ is small.

Lemma 2. *Consider an arbitrary W. Then the following holds:*

(i) *For every sequence of open intervals (ω_ε) in $(0,1)$ $\lim_\varepsilon \varepsilon^{-\frac{1}{3}} \lambda(\omega_\varepsilon) = +\infty$ implies $\lim_\varepsilon \inf_{\omega_\varepsilon} |v_\varepsilon'| = 0$,*

(ii) $\limsup_{\varepsilon \longrightarrow 0} \|\varepsilon^{\frac{2}{3}} v_\varepsilon'\|_{\mathrm{L}^\infty(0,1)} < +\infty$.

Proof. To establish (i), we consider an arbitrary subsequence of (v_ε) (not relabeled). Since the sequence $(\varepsilon^{-\frac{1}{3}} v_\varepsilon)$ is bounded in $\mathrm{L}^1(0,1)$, up to a subsequence, we have $\rho_\varepsilon^{-1} v_\varepsilon(s) \longrightarrow 0$ (a.e. $s \in (0,1)$), where $\rho_\varepsilon := \lambda(\omega_\varepsilon)$. Then $\rho_\varepsilon^{-1} v_\varepsilon(s) \chi_{\omega_\varepsilon^0}(s) \longrightarrow 0$ for every $s \in (0,1)$, where $\omega_\varepsilon^0 \subseteq \omega_\varepsilon$ is a conveniently chosen measurable set such that

$\lambda(\omega_\varepsilon \backslash \omega_\varepsilon^0) = 0$. Consider $s_1, s_2 \in \omega_\varepsilon^0$ such that $|s_2 - s_1| \geq \frac{1}{2}\rho_\varepsilon$. By the Lagrange mean value theorem we have $v_\varepsilon(s_2) - v_\varepsilon(s_1) = v_\varepsilon'(\theta_\varepsilon)(s_2 - s_1)$, where $\theta_\varepsilon \in (s_1, s_2)$. It gives $|v_\varepsilon(s_2) - v_\varepsilon(s_1)| \geq \frac{1}{2}\rho_\varepsilon|v_\varepsilon'(\theta_\varepsilon)|$, so that $\rho_\varepsilon^{-1}|v_\varepsilon(s_2) - v_\varepsilon(s_1)| \geq \frac{1}{2}|v_\varepsilon'(\theta_\varepsilon)|$, i.e., $\lim_{\varepsilon \longrightarrow 0} v_\varepsilon'(\theta_\varepsilon) = 0$. The assertion for the whole sequence follows from the argument of the unique feature of the cluster point as we let $\varepsilon \longrightarrow 0$. Next, we apply the area formula for absolutely continuous functions (cf. [17], Theorem 3.65), getting $M^{\frac{1}{2}} \geq \varepsilon^{\frac{2}{3}} \int_0^1 |v_\varepsilon''|ds \geq \varepsilon^{\frac{2}{3}}(M_\varepsilon - m_\varepsilon)$, which gives $\varepsilon^{\frac{2}{3}} M_\varepsilon \leq M^{\frac{1}{2}} + \varepsilon^{\frac{2}{3}} m_\varepsilon$, where $m_\varepsilon := \min_{[0,1]} |v_\varepsilon'|$, $M_\varepsilon := \max_{[0,1]} |v_\varepsilon'|$. By (i) we obtain that the sequence $(\varepsilon^{\frac{2}{3}} m_\varepsilon)$ is bounded, and so is $(\varepsilon^{\frac{2}{3}} M_\varepsilon)$. □

Lemma 3. *Consider an arbitrary W and an arbitrary sequence of measurable sets (A_ε) in $(0,1)$. Then the following conclusions hold:*

(i) $\liminf_{\varepsilon \longrightarrow 0} \varepsilon^{-\frac{2}{3}}\lambda(A_\varepsilon) = +\infty$ *implies* $1 \leq \liminf_{\varepsilon \longrightarrow 0} \|v_\varepsilon'\|_{L^\infty(A_\varepsilon)}$. *In particular, it holds that* $1 \leq \liminf_{\varepsilon \longrightarrow 0} \|v_\varepsilon'\|_{L^\infty(0,1)}$,

(ii) $1 \leq \liminf_{\varepsilon \longrightarrow 0} \sum_{j=1}^{N_\varepsilon} \frac{1}{N_\varepsilon}\|v_\varepsilon'\|_{L^\infty(I_j^\varepsilon)}$.

Proof. We first prove (i) by contradiction. If there exists $0 < \delta << 1$ such that $\limsup_{\varepsilon \longrightarrow 0} \|v_\varepsilon'\|_{L^\infty(A_\varepsilon)} \leq 1 - 2\delta$, we have $-1 + \delta \leq v_\varepsilon'(s) \leq 1 - \delta$ (a.e. $s \in A_\varepsilon$) for every $0 < \varepsilon < \varepsilon_0(\delta)$. Hence, $\varepsilon^{-\frac{2}{3}} \int_0^1 W(v_\varepsilon'(s))ds \geq \varepsilon^{-\frac{2}{3}}\lambda(A_\varepsilon) \min_{-1+\delta \leq \zeta \leq 1-\delta} W(\zeta)$, which provides the conclusion as we pass to the limit as $\varepsilon \longrightarrow 0$. Second, to prove (ii), we consider $j_0^\varepsilon \in \{1, \ldots, N_\varepsilon\}$ such that $\|v_\varepsilon'\|_{L^\infty(I_j^\varepsilon)} \geq \|v_\varepsilon'\|_{L^\infty(I_{j_0^\varepsilon}^\varepsilon)}$ for every $j \in \{1, \ldots, N_\varepsilon\}$, and so $\liminf_{\varepsilon \longrightarrow 0} \sum_{j=1}^{N_\varepsilon} \frac{1}{N_\varepsilon}\|v_\varepsilon'\|_{L^\infty(I_j^\varepsilon)} \geq \liminf_{\varepsilon \longrightarrow 0} \|v_\varepsilon'\|_{L^\infty(I_{j_0^\varepsilon}^\varepsilon)} \geq 1$. □

The last lemma of this section establishes an estimate which is the reverse of estimate (ii) of Lemma 3.

Lemma 4. *Consider an arbitrary W. Then for every $0 < m \leq 1$ it holds that*

$$\limsup_{\varepsilon \longrightarrow 0} \sum_{j=1}^{N_\varepsilon} \frac{1}{N_\varepsilon}\phi_m(\|v_\varepsilon'\|_{L^\infty(I_j^\varepsilon)}) \leq C < +\infty, \tag{6}$$

where $\phi_m(t) := \int_0^t \sqrt{V_m(\xi)}d\xi$, $t \geq 0$, $V_m(\xi) := \min\{V(\xi), m\}$, and where $C > 0$ is independent of $m \in (0,1]$. If $m > 1$, we can recover (6) with C replaced by $C \cdot \sqrt{m}$.

Proof. For a sufficiently small $\varepsilon_0 > 0$ and for every $\varepsilon \in (0, \varepsilon_0)$ we have $M \geq \varepsilon^{-\frac{2}{3}} I_a^\varepsilon(v_\varepsilon) \geq \rho_{\varepsilon,\star}\varepsilon_\star^{-\frac{2}{3}} I_a^{\varepsilon\star}(v_\varepsilon)$. Since $\rho_{\varepsilon,\star} \nearrow 1$ as $\varepsilon \longrightarrow 0$, for a sufficiently small $\varepsilon_1 > 0$ and for every $\varepsilon \in (0, \varepsilon_1)$ we get $\varepsilon_\star^{-\frac{2}{3}} I_a^{\varepsilon\star}(v_\varepsilon) \leq 2M$. On the other hand, by the same argument as in Theorem 1.3 in [15], we can write

$$\int_{I_j^\varepsilon} \left(\varepsilon_\star^{\frac{4}{3}} v_\varepsilon''^2 + \varepsilon_\star^{-\frac{2}{3}} V_m(|v_\varepsilon'|)\right) \geq \varepsilon_\star^{\frac{1}{3}} \int_{I_j^\varepsilon} \sqrt{V_m(|v_\varepsilon'|)}|v_\varepsilon''| \geq \varepsilon_\star^{\frac{1}{3}} \int_{m_{\varepsilon,j}}^{M_{\varepsilon,j}} \sqrt{V_m}, \tag{7}$$

where $m_{\varepsilon,j} := \inf\{|v'_\varepsilon(s)| : s \in I^\varepsilon_j\}$, $M_{\varepsilon,j} := \sup\{|v'_\varepsilon(s)| : s \in I^\varepsilon_j\}$ and so

$$\sum_{j=1}^{N_\varepsilon} \frac{1}{N_\varepsilon} \phi_m(M_{\varepsilon,j}) \leq 2M + \sum_{j=1}^{N_\varepsilon} \frac{1}{N_\varepsilon} \phi_m(m_{\varepsilon,j}), \tag{8}$$

where $\phi_m(t) := \int_0^t \sqrt{V_m(\xi)} d\xi$, $t \geq 0$. Since for every $m \in (0,1]$ and every $\xi \in [0,+\infty)$ it holds that $\sqrt{V_m(\xi)} \leq \sqrt{m} \leq 1$, we get $\phi_m(t) \leq \sqrt{m}t \leq t$. By Lemma 1 the sum on the right-hand side in (8) is bounded by $16\alpha_0^{-\frac{1}{2}} M^{\frac{1}{2}}$ as $\varepsilon \longrightarrow 0$ and we get (6) with $C := 16\alpha_0^{-\frac{1}{2}} M^{\frac{1}{2}} + 2M$.   □

5. The case of slow decay of W at $\pm\infty$

In this section, we address the case of W which satisfies (4). We recover precise estimates in terms of the parameter $q \in [0,1)$. In the results below we utilized only estimate (4), but typical examples we have in mind are functions W which satisfy both (4) and (5). In this more specific case we can obtain additional information about the asymptotic behavior of FE sequences. In the case of $q \in [1,2]$, Lemma 2, (ii), provides more precise estimates in comparison to the following proposition.

Proposition 1. *Consider W which satisfies (4). If $0 \leq q < 1$, then it holds that*

$$\limsup_{\varepsilon \longrightarrow 0} \|\varepsilon^{\frac{1}{3}\frac{2}{2-q}} v'_\varepsilon\|_{\mathrm{L}^\infty(0,1)} < +\infty. \tag{9}$$

If (A_ε) are measurable sets in $(0,1)$ such that $\liminf_{\varepsilon \longrightarrow 0} \operatorname{ess\,inf}_{A_\varepsilon} |v'_\varepsilon| = +\infty$, then for $0 \leq q < 1$ the following holds:

$$\limsup_{\varepsilon \longrightarrow 0} \varepsilon^{\frac{1}{3}\frac{2}{2-q}} \operatorname{ess\,inf}_{A_\varepsilon} |v'_\varepsilon| < +\infty. \tag{10}$$

Proof. To prove (9), we note that without loss of generality we can assume that $\lim_{\varepsilon \longrightarrow 0} \|v'_\varepsilon\|_{\mathrm{L}^\infty(0,1)} = +\infty$ (otherwise the assertion is obvious), so that for $\varepsilon \in (0,\varepsilon_0)$ it holds that $\|v'_\varepsilon\|_{\mathrm{L}^\infty(0,1)} > R_0 > 1$. Similarly to Theorem 1.3 in [15] we get $M \geq \varepsilon^{\frac{1}{3}} \int_{m_\varepsilon}^{M_\varepsilon} \sqrt{V(\xi)} d\xi$, where $m_\varepsilon := \min_{[0,1]} |v'_\varepsilon|$, $M_\varepsilon := \max_{[0,1]} |v'_\varepsilon|$. By Lemma 2, (i), it follows that $\lim_{\varepsilon \longrightarrow 0} m_\varepsilon = 0$. Then we have

$$M \geq \varepsilon^{\frac{1}{3}} \int_{R_0}^{\|v'_\varepsilon\|_{\mathrm{L}^\infty(0,1)}} \sqrt{V(\xi)} d\xi \geq \varepsilon^{\frac{1}{3}} \sqrt{c_0}(\|v'_\varepsilon\|_{\mathrm{L}^\infty(0,1)}^{1-\frac{q}{2}} - R_0^{1-\frac{q}{2}}),$$

whereby (9) (and, consequently, (10)) follows.   □

In the case of $0 < q < 1$ we can derive further conclusions.

Lemma 5. *Consider arbitrary $\delta > 0$. Suppose that W satisfies (4) with $0 < q < 1$. Then it holds that*

$$\limsup_{\varepsilon \longrightarrow 0} \varepsilon^{-\frac{2}{3}+\frac{1}{3}\frac{2q}{2-q}} \lambda\{s \in (0,1) : |v_\varepsilon'(s)| > 1+\delta\} < +\infty, \tag{11}$$

$$|v_\varepsilon'| \xrightarrow{\lambda} 1 \ on \ (0,1) \ as \ \varepsilon \longrightarrow 0, \tag{12}$$

$$\limsup_{\varepsilon \longrightarrow 0} \int_0^1 |v_\varepsilon'(s)|^{2-2q} ds < +\infty, \tag{13}$$

$$\limsup_{\varepsilon \longrightarrow 0} \varepsilon^{-\frac{2}{3}+\frac{2}{3}\frac{q+p}{2-q}} \int_{\{|v_\varepsilon'|>1+\delta\}} |v_\varepsilon'(s)|^p ds < +\infty, \ where \ p \in (0, 2-2q). \tag{14}$$

If $0 < q < \frac{1}{2}$, then for every $p \in [1, 2-2q)$ we have

$$|v_\varepsilon'| \xrightarrow{\mathrm{L}^p(0,1)} 1, \tag{15}$$

$$\lim_{\varepsilon \longrightarrow 0} \|v_\varepsilon\|_{\mathrm{L}^\infty(0,1)} = 0. \tag{16}$$

If $q = \frac{1}{2}$, then we have

$$\limsup_{\varepsilon \longrightarrow 0} \|v_\varepsilon\|_{\mathrm{L}^\infty(0,1)} < +\infty. \tag{17}$$

Proof. If $R_0 > 1$ is chosen as in (4), it follows that there exists $M > 0$ such that

$$\varepsilon^{-\frac{2}{3}} \lambda\{|v_\varepsilon'| > R_0\} \|v_\varepsilon'\|_{\mathrm{L}^\infty(0,1)}^{-q} c_0 \le \varepsilon^{-\frac{2}{3}} \int_{\{|v_\varepsilon'|>R_0\}} \frac{c_0}{|v_\varepsilon'(s)|^q} ds \le M,$$

which, in turn, by (9), provides $\limsup_{\varepsilon \longrightarrow 0} \varepsilon^{-\frac{2}{3}} \varepsilon^{\frac{1}{3}\frac{2q}{2-q}} \lambda\{|v_\varepsilon'| > R_0\} < +\infty$. On the other hand, for arbitrary $\delta > 0$ such that $R_0 > 1 + \delta$ we have

$$M \ge \varepsilon^{-\frac{2}{3}} \int_{\{1+\delta<|v_\varepsilon'|<R_0\}} W(v_\varepsilon') \ge \varepsilon^{-\frac{2}{3}} \lambda\{1 + \delta < |v_\varepsilon'| < R_0\} \min_{1+\delta\le|\zeta|\le R_0} W(\zeta),$$

and so (11) holds true. To proceed, we note that the inequality

$$M \ge \varepsilon^{-\frac{2}{3}} \int_{\{|v_\varepsilon'|\le 1-\delta\}} W(v_\varepsilon') \ge \varepsilon^{-\frac{2}{3}} \lambda\{|v_\varepsilon'| \le 1-\delta\} \min_{|\zeta|\le 1-\delta} W(\zeta), \tag{18}$$

coupled with (11), provides (12). Next, by (9) and (11), for $0 < q < 1$ we estimate

$$\int_{\{|v_\varepsilon'|>1+\delta\}} |v_\varepsilon'(s)|^{2-2q} ds \le C_1 \varepsilon^{-\frac{2}{3}\frac{1}{2-q}(2-2q)} \lambda\{|v_\varepsilon'| > 1+\delta\} \le C_2.$$

In effect, (13) holds. (14) also follows directly from (9) and (11). By (12), as we pass to an appropriate subsequence (not relabeled), we get $|v_\varepsilon'(s)| \longrightarrow 1$ (a.e. $s \in (0,1)$) as $\varepsilon \longrightarrow 0$. If $0 < q < \frac{1}{2}$ and $p \in [1, 2-2q)$, we argue as follows. By the Hölder inequality we estimate

$$\int_0^1 ||v_\varepsilon'| - 1|^p \le \int_{\{|v_\varepsilon'|\le 1+\delta\}} ||v_\varepsilon'| - 1|^p + \||v_\varepsilon'| - 1|^p\|_{\mathrm{L}^{\frac{2-2q}{p}}(0,1)} \lambda\{|v_\varepsilon'| > 1+\delta\}^{\frac{2-2q-p}{2-2q}}.$$

Finally, we obtain (15) by combining the dominated convergence theorem, (11) and (13). Consequently, (16) ((17), resp.) follows from $v_\varepsilon \longrightarrow 0$ in $\mathrm{L}^2(0,1)$ and from the Rellich compactness theorem (continuous embedding $\mathrm{W}^{1,1}(0,1) \hookrightarrow \mathrm{L}^\infty(0,1)$, resp.). $\qquad\square$

In the next proposition, provided that $0 < q < 1$, we estimate the sizes of sets where v'_ε exhibits singular behavior in terms of the small parameter ε.

Proposition 2. *Suppose that W satisfies (4) with $0 < q < 1$ and $R_0 > 1$. Consider an arbitrary sequence of measurable sets (A_ε) in $(0,1)$. Then the following holds:*

(i) $\liminf_\varepsilon \varepsilon^{-\frac{2}{3}\frac{2-2q}{2-q}} \lambda(A_\varepsilon) = +\infty$ *implies* $\limsup_\varepsilon \operatorname{ess\,inf}_{A_\varepsilon} |v'_\varepsilon| < +\infty$,

(ii) $+\infty \geq \liminf_\varepsilon \operatorname{ess\,inf}_{A_\varepsilon} |v'_\varepsilon| \geq R_0$ *implies* $\limsup_\varepsilon \varepsilon^{-\frac{2}{3}\frac{2-2q}{2-q}} \lambda(A_\varepsilon) < +\infty$,

(iii) *Under assumption* $+\infty > \limsup_\varepsilon \operatorname{ess\,sup}_{A_\varepsilon} |v'_\varepsilon| \geq \liminf_\varepsilon \operatorname{ess\,inf}_{A_\varepsilon} |v'_\varepsilon| \geq R_0$ *it follows that* $\limsup_\varepsilon \varepsilon^{-\frac{2}{3}} \lambda(A_\varepsilon) < +\infty$.

Also, if $\liminf_{\varepsilon \longrightarrow 0} \operatorname{ess\,inf}_{A_\varepsilon} |v'_\varepsilon| = +\infty$, *then it holds that*

(iv) $\limsup_{\varepsilon \longrightarrow 0} \lambda(A_\varepsilon)(\operatorname{ess\,inf}_{A_\varepsilon} |v'_\varepsilon|)^{2-2q} < +\infty$.

Proof. To prove (ii) ((iii), resp.), we note that, by assumption, for sufficiently small ε it holds that $\operatorname{ess\,inf}_{A_\varepsilon} |v'_\varepsilon| \geq R_0$, and so

$$M \geq \varepsilon^{-\frac{2}{3}} \int_{A_\varepsilon} W(v'_\varepsilon(s)) ds \geq \varepsilon^{-\frac{2}{3}} \int_{A_\varepsilon} \frac{c_0}{|v'_\varepsilon(s)|^q} ds \geq \varepsilon^{-\frac{2}{3}} \int_{A_\varepsilon} \frac{c_0}{\|v'_\varepsilon\|^q_{\mathrm{L}^\infty(A_\varepsilon)}} ds.$$

Hence, $\lambda(A_\varepsilon) \leq M c_0^{-1} \varepsilon^{\frac{2}{3}} \|v'_\varepsilon\|^q_{\mathrm{L}^\infty(A_\varepsilon)}$ ($M \geq \varepsilon^{-\frac{2}{3}} c_0 \lambda(A_\varepsilon) \|v'_\varepsilon\|^{-q}_{\mathrm{L}^\infty(A_\varepsilon)}$, resp.). On the other hand, from (9) we have $\|\varepsilon^{\frac{1}{3}\frac{2}{2-q}} v'_\varepsilon\|_{\mathrm{L}^\infty(0,1)} \leq C_1$ (by assumption in (iii) it holds that $0 < C_2 \leq \|v'_\varepsilon\|^{-q}_{\mathrm{L}^\infty(A_\varepsilon)} \leq C_3 < +\infty$, resp.), so that $\lambda(A_\varepsilon) \leq C_4 \varepsilon^{\frac{2}{3}(1-\frac{q}{2-q})}$, which gives (ii) ((iii), resp.). Assertion (i) follows from a contraposition of (ii). Finally, to verify (iv), we note that from (13) it follows that $C_5 \geq \int_{A_\varepsilon} |v'_\varepsilon(s)|^{2-2q} ds \geq \lambda(A_\varepsilon)(\operatorname{ess\,inf}_{A_\varepsilon} |v'_\varepsilon|)^{2-2q}$. $\qquad\square$

Corollary 1. *If W satisfies (4) with $0 < q < 2$, then the following holds:*

$$\limsup_{\varepsilon \longrightarrow 0} \sum_{j=1}^{N_\varepsilon} \frac{1}{N_\varepsilon} \inf_{I_j^\varepsilon} |v'_\varepsilon|^{1-\frac{q}{2}} < +\infty, \tag{19}$$

$$\limsup_{\varepsilon \longrightarrow 0} \sum_{j=1}^{N_\varepsilon} \frac{1}{N_\varepsilon} \|v'_\varepsilon\|^{1-\frac{q}{2}}_{\mathrm{L}^\infty(I_j^\varepsilon)} < +\infty. \tag{20}$$

Proof. We set $B_\varepsilon^{R_0} := \{j = 1, \ldots, N_\varepsilon : m_{\varepsilon,j} > R_0\}$, $G_\varepsilon^{R_0} := \{j = 1, \ldots, N_\varepsilon : m_{\varepsilon,j} \leq R_0\}$, $\tilde{B}_\varepsilon^{R_0} := \{j = 1, \ldots, N_\varepsilon : M_{\varepsilon,j} > R_0\}$, $\tilde{G}_\varepsilon^{R_0} := \{j = 1, \ldots, N_\varepsilon : M_{\varepsilon,j} \leq R_0\}$, where $m_{\varepsilon,j}$ and $M_{\varepsilon,j}$ are defined as in Lemma 4. Then we have $\sum_{j \in G_\varepsilon^{R_0}} \frac{1}{N_\varepsilon} \inf_{I_j^\varepsilon} |v'_\varepsilon|^{1-\frac{q}{2}} \leq R_0^{1-\frac{q}{2}}$. Since it holds that $R_0 > 1$, by Lemma 1, it follows

that $\sum_{j \in B_\varepsilon^{R_0}} \frac{1}{N_\varepsilon} \inf_{I_j^\varepsilon} |v_\varepsilon'|^{1-\frac{q}{2}} \leq \sum_{j=1}^{N_\varepsilon} \frac{1}{N_\varepsilon} \inf_{I_j^\varepsilon} |v_\varepsilon'| \leq 16\alpha_0^{-\frac{1}{2}} M^{\frac{1}{2}}$. Hence, we get (19). To prove (20), we note that, similarly to the proof of Lemma 4, by (4) we have

$$2M \geq \sum_{j \in \tilde{B}_\varepsilon^{R_0} \cap G_\varepsilon^{R_0}} \frac{1}{N_\varepsilon} \int_{m_{\varepsilon,j}}^{M_{\varepsilon,j}} \sqrt{V} + \sum_{j \in \tilde{B}_\varepsilon^{R_0} \cap B_\varepsilon^{R_0}} \frac{1}{N_\varepsilon} \int_{m_{\varepsilon,j}}^{M_{\varepsilon,j}} \sqrt{V}$$

$$\geq \sum_{j \in \tilde{B}_\varepsilon^{R_0}} \frac{1}{N_\varepsilon} \sqrt{c_0} M_{\varepsilon,j}^{1-\frac{q}{2}} - \sum_{j \in \tilde{B}_\varepsilon^{R_0} \cap G_\varepsilon^{R_0}} \frac{1}{N_\varepsilon} \sqrt{c_0} R_0^{1-\frac{q}{2}} - \sum_{j \in \tilde{B}_\varepsilon^{R_0} \cap B_\varepsilon^{R_0}} \frac{1}{N_\varepsilon} \sqrt{c_0} m_{\varepsilon,j}^{1-\frac{q}{2}}.$$

By Lemma 1 we deduce $\sum_{j \in \tilde{B}_\varepsilon^{R_0}} \frac{1}{N_\varepsilon} M_{\varepsilon,j}^{1-\frac{q}{2}} \leq 2c_0^{-\frac{1}{2}} M + 16\alpha_0^{-\frac{1}{2}} M^{\frac{1}{2}} + R_0^{1-\frac{q}{2}}$. On the other hand, we have $\sum_{j \in \tilde{G}_\varepsilon^{R_0}} \frac{1}{N_\varepsilon} M_{\varepsilon,j}^{1-\frac{q}{2}} \leq R_0^{1-\frac{q}{2}}$, getting (20). $\qquad\square$

We immediately get an improvement of Proposition 1:

Corollary 2. *If W satisfies (4) with $0 < q < 2$ ($q = 2$, resp.), then the following holds:* $\limsup_\varepsilon \sum_{j=1}^{N_\varepsilon} \frac{1}{N_\varepsilon} \|\varepsilon^{\frac{1}{3}\frac{q}{2-q}} v_\varepsilon'\|_{\mathrm{L}^\infty(I_j^\varepsilon)} < +\infty$ ($\limsup_\varepsilon |\sum_{j=1}^{N_\varepsilon} \frac{1}{N_\varepsilon} \ln\|v_\varepsilon'\|_{\mathrm{L}^\infty(I_j^\varepsilon)}| < +\infty$, *resp.*).

Proof. By rewriting (20), we get $\limsup_\varepsilon \sum_{j=1}^{N_\varepsilon} \|\varepsilon^{\frac{1}{3}\frac{2}{2-q}} v_\varepsilon'\|_{\mathrm{L}^\infty(I_j^\varepsilon)}^{1-\frac{q}{2}} < +\infty$, and the assertion follows from the elementary inequality $\left(\sum_{i=1}^n |a_i|\right)^{1-\frac{q}{2}} \leq \sum_{i=1}^n |a_i|^{1-\frac{q}{2}}$. The assertion in the case $q = 2$ follows as in Corollary 1. $\qquad\square$

6. The case of steady behavior of W at $\pm\infty$

In this section we gather the corresponding results in the case $q = 0$. The proofs are left to the interested reader. We note that condition (4) with $q = 0$ reads

$$C(L) := \inf\{\sqrt{V(\xi)} : \xi > L\} > 0 \text{ for every } L > 1. \tag{21}$$

We begin with a counterpart of Lemma 5.

Lemma 6. *Consider an arbitrary $\delta > 0$. Suppose that W satisfies (21). Then the following holds:*

$$\limsup_{\varepsilon \longrightarrow 0} \varepsilon^{-\frac{2}{3}} \lambda\{s \in (0,1) : |v_\varepsilon'(s)| > 1 + \delta\} < +\infty, \tag{22}$$

$$\limsup_{\varepsilon \longrightarrow 0} \varepsilon^{-\frac{2}{3}(1-\frac{p}{2})} \int_{\{|v_\varepsilon'|>1+\delta\}} |v_\varepsilon'(s)|^p ds < +\infty \text{ for every } p \in (0,2), \tag{23}$$

$$\limsup_{\varepsilon \longrightarrow 0} \|v_\varepsilon'\|_{\mathrm{L}^2(0,1)} < +\infty, \tag{24}$$

$$|v_\varepsilon'| \xrightarrow{\mathrm{L}^p(0,1)} 1 \quad \text{for every} \quad p \in [1,2) \quad \text{as } \varepsilon \longrightarrow 0, \tag{25}$$

$$\lim_{\varepsilon \longrightarrow 0} \|v_\varepsilon\|_{\mathrm{L}^\infty(0,1)} = 0. \tag{26}$$

Next, we recover the analogue of Proposition 2 in the case $q = 0$.

Proposition 3. *Consider W which satisfies (21) and an arbitrary sequence of measurable sets (A_ε) in $(0,1)$. Then the following conclusions hold:*

(i) *$\liminf_{\varepsilon \longrightarrow 0} \varepsilon^{-\frac{2}{3}} \lambda(A_\varepsilon) = +\infty$ implies $\limsup_{\varepsilon \longrightarrow 0} \operatorname{ess\,inf}_{A_\varepsilon} |v'_\varepsilon| \leq 1$,*

(ii) *$\limsup_{\varepsilon \longrightarrow 0} \sum_{j=1}^{N_\varepsilon} \frac{1}{N_\varepsilon} \inf_{I_j^\varepsilon} |v'_\varepsilon| \leq 1$.*
   Moreover, $\liminf_{\varepsilon \longrightarrow 0} \operatorname{ess\,inf}_{A_\varepsilon} |v'_\varepsilon| = +\infty$ implies

(iii) *$\limsup_{\varepsilon \longrightarrow 0} \lambda(A_\varepsilon)(\operatorname{ess\,inf}_{A_\varepsilon} |v'_\varepsilon|)^2 < +\infty$.*

Now we state the analogue of Lemma 4 in the case $q = 0$. Also, the result below sharpens the estimate $\|\varepsilon^{\frac{1}{3}} v'_\varepsilon\|_{L^\infty(0,1)} \leq C$, which is obtained in Proposition 1 for $q = 0$.

Corollary 3. *If W satisfies (21), then it holds that*

$$\limsup_{\varepsilon \longrightarrow 0} \sum_{j=1}^{N_\varepsilon} \frac{1}{N_\varepsilon} \|v'_\varepsilon\|_{L^\infty(I_j^\varepsilon)} < +\infty. \tag{27}$$

7. Main results

In this section we provide some applications of the a priori estimates from the previous sections. More precisely, in the proofs below we essentially use Lemma 1 and Lemma 3. We begin with a priori L^∞-bounds for FE sequences under the assumption (3). Such result is a kind of biting property for FE sequences (v_ε) such that the sequence (v'_ε) is not necessarily bounded in $L^1(0,1)$.

Lemma 7. *Consider W which satisfies (3), a strictly increasing sequence of strictly positive real numbers (a_k) such that $\lim_{k \longrightarrow +\infty} a_k = +\infty$, and an arbitrary FE sequence (v_ε) for $(\varepsilon^{-\frac{2}{3}} I_a^\varepsilon)$ in $H^2(0,1)$. Then every subsequence of (v_ε) allows a further subsequence (not relabeled), which satisfies the following: there exists a subsequence (a_{k_n}) such that $\lim_{n \longrightarrow +\infty} a_{k_n} = +\infty$ and a sequence of non-decreasing open sets $(\omega_\varepsilon^{k_n})_{n \in \mathbf{N}}$ in $(0,1)$ with the following properties:*

$$\limsup_{\varepsilon \longrightarrow 0} \|v'_\varepsilon\|_{L^\infty(\omega_\varepsilon^{k_n})} \leq a_{k_{n+1}} , \quad \lim_{n \longrightarrow +\infty} \limsup_{\varepsilon \longrightarrow 0} \lambda((0,1) \backslash \omega_\varepsilon^{k_n}) = 0; . \tag{28}$$

Proof. Consider an arbitrary FE sequence (v_ε) in $H^2(0,1)$ for $(\varepsilon^{-\frac{2}{3}} I_a^\varepsilon)$ such that for sufficiently small ε it holds that $M \geq \varepsilon^{-\frac{2}{3}} I_a^\varepsilon(v_\varepsilon)$. For arbitrarily chosen subsequence (v_ε) (not relabeled) we estimate $M \geq \varepsilon^{-\frac{2}{3}} I_a^\varepsilon(v_\varepsilon) \geq \rho_{\varepsilon,\star} \varepsilon_\star^{-\frac{2}{3}} I_{a^\star}^{\varepsilon_\star}(v_\varepsilon)$. By (7) we get $\limsup_{\varepsilon \longrightarrow 0} \int_0^{+\infty} \sum_{j=1}^{N_\varepsilon} \frac{1}{N_\varepsilon} \sqrt{V(\xi)} \chi_{[m_{\varepsilon,j}, M_{\varepsilon,j}]}(\xi) d\xi \leq M$, where $m_{\varepsilon,j} := \inf_{I_j^\varepsilon} |v'_\varepsilon|$, $M_{\varepsilon,j} := \sup_{I_j^\varepsilon} |v'_\varepsilon|$. On the other hand, by the elementary inequality

$$\int_{a_k}^{a_{k+1}} \sqrt{V(\xi)} \chi_{[c,d]}(\xi) d\xi \geq \int_{a_k}^{a_{k+1}} \sqrt{V(\xi)} d\xi \min\{\chi_{[c,d]}(a_k), \chi_{[c,d]}(a_{k+1})\}$$

we get $\limsup_\varepsilon \sum_{k=1}^{+\infty} \chi_\varepsilon(k) \int_{a_k}^{a_{k+1}} \sqrt{V(\xi)} d\xi \leq M$, where $\chi_\varepsilon(k) := \sum_{j=1}^{N_\varepsilon} \frac{1}{N_\varepsilon} \chi_{\varepsilon,j}(k)$ and $\chi_{\varepsilon,j}(k) := \min\{\chi_{[m_{\varepsilon,j}, M_{\varepsilon,j}]}(a_k), \chi_{[m_{\varepsilon,j}, M_{\varepsilon,j}]}(a_{k+1})\}$. Next, by Fatou's Lemma,

we recover the estimate $\sum_{k=1}^{+\infty} \liminf_{\varepsilon \longrightarrow 0} \chi_\varepsilon(k) \int_{a_k}^{a_{k+1}} \sqrt{V(\xi)}d\xi \leq M$. We note that $\sum_{k=1}^{+\infty} \int_{a_k}^{a_{k+1}} \sqrt{V(\xi)}d\xi b_{k,\infty} < +\infty$ and (3) imply $\lim_{k \longrightarrow +\infty} b_{k,\infty} = 0$, where $b_{k,\infty} := \liminf_{\varepsilon \longrightarrow 0} \chi_\varepsilon(k)$. Therefore, for a given $n \in \mathbf{N}$ there exists a $k_0 = k_0(n) \in \mathbf{N}$ such that for every $k \geq k_0(n)$ it holds that $|b_{k,\infty}| \leq \frac{1}{n}$. Then, up to a subsequence, for every such k there exists an $\varepsilon_0 = \varepsilon_0(k,n)$ such that for every $0 < \varepsilon \leq \varepsilon_0$ it holds that $\sum_{j=1}^{N_\varepsilon} \frac{1}{N_\varepsilon} \chi_{\varepsilon,j}(k) \leq \frac{2}{n}$. We set $B_\varepsilon^k := \{j : \chi_{\varepsilon,j}(k) = 1\}$, $G_\varepsilon^k := \{j : \chi_{\varepsilon,j}(k) = 0\}$, whereby $\{1, \ldots, N_\varepsilon\} = B_\varepsilon^k \cup G_\varepsilon^k$, $B_\varepsilon^k \cap G_\varepsilon^k = \emptyset$, and $\sum_{j \in B_\varepsilon^k} \frac{1}{N_\varepsilon} \chi_{\varepsilon,j}(k) \leq \frac{2}{n}$, i.e., $\mathrm{card}B_\varepsilon^k \varepsilon_\star^{\frac{1}{3}} \leq \frac{2}{n}$. As we pass to the limit (first as $\varepsilon \longrightarrow 0$, then as $k \longrightarrow +\infty$, and finally as $n \longrightarrow +\infty$), we get $\lim_{n \longrightarrow +\infty} \lim_{k \longrightarrow +\infty} \limsup_{\varepsilon \longrightarrow 0} \mathrm{card}B_\varepsilon^k \varepsilon_\star^{\frac{1}{3}} = 0$, and so (by Fatou's Lemma) $\lim_{n \longrightarrow +\infty} \lim_{k \longrightarrow +\infty} \limsup_{\varepsilon \longrightarrow 0} \mathrm{card}G_\varepsilon^k \varepsilon_\star^{\frac{1}{3}} = 1$. We decompose G_ε^k into a pairwise disjoint union of sets as follows: $G_\varepsilon^k = \tilde{G}_\varepsilon^k \cup \tilde{B}_\varepsilon^k \cup D_\varepsilon^k$, where $D_\varepsilon^k := \{j \in G_\varepsilon^k : a_k < m_{\varepsilon,j} \leq M_{\varepsilon,j} < a_{k+1}\}$, $\tilde{G}_\varepsilon^k := \{j \in G_\varepsilon^k : M_{\varepsilon,j} < a_{k+1}\}\backslash D_\varepsilon^k$, $\tilde{B}_\varepsilon^k := \{j \in G_\varepsilon^k : a_k < m_{\varepsilon,j}\}\backslash D_\varepsilon^k$. At this point we use the estimate of Lemma 1, whereby for sufficiently small ε it follows that $\sum_{j=1}^{N_\varepsilon} \frac{1}{N_\varepsilon} m_{\varepsilon,j} \leq C$. Hence, for a suitable choice of the subsequence (a_{k_n}) (for instance, we can choose $k_n := k_0(n)$) we have $\sum_{j \in \tilde{B}_\varepsilon^{k_n}} \frac{1}{N_\varepsilon} a_{k_n} \leq \sum_{j \in \tilde{B}_\varepsilon^{k_n}} \frac{1}{N_\varepsilon} m_{\varepsilon,j} \leq C$, $\limsup_{\varepsilon \longrightarrow 0} \mathrm{card}\tilde{B}_\varepsilon^{k_n} \varepsilon_\star^{\frac{1}{3}} \leq \frac{C}{a_{k_n}}$, and $\lim_{n \longrightarrow +\infty} \limsup_{\varepsilon \longrightarrow 0} \mathrm{card}\tilde{B}_\varepsilon^{k_n} \varepsilon_\star^{\frac{1}{3}} = 0$. Quite in the similar way it results that $\lim_{n \longrightarrow +\infty} \limsup_{\varepsilon \longrightarrow 0} \mathrm{card}D_\varepsilon^{k_n} \varepsilon_\star^{\frac{1}{3}} = 0$, whereby we conclude that it holds $\lim_{n \longrightarrow +\infty} \limsup_{\varepsilon \longrightarrow 0} \mathrm{card}\tilde{G}_\varepsilon^{k_n} \varepsilon_\star^{\frac{1}{3}} = 1$. Finally, since $\lim_{n \longrightarrow +\infty} a_{k_n} = +\infty$, if we set $\omega_\varepsilon^{k_n} := \cup_{j \in \tilde{G}_\varepsilon^{k_n} \cup D_\varepsilon^{k_n}} I_j^\varepsilon$, we obtain (28). $\square$

Corollary 4. *Under the assumptions of Lemma 7, every subsequence of (v_ε) allows a further subsequence (not relabeled), which satisfies the following: there exists $\varepsilon_0 > 0$ such that it holds*

$$\lim_{R \longrightarrow +\infty} \sup_{0 < \varepsilon \leq \varepsilon_0} \lambda(B_R^\varepsilon) = 0 , \quad \text{where } B_R^\varepsilon := \lambda\{s \in (0,1) : |v_\varepsilon'(s)| > R\}. \qquad (29)$$

Proof. We consider the sequence $u_j := v_{\varepsilon_j}'$, where (v_{ε_j}) is chosen in such a way that the conclusions of Lemma 7 hold true, i.e., $\lim_{R \longrightarrow +\infty} \lim_{j \longrightarrow +\infty} \sup_{i \geq j} \lambda(B_R^{\varepsilon_i}) = 0$. We infer that there exists a sequence $(j_0(R_m))_{m \in \mathbf{N}}$ such that $\lim_{m \longrightarrow +\infty} j_0(R_m) = +\infty$ and $\lim_{m \longrightarrow +\infty} \sup_{j \geq j_0(R_m)} \lambda(B_{R_m}^{\varepsilon_j}) = 0$, where (R_m) is an increasing sequence which satisfies $\lim_{m \longrightarrow +\infty} R_m = +\infty$. To prove (29), it is enough to show that there holds $\lim_{m \longrightarrow +\infty} \sup_{j \in \mathbf{N}} \lambda(B_{R_m}^{\varepsilon_j}) = 0$. For any given $m \in \mathbf{N}$ there are only two possibilities: either $\sup_{j > j_0(R_m)} \lambda(B_{R_m}^{\varepsilon_j}) = \sup_{j \in \mathbf{N}} \lambda(B_{R_m}^{\varepsilon_j})$ or $\max_{1 \leq j \leq j_0(R_m)} \lambda(B_{R_m}^{\varepsilon_j}) = \sup_{j \in \mathbf{N}} \lambda(B_{R_m}^{\varepsilon_j})$. If the first case occurs for all but finitely many indices m, we immediately get (29). Otherwise, the second case occurs for infinitely many indices $(m_n)_{n \in \mathbf{N}}$, whereby $\lim_{n \longrightarrow +\infty} m_n = +\infty$. Then it suffices to recover a sequence $(\theta_n)_{n \in \mathbf{N}}$, $\lim_{n \longrightarrow +\infty} \theta_n = +\infty$, with the property

$$\lim_{n \longrightarrow +\infty} \max\{\sup_{j \geq j_0(R_{m_n})} \lambda(B_{\theta_n}^{\varepsilon_j}), \max_{1 \leq j \leq j_0(R_{m_n})} \lambda(B_{\theta_n}^{\varepsilon_j})\} = 0. \qquad (30)$$

To this end, we argue as follows. Since it holds that $u_j \in \mathrm{L}^1(0,1)$, we conclude that for every $j \in \{1, \ldots, j_0(R_{m_n})\}$ we have $\lim_{L \longrightarrow +\infty} \lambda(B_L^{\varepsilon_j}) = 0$. Moreover, by

induction it follows that $\max\{|u_1|, \ldots, |u_{j_0(R_{m_n})}|\} \in \mathrm{L}^1(0,1)$, and so we get

$$\lim_{L \longrightarrow +\infty} \lambda(s \in (0,1) : \max\{|u_1(s)|, \ldots, |u_{j_0(R_{m_n})}(s)|\} \geq L) = 0.$$

Hence, for every $\delta > 0$ there exists $L_0 = L_0(R_{m_n}, \delta) > 0$ such that for every $L \geq L_0(R_{m_n}, \delta)$ it holds that $\max_{1 \leq j \leq j_0(R_{m_n})} \lambda(B_L^{\varepsilon j}) \leq \delta$. If we choose $\delta := \frac{1}{n}$, we get $\max_{1 \leq j \leq j_0(R_{m_n})} \lambda(B_{L_n}^{\varepsilon j}) \leq \frac{1}{n}$, where $L_n := L(R_{m_n}, \frac{1}{n})$. Finally, if we define $\theta_n := \max\{L_n, R_{m_n}\}$, it results that there holds $\lim_{n \longrightarrow +\infty} \sup_{j \geq j_0(R_{m_n})} \lambda(B_{\theta_n}^{\varepsilon j}) = 0$ and $\lim_{n \longrightarrow +\infty} \max_{1 \leq j \leq j_0(R_{m_n})} \lambda(B_{\theta_n}^{\varepsilon j}) = 0$, which yields (30). $\qquad\square$

For $a \in \mathrm{L}^1_{loc}(\mathbf{R})$, which satisfies $a(s) \geq \alpha_0 > 0$ (a.e. $s \in \mathbf{R}$), we introduce $f_s^\varepsilon, f_s : K \longrightarrow [0, +\infty]$ as follows:

$$f_s^\varepsilon(x) := \begin{cases} \fint_{-r}^{r} \left(\varepsilon^{\frac{2}{3}} x''^2(t) + \varepsilon^{-\frac{2}{3}} W(x'(t)) + a_s^\varepsilon(t) x^2(t) \right) dt, & \text{if } x \in \mathrm{H}^2(-r, r) \\ +\infty, & \text{otherwise,} \end{cases}$$

$$f_s(x) := \begin{cases} \frac{A_0}{2r} \mathrm{card}(Sx' \cap (-r, r)) + a(s) \fint_{-r}^{r} x^2(t) dt, & \text{if } x \in \mathcal{S}(-r, r) \\ +\infty, & \text{otherwise,} \end{cases}$$

where $A_0 := 2 \int_{-1}^{1} \sqrt{W(\xi)} d\xi$, $a_s^\varepsilon(t) := a(s + \varepsilon^{\frac{1}{3}} t)$, $t \in (-r, r)$ and $s \in \mathbf{R}$. We recall that, by the approach in [2], the corresponding relaxed functionals $F_a^\varepsilon, F_a : YM((0,1); K) \longrightarrow [0, +\infty]$ are defined by

$$F_a^\varepsilon(\boldsymbol{\nu}) := \begin{cases} \int_{r\varepsilon^{\frac{1}{3}}}^{1 - r\varepsilon^{\frac{1}{3}}} \langle \nu_s, f_s^\varepsilon \rangle ds, & \text{if } \boldsymbol{\nu} = \underline{\delta}_{R^\varepsilon v} \text{ for some } v \in \mathrm{H}^2(0,1) \\ +\infty, & \text{otherwise,} \end{cases}$$

$$F_a(\boldsymbol{\nu}) := \begin{cases} \int_0^1 \langle \nu_s, f_s \rangle ds, & \text{if } \nu_s \in \mathcal{I}(K) \text{ for a.e. } s \in (0,1) \\ +\infty, & \text{otherwise,} \end{cases}$$

where $R_s^\varepsilon v(\tau) := \varepsilon^{-\frac{1}{3}} v(s + \varepsilon^{\frac{1}{3}} \tau)$, $s \in (r\varepsilon^{\frac{1}{3}}, 1 - r\varepsilon^{\frac{1}{3}})$, $\tau \in (-r, r)$. Indeed, for $v \in \mathrm{H}^2(0,1)$ we have

$$\varepsilon^{-\frac{2}{3}} I_a^{\varepsilon,r}(v_\varepsilon) \leq F_a^\varepsilon(\underline{\delta}_{R^\varepsilon v}) \leq \varepsilon^{-\frac{2}{3}} I_a^\varepsilon(v), \tag{31}$$

where $I_a^{\varepsilon,r}$ is defined as I_a^ε, but with the domain of integration, $(0,1)$, replaced by $(2r\varepsilon^{\frac{1}{3}}, 1 - 2r\varepsilon^{\frac{1}{3}})$. In particular, if $v_\varepsilon \in \mathrm{H}^2_{per}(0,1)$, in the definition of F_a^ε we can replace the domain of integration $(r\varepsilon^{\frac{1}{3}}, 1 - r\varepsilon^{\frac{1}{3}})$ by $(0,1)$, getting $F_a^\varepsilon(\underline{\delta}_{R^\varepsilon v}) = \varepsilon^{-\frac{2}{3}} I_a^\varepsilon(v)$ (cf. [2], p. 781). Thus, every FE sequence (v_ε) for $(\varepsilon^{-\frac{2}{3}} I_a^\varepsilon)$ satisfies $\limsup_{\varepsilon \longrightarrow 0} F_a^\varepsilon(\underline{\delta}_{R^\varepsilon v_\varepsilon}) < +\infty$. To extract relevant geometric information regarding asymptotic behavior of ε-blowups of FE sequences (v_ε) for $(\varepsilon^{-\frac{2}{3}} I_a^\varepsilon)$, we aim to prove Γ-convergence of relaxed functionals (F_a^ε) as $\varepsilon \longrightarrow 0$. As the first step, we obtain Γ-convergence of the integrands (f_s^ε) for almost every $s \in (0,1)$. The following proposition is a consequence of Theorem 1.3 in [15]. This result is most likely known (compare Proposition 3.3 in [2], [18], [19]). We present the proof of the related compactness property in full for the convenience of the reader. We only sketch the argument which gives the lower bound since, once compactness is ensured, the proof is classical (cf. [16]).

Proposition 4. *If W satisfies (3), then it holds that $f_s^\varepsilon \xrightarrow{\Gamma} f_s$ on K as $\varepsilon \longrightarrow 0$ (a.e. $s \in (0,1)$). Moreover, all FE sequences (x_ε) for (f_s^ε) are equi-Lipschitz and satisfy $\liminf_{\varepsilon \longrightarrow 0} \|x_\varepsilon'\|_{\mathrm{L}^\infty(-r,r)} \geq 1$ (a.e. $s \in (0,1)$).*

Proof. The only non-trivial part of the proof is the lower-bound inequality (the upper-bound inequality can be obtained by repeating the proof of Theorem 1.13 in [16]). To prove the lower bound, we argue as follows. First, we prove equi-coercivity of (f_s^ε) on $\mathrm{W}^{1,1}(-r,r)$ for a fixed $s \in (0,1)$. As in Theorem 1.3 in [15], we estimate

$$\tilde{M} \geq f_s^\varepsilon(x_\varepsilon) \geq \int_{m_\varepsilon}^{M_\varepsilon} \sqrt{V(\xi)}d\xi, \tag{32}$$

where $m_\varepsilon := \min_{[-r,r]} |x_\varepsilon'|$, $M_\varepsilon := \max_{[-r,r]} |x_\varepsilon'|$. We claim that there exists $R > 0$ such that for sufficiently small ε it holds that $M_\varepsilon \leq R$. Let us assume the opposite, i.e., $\limsup_{\varepsilon \longrightarrow 0} M_\varepsilon = +\infty$. We show that there exists $L \geq 0$ and $c_\varepsilon \in (-r,r)$ such that $|x_\varepsilon'(c_\varepsilon)| \leq L$. Indeed, since $\|x_\varepsilon\|_{\mathrm{L}^1(-r,r)} \leq C_0$ (by (32) and by the Hölder inequality, we can choose any C_0 such that $C_0 > 2r\alpha_0^{-\frac{1}{2}}\tilde{M}^{\frac{1}{2}}$), by the integral mean value theorem there exists a $c_\varepsilon^{(1)} \in (0, \frac{r}{4})$ ($c_\varepsilon^{(2)} \in (\frac{3r}{4}, r)$, resp.) such that $|x_\varepsilon(c_\varepsilon^{(1)})| \leq C_0 \frac{4}{r}$ ($|x_\varepsilon(c_\varepsilon^{(2)})| \leq C_0 \frac{4}{r}$, resp.). By the Lagrange mean value theorem we get

$$C_0 \frac{8}{r} \geq \left||x_\varepsilon(c_\varepsilon^{(1)})| - |x_\varepsilon(c_\varepsilon^{(2)})|\right| \geq \mathrm{ess\,inf}_{\theta \in (c_\varepsilon^{(1)}, c_\varepsilon^{(2)})} |x_\varepsilon'(\theta)| \cdot |c_\varepsilon^{(1)} - c_\varepsilon^{(2)}|,$$

and so $C_0 \frac{8}{r} \geq \mathrm{ess\,inf}_{\theta \in (c_\varepsilon^{(1)}, c_\varepsilon^{(2)})} |x_\varepsilon'(\theta)| \frac{r}{2}$. Thus, provided $L > C_0 \frac{16}{r^2} > 0$, there exists a $c_\varepsilon \in (c_\varepsilon^{(1)}, c_\varepsilon^{(2)})$ such that $|x_\varepsilon'(c_\varepsilon)| \leq L$, and from (32) it follows that $M \geq \int_L^{M_\varepsilon} \sqrt{V(\xi)}d\xi$. As we pass to the limit as $\varepsilon \longrightarrow 0$, it follows that $M \geq \int_L^\infty \sqrt{V(\xi)}d\xi$, which contradicts assumption (3). Therefore, $\sup_{\varepsilon \in (0,\varepsilon_0)} \|x_\varepsilon'\|_{\mathrm{L}^\infty(-r,r)} \leq R$, and, if $W_R \in \mathrm{C}_0(\mathbf{R})$ is chosen such that $W_R(\zeta) := W(\zeta)$ for $\zeta \in [-R,R]$ and $W_R(\zeta) := 0$ for $|\zeta| > R + 1$, we get $\lim_{\varepsilon \longrightarrow 0} \int_{-r}^r W_R(x_\varepsilon'(t))dt = 0$. By the fundamental theorem of Young measures (cf. [4]) it follows that there exist a subsequence (not relabeled) such that $\underline{\delta}_{x_\varepsilon'} \xrightarrow{*} \boldsymbol{\nu}$ in $\mathrm{L}_{w*}^\infty((-r,r); \mathcal{P}(\mathbf{R}))$ as $\varepsilon \longrightarrow 0$, where $\mathrm{supp}\,\nu_t \subseteq \{-1,1\}$ (a.e. $t \in (-r,r)$). Hence, $x_\varepsilon' \xrightarrow{X*} \{-1,1\}$ on $(-r,r)$ as $\varepsilon \longrightarrow 0$. By compactness of the Sobolev imbedding (cf. [8]) it follows that there exists a further subsequence (not relabeled) such that $x_\varepsilon \longrightarrow x$ strongly in $\mathrm{L}^p(-r,r)$ for every $p \in [1,+\infty]$ as $\varepsilon \longrightarrow 0$. Moreover, $x' \in \mathrm{BV}(-r,r)$ (where $\mathrm{BV}(-r,r)$ stands for the set of all functions of bounded variation on $(-r,r)$), $x_\varepsilon' \longrightarrow x'$ strongly in $\mathrm{L}^1(-r,r)$ (cf. [11], Theorem 4, Subsection 5.2.3), and so $x \in \mathcal{S}(-r,r)$. Since (arguing by contradiction, as in the proof of Lemma 3, (i)), we have $R \geq 1$, we eventually get $\liminf_{\varepsilon \longrightarrow 0} f_s^\varepsilon(x_\varepsilon) \geq f_s(x)$ (cf. Theorem 1.12 in [16]). Quite in the same way as in the proof of Lemma 3, (i), we obtain the reverse estimate $\liminf_{\varepsilon \longrightarrow 0} \|x_\varepsilon'\|_{\mathrm{L}^\infty(-r,r)} \geq 1$. $\square$

The proof of our main result relies on two key ingredients. The first one is a remark subsequent to the statement (iv) of Theorem 2.12 in [2], which, in essence, says that, for the proof of the lower bound corresponding to Γ-convergence of relaxed functionals (F_a^ε), it is enough to obtain the lower bound corresponding to Γ-convergence of the integrands (f_s^ε) on K for almost every $s \in (0,1)$. The second one is an improvement of Theorem 1.3 in [15] (cf. Proposition 4). As such, the

following result is a technical extension of Theorem 3.4 in [2] to the case of non-coercive W and a generalization to the case of absent boundary conditions on (v_ε) (cf. [2], Section 6, Subsection 6.1).

Theorem 1. *If W satisfies (3), then it holds that*

$$F_a^\varepsilon \xrightarrow{\Gamma} F_a \text{ in } YM((0,1);K) \text{ as } \varepsilon \longrightarrow 0. \tag{33}$$

For any FE sequence (v_ε) for $(\varepsilon^{-\frac{2}{3}} I_a^\varepsilon)$ in $\mathrm{H}^2(0,1)$ we have $\underline{\delta}_{v'_\varepsilon} \xrightarrow{\ *\ } \frac{1}{2}\delta_{-1} + \frac{1}{2}\delta_1$ in $\mathrm{L}^\infty_{w*}((0,1);\mathcal{P}(\mathbf{R}))$ as $\varepsilon \longrightarrow 0$ and $v'_\varepsilon \xrightarrow{\lambda} \{-1,1\}$ on $(0,1)$ as $\varepsilon \longrightarrow 0$. In particular, (12) holds true.

Proof. First, we note that the proof of the upper bound associated with (33) can be carried out exactly as in Theorem 3.4 in [2]. The proof of the lower bound associated with (33) is more involved, and relies on Lemma 7. We claim that for every sequence $(\boldsymbol{\nu}^\varepsilon)$ such that $\boldsymbol{\nu}^\varepsilon \xrightarrow{\ *\ } \boldsymbol{\nu}$ in $YM((0,1);K)$ as $\varepsilon \longrightarrow 0$ it holds that $\liminf_{\varepsilon \longrightarrow 0} F_a^\varepsilon(\boldsymbol{\nu}^\varepsilon) \geq F_a(\boldsymbol{\nu})$. Without loss of generality, we can assume that $+\infty > \liminf_{\varepsilon \longrightarrow 0} F_a^\varepsilon(\boldsymbol{\nu}^\varepsilon)$ holds (otherwise there is nothing to prove). Thus (up to a subsequence which we do not relabel) we can assume that the limit inferior is actually a limit). Then for a sufficiently small ε it holds that $\nu_s^\varepsilon = \delta_{R_s^\varepsilon v_\varepsilon}$ (a.e. $s \in (r\varepsilon^{\frac{1}{3}}, 1 - r\varepsilon^{\frac{1}{3}})$), where $v_\varepsilon \in \mathrm{H}^2(0,1)$. Hence, by (31), it follows that (v_ε) is an FE sequence for $\varepsilon^{-\frac{2}{3}} I_a^{\varepsilon,r}(v_\varepsilon)$. We consider an arbitrary subsequence of (v_ε) (not relabeled). Then for every $\delta \in (0,1)$ there exists an $\varepsilon_0 = \varepsilon_0(\delta) > 0$ such that for every $0 < \varepsilon \leq \varepsilon_0$ it holds that

$$F_a^\varepsilon(\boldsymbol{\nu}^\varepsilon) = \int_{r\varepsilon^{\frac{1}{3}}}^{1-r\varepsilon^{\frac{1}{3}}} f_s^\varepsilon(R_s^\varepsilon v_\varepsilon)ds \geq \int_\delta^{1-\delta} f_s^\varepsilon(R_s^\varepsilon v_\varepsilon)ds \ .$$

Next, we invoke a remark subsequent to Theorem 2.12 in [2], which states that, for the proof of the lower-bound inequality of relaxed functionals, it is enough to have a lower-bound inequality for integrands. Now, by Proposition 4, we pass to the limit (first as $\varepsilon \longrightarrow 0$, then as $\delta \longrightarrow 0$), and it results $\liminf_{\varepsilon \longrightarrow 0} F_a^\varepsilon(\boldsymbol{\nu}^\varepsilon) \geq F_a(\boldsymbol{\nu})$. Since the argument above can be carried out for any subsequence of (v_ε), we proved (33). In the second part of our consideration, we note that by (33) it holds that $F_{\alpha_0}(\boldsymbol{\nu}) < +\infty$, and we get $\langle \nu_s, f_{\alpha_0} \rangle < +\infty$ (a.e. $s \in (0,1)$). Starting from the linear operator $D : \mathrm{W}_{loc}^{1,1}(\mathbf{R}) \longrightarrow K$ defined by $D(x)(t) := x'(t)$, we construct the corresponding push-forward operator (cf. [2], p. 795) $D^\# : \mathcal{P}(K) \longrightarrow \mathcal{P}(K)$ (as usual, in the first step we set $D^\#\delta_x := \delta_{D(x)}$ for $x \in \mathrm{W}_{loc}^{1,1}(\mathbf{R})$; in the second step we extend $D^\#$ onto the convex hull $\mathrm{conv}\{\delta_x : x \in \mathrm{W}_{loc}^{1,1}(\mathbf{R})\}$ by linearity; in the final step we recover the unique continuous extension of $D^\#$ (not relabeled) defined on $\mathcal{P}(K)$ (since the convex hull is ϕ-dense in $\mathcal{P}(K)$), and then extend it to $YM((0,1);K)$ via relation $(D^\#\boldsymbol{\nu})(s) := D^\#\nu_s$ (a.e. $s \in (0,1)$). We deduce that $D^\# : \mathcal{P}(K) \longrightarrow \mathcal{P}(K)$ ($D^\# : YM((0,1);K) \longrightarrow YM((0,1);K)$, resp.) is uniformly continuous. By Corollary 5.11 in [2], for every $\boldsymbol{\nu} \in \mathcal{I}(K)$ such that $F_{\alpha_0}(\boldsymbol{\nu}) < +\infty$ and every $\eta > 0$ there exists $x_s^\eta \in \mathcal{S}_{per,0}(0, h_\eta(s))$ such that $\Phi(\boldsymbol{\nu}, \mathcal{E}_{x^\eta}) \leq \eta$ and $|F_{\alpha_0}(\boldsymbol{\nu}) - F_{\alpha_0}(\mathcal{E}_{x^\eta})| \leq \eta$, where $\mathcal{E}_{x^\eta}(s) := \epsilon_{x_s^\eta}$ (a.e. $s \in (0,1)$). We define $\overline{\delta}_{R^\varepsilon v_\varepsilon}(s) := \underline{\delta}_{R^\varepsilon v_\varepsilon}(s)\chi_{(r\varepsilon^{\frac{1}{3}}, 1-r\varepsilon^{\frac{1}{3}})}(s)$,

$s \in (0,1)$. Then we have

$$\limsup_{\varepsilon} \Phi(D^{\#}\overline{\delta}_{R^{\varepsilon}v_{\varepsilon}}, D^{\#}\mathcal{E}_{x^{\eta}}) \leq \limsup_{\varepsilon} \omega\Big(\Phi(\overline{\delta}_{R^{\varepsilon}v_{\varepsilon}}, \mathcal{E}_{x^{\eta}})\Big) \leq \limsup_{\varepsilon} \omega\Big(\eta + e_{\varepsilon}\Big) = \omega(\eta),$$

where we chose ω to be a strictly increasing and continuous modulus of continuity of $D^{\#}$ such that $\omega(0) = 0$, and where, by construction, we have $\Phi(\overline{\delta}_{R^{\varepsilon}v_{\varepsilon}}, \boldsymbol{\nu}) \leq e_{\varepsilon}$ with $\lim_{\varepsilon \longrightarrow 0} e_{\varepsilon} = 0$. Taking into account $D^{\#}\mathcal{E}_{x^{\eta}} = \frac{1}{2}\delta_{-1} + \frac{1}{2}\delta_1$, as we pass to the limit as $\eta \longrightarrow 0$, it follows that $\lim_{\varepsilon \longrightarrow 0} \Phi(D^{\#}\overline{\delta}_{R^{\varepsilon}v_{\varepsilon}}, \frac{1}{2}\delta_{-1} + \frac{1}{2}\delta_1) = 0$. This means that the sequence $\Big(\overline{\delta}_{(R^{\varepsilon}v_{\varepsilon})'}\Big)$ generates a homogeneous Young measure $\frac{1}{2}\delta_{-1} + \frac{1}{2}\delta_1$ in $YM((0,1);K)$ as $\varepsilon \longrightarrow 0$. Since for $s \in (r\varepsilon^{\frac{1}{3}}, 1 - r\varepsilon^{\frac{1}{3}})$ and $\tau \in (-r, r)$ it holds that $(R^{\varepsilon}_s v_{\varepsilon})'(\tau) = T_{-\varepsilon^{\frac{1}{3}}\tau} v'_{\varepsilon}(s)$, where $\varepsilon^{\frac{1}{3}}\tau \longrightarrow 0$, by Lemma 2.1 in [2] we deduce that $\underline{\delta}_{v'_{\varepsilon}} \overset{*}{\longrightarrow} \frac{1}{2}\delta_{-1} + \frac{1}{2}\delta_1$ in $YM((0,1);K)$ as $\varepsilon \longrightarrow 0$. To proceed, we consider $\xi \in (0, \frac{1}{2})$, and we recall that, by (31), (v_{ε}) is also an FE sequence for $(\varepsilon^{-\frac{2}{3}}I_a^{\varepsilon,(\xi)})$ (where $I_a^{\varepsilon,(\xi)}$ is defined as I_a^{ε} but with the domain of integration, $(0,1)$, replaced by $(\xi, 1-\xi)$). By Corollary 4, arbitrary subsequence of such FE sequence (v_{ε}) allows a further subsequence (which depends on ξ and which is not relabeled) which satisfies Ball's condition $\lim_{R \longrightarrow +\infty} \sup_{0 < \varepsilon \leq \varepsilon_1} \lambda(B_R^{\varepsilon,(\xi)}) = 0$, where $B_R^{\varepsilon,(\xi)} := (\xi, 1-\xi) \cap B_R^{\varepsilon}$, i.e., we deduce that the latter subsequence of (v'_{ε}) "does not escape" to infinity on $(\xi, 1-\xi)$. Hence, by the fundamental theorem of Young measures (cf. [21], Theorem 3.1) there exists a further subsequence of $(\underline{\delta}_{v'_{\varepsilon}})$ (not relabeled) which generates a Young measure $\boldsymbol{\mu}^{(\xi)} = (\mu_s^{(\xi)})_s$ as $\varepsilon \longrightarrow 0$ in $L_{w*}^{\infty}((\xi, 1-\xi); \mathcal{P}(\mathbf{R}))$ such that $\|\mu_s^{(\xi)}\| = 1$ (a.e. $s \in (\xi, 1-\xi)$). Moreover, since for every $m > 0$ it holds that $[-m, m] \hookrightarrow K$, we have $C(K) \hookrightarrow C[-m, m] \hookrightarrow C_0(\mathbf{R})$. Therefore, it results $L^1((\xi, 1-\xi); C(K)) \hookrightarrow L^1((\xi, 1-\xi); C_0(\mathbf{R}))$, so that $L_{w*}^{\infty}((\xi, 1-\xi); \mathcal{P}(\mathbf{R})) \hookrightarrow L_{w*}^{\infty}((\xi, 1-\xi); \mathcal{P}(K))$, which gives $\mu_s^{(\xi)} = \frac{1}{2}\delta_{-1} + \frac{1}{2}\delta_1$ (a.e. $s \in (\xi, 1-\xi)$). Consequently, the arbitrariness of $\xi > 0$ yields $\mu_s^{(\xi)} = \frac{1}{2}\delta_{-1} + \frac{1}{2}\delta_1$ (a.e. $s \in (0,1)$), whereby $\boldsymbol{\mu}^{(\xi)}$ is independent of ξ. In particular, by the fundamental theorem of Young measures (cf. [21]) we get $v'_{\varepsilon} \overset{\lambda}{\longrightarrow} \{-1, 1\}$ on $(0,1)$ as $\varepsilon \longrightarrow 0$. Finally, we furnish the proof by an application of the argument of the unique feature of the cluster point. $\square$

Corollary 5. *For $a \in L^1(0,1)$ such that $a(s) \geq \alpha_0 > 0$ (a.e. $s \in (0,1)$) and W which satisfies (3), we have $\mathcal{E}_a = \mathcal{E}_{a,per} = E_0 \int_0^1 a^{\frac{1}{3}}(s)ds$, where $E_0 := (\frac{3}{4})^{\frac{2}{3}} A_0^{\frac{2}{3}}$, $A_0 := 2\int_{-1}^1 \sqrt{W(\xi)}d\xi$.*

Proof. We combine Theorem 1 and Theorem 3.12 in [2]. $\square$

A posteriori we deduce the following improvement of Lemma 7.

Corollary 6. *Consider W which satisfies (3) and an arbitrary FE sequence (v_{ε}) for $(\varepsilon^{-\frac{2}{3}}I_a^{\varepsilon})$ in $H^2(0,1)$. Then there exists a sequence of measurable sets (ω_{ε}) in $(0,1)$ such that $\lim_{\varepsilon \longrightarrow 0} \|v'_{\varepsilon}\|_{L^{\infty}(\omega_{\varepsilon})} = 1$, $\lim_{\varepsilon \longrightarrow 0} \lambda((0,1) \backslash \omega_{\varepsilon}) = 0$.*

Proof. By Theorem 1 it follows that $|v'_{\varepsilon}| \overset{\lambda}{\longrightarrow} 1$ on $(0,1)$. By Egoroff's theorem there exists a subsequence (not relabeled) such that $|v'_{\varepsilon}(s)| \longrightarrow 1$ uniformly over $s \in \Omega_k$, where $\Omega_k \subseteq (0,1)$ are measurable sets which satisfy $\lim_{k \longrightarrow +\infty} \lambda((0,1) \backslash \Omega_k) = 0$.

Hence, $\lim_{\varepsilon \longrightarrow 0} \|v'_\varepsilon\|_{\mathrm{L}^\infty(\Omega_k)} = 1$. By the usual diagonal argument we pass to the limit as $k \longrightarrow +\infty$, proving the assertion for a suitable subsequence of (v_ε), and, ultimately, for the whole sequence (v_ε).   □

Also, the following improvement of Lemma 2, (i), is available:

Corollary 7. *If W satisfies (3), then for an arbitrary FE sequence (v_ε) in $\mathrm{H}^2(0,1)$ and for an arbitrary open interval $\omega \subset (0,1)$, the following holds: for a sufficiently small $\varepsilon_0 > 0$ we have $\min_\omega |v'_\varepsilon| = 0$ for every $\varepsilon \in (0, \varepsilon_0]$.*

Proof. We set $A_\varepsilon^+ := \{s \in \omega : v'_\varepsilon(s) > 0\}$ ($A_\varepsilon^- := \{s \in \omega : v'_\varepsilon(s) < 0\}$, resp.). Since it holds that $\underline{\delta}_{v'_\varepsilon} \overset{*}{\longrightarrow} \frac{1}{2}\delta_{-1} + \frac{1}{2}\delta_1$ in $\mathrm{L}^\infty_{w*}(\omega; \mathcal{P}(\mathbf{R}))$ as $\varepsilon \longrightarrow 0$, we recover $|v'_\varepsilon \chi_{A_\varepsilon^+} - \chi_{A_\varepsilon^+}| \overset{\lambda}{\longrightarrow} 0$ ($|v'_\varepsilon \chi_{A_\varepsilon^-} + \chi_{A_\varepsilon^-}| \overset{\lambda}{\longrightarrow} 0$, resp.) on ω as $\varepsilon \longrightarrow 0$. Hence, there exist $1 >> \delta^+ > 0$ and $s_\delta^{\varepsilon,+} \in A_\varepsilon^+$ ($1 >> \delta^- > 0$ and $s_\delta^{\varepsilon,-} \in A_\varepsilon^-$, resp.) such that $v'_\varepsilon(s_\delta^{\varepsilon,+}) \geq 1 - \delta^+$ ($v'_\varepsilon(s_\delta^{\varepsilon,-}) \leq -1 + \delta^-$, resp.). By the intermediate value property of continuous functions, there exists a $\theta_\varepsilon \in \omega$ such that $v'_\varepsilon(\theta_\varepsilon) = 0$.   □

If W satisfies condition (3), then all FE sequences (v_ε) for $(\varepsilon^{-\frac{2}{3}} I_a^\varepsilon)$ share the properties $|v'_\varepsilon| \overset{\lambda}{\longrightarrow} 1$ and $\underline{\delta}_{|v'_\varepsilon|} \overset{*}{\longrightarrow} \delta_1$ in $\mathrm{L}^\infty_{w*}((0,1); \mathcal{P}(\mathbf{R}))$ as $\varepsilon \longrightarrow 0$. If we impose a q-rate of decay at infinity on W with non-negative q, we can obtain further a priori estimates for the ratio of the "minus"-phase and the "plus"-phase of v'_ε for small but strictly positive ε. To this end, we introduce the following terminology. We say that an FE sequence (v_ε) is an m_ε-FE sequence if it holds that $m_\varepsilon = \int_0^1 v'_\varepsilon(s)ds$. Note that the quantity m_ε indeed can be interpreted as a measure of the ratio of the two aforementioned phases. In the last proposition we provide necessary conditions for the existence of m_ε-FE sequences for $(\varepsilon^{-\frac{2}{3}} I_a^\varepsilon)$ in $\mathrm{H}^2(0,1)$. A typical constraint found in the literature is $m_\varepsilon := m$ for some $m \in \mathbf{R}$ (cf. [15]).

Proposition 5. *Consider W which satisfies (4) with $q \in [0,2]$, and suppose that (v_ε) is an m_ε-FE sequence for $(\varepsilon^{-\frac{2}{3}} I_a^\varepsilon)$ in $\mathrm{H}^2(0,1)$. Then the following holds:*

(i) *If $0 \leq q < \frac{1}{2}$, then $\lim_{\varepsilon \longrightarrow 0} m_\varepsilon = 0$,*

(ii) *If $q = \frac{1}{2}$, then (m_ε) is bounded,*

(iii) *If $\frac{1}{2} < q \leq \frac{4}{3}$, then $(\varepsilon^{\frac{1}{3}\frac{q}{2-q}} m_\varepsilon)$ is bounded,*

(iv) *If $\frac{4}{3} < q \leq 2$, then $(\varepsilon^{\frac{2}{3}} m_\varepsilon)$ is bounded.*

Proof. First, we prove (i). If $0 < q < \frac{1}{2}$ ($q = 0$, resp.), by (13) ((24), resp.) we have $\|v'_\varepsilon\|_{\mathrm{W}^{1,p}(0,1)} \leq C_1$ for every $p \in [1, 2 - 2q]$. Since $0 \leq q < \frac{1}{2}$ implies $2 - 2q > 1$, the Rellich compactness theorem gives (16), so that Helly's selection theorem (cf. [8], p. 130) yields $v_\varepsilon(s) \longrightarrow 0$ for every $s \in [0,1]$. Hence, $m_\varepsilon = \int_0^1 v'_\varepsilon(s)ds = v_\varepsilon(1) - v_\varepsilon(0)$ tends to zero as $\varepsilon \longrightarrow 0$. Second, we note that (ii) is an immediate consequence of (13). Next, we compute $|\varepsilon^{\frac{1}{3}\frac{q}{2-q}} m_\varepsilon| \leq \sum_{j=1}^{N_\varepsilon} \varepsilon^{\frac{1}{3}\frac{q}{2-q}} \int_{I_j^\varepsilon} |v'_\varepsilon| \leq \sum_{j=1}^{N_\varepsilon} \frac{1}{N_\varepsilon} \|\varepsilon^{\frac{1}{3}\frac{q}{2-q}} v'_\varepsilon\|_{\mathrm{L}^\infty(I_j^\varepsilon)}$, whereby Corollary 2 gives (iii). Finally, (iv) follows from Lemma 2, (ii).   □

Corollary 8. *Consider W which satisfies (4) with $0 \leq q < \frac{1}{2}$ and an arbitrary FE sequence (v_ε) for $(\varepsilon^{-\frac{2}{3}} I_a^\varepsilon)$ in $\mathrm{H}^2(0,1)$. Then for every $p \in [1, 2 - 2q]$ it holds that $v'_\varepsilon \longrightarrow 0$ in $\mathrm{L}^p(0,1)$ as $\varepsilon \longrightarrow 0$.*

Proof. We recall that (13) and (24) provide boundedness of (v'_ε) in $\mathrm{L}^p(0,1)$ for arbitrary $p \in [1, 2 - 2q]$. On the other hand, by Proposition 5, (i), it follows that for every open interval $\omega \subseteq (0,1)$ we have $\lim_{\varepsilon \longrightarrow 0} \int_\omega v'_\varepsilon(s) ds = 0$. Then the assertion follows from Proposition 2.81 in [12], p. 198. $\qquad\square$

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