MATHEMATICAL COMMUNICATIONS
Math. Commun. **24**(2019), 83–90

Logarithms of a binomial series: Extension of a series of Knuth

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Received May 17, 2018; accepted July 24, 2018

Abstract. The Lagrange expansion formula is employed to determine the Maclaurin series for the logarithms of Lambert series of binomial coefficients, extending the log–squared of the Catalan generating function due to Knuth (2015).

AMS subject classifications: Primary 05A15; Secondary 05A10

Key words: Catalan numbers, harmonic numbers, elementary symmetric function, Bell polynomial, Lagrange expansion formula

1. Introduction and motivation

Let $\mathbb{N}$ be the set of natural numbers with $\mathbb{N}_0 = \{0\} \cup \mathbb{N}$. Catalan numbers (cf. [12]) defined by

$$C_n = \frac{1}{n+1}\binom{2n}{n} \quad \text{with} \quad n \in \mathbb{N}_0 \tag{1}$$

are one of the fascinating sequences in classical combinatorics with more than fifty significant combinatorial interpretations [19,20]. Their ordinary generating function is given by

$$C(x) = \sum_{n\geq 0} C_n x^n = \frac{1-\sqrt{1-4x}}{2x}.$$

Recently, Knuth [13] discovered the following remarkable log–squared series

$$\log^2 C(x) = \sum_{m\geq 2} \frac{x^m}{m}\binom{2m}{m}\big\{H_{2m-1} - H_m\big\}, \tag{2}$$

where harmonic numbers are defined by

$$H_m := H_m^{\langle 1\rangle} \qquad \text{with} \qquad H_m^{\langle n\rangle} = \sum_{k=1}^{m} \frac{1}{k^n}.$$

The aim of this short paper is to extend Knuth's identity to Lambert's binomial series (see [15,16]) with higher powers. The main theorem will be proved in the next

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section by means of the Lagrange expansion formula. Then the paper will end with a few applications to convolution identities involving harmonic numbers.

In order to facilitate the subsequent references, we record briefly the elementary symmetric functions and their representation in terms of power sum functions. For an indeterminate y and $m, n \in \mathbb{N}$, define the power sums

$$\mathcal{P}_n(y|m) := \sum_{i=1}^{m} \frac{1}{(y+i)^n}$$

and the elementary symmetric functions

$$\Omega_0(y|m) = 1 \quad \text{and} \quad \Omega_n(y|m) := \sum_{1 \le k_1 < k_2 < \cdots < k_n \le m} \prod_{i=1}^{n} \frac{1}{y+k_i}.$$

Their generating function can be manipulated as follows:

$$\begin{aligned}
\sum_{n=0}^{m} x^n \Omega_n(y|m) &= \prod_{k=1}^{m} \Big(1 + \frac{x}{y+k}\Big) \\
&= \exp\Big\{ \sum_{k=1}^{m} \log\Big(1 + \frac{x}{y+k}\Big) \Big\} \\
&= \exp\Big\{ \sum_{k=1}^{m} \sum_{i=1}^{\infty} \frac{(-1)^{i-1} x^i}{i(y+k)^i} \Big\} \\
&= \exp\Big\{ \sum_{i=1}^{\infty} x^i \frac{(-1)^{i-1} \mathcal{P}_i(y|m)}{i} \Big\}.
\end{aligned}$$

By extracting the coefficient of x^n from the two extreme members of these equations, we obtain the expression in terms of power sums (cf. [17, §1.2] and [2, 4, 6])

$$\Omega_n(y|m) = \sum_{\sigma(n)} (-1)^n \prod_{i=1}^{n} \frac{\mathcal{P}_i^{j_i}(y|m)}{j_i!} \Big(\frac{-1}{i}\Big)^{j_i}, \tag{3}$$

where the multiple sum runs over all the partitions $\sigma(n)$ of n indexed by nonnegative integers $\{j_1, j_2, \cdots, j_n\}$ subject to the condition $j_1 + 2j_2 + \cdots + nj_n = n$. Denote further by $\sigma_k(n)$ the subset of partitions of n with exactly k parts, characterized by $\{j_1, j_2, \cdots, j_n\}$ subject to $j_1 + 2j_2 + \cdots + nj_n = n$ and $j_1 + j_2 + \cdots + j_n = k$. Then the exponential Bell polynomials (see [9, §3.3]) read:

$$\mathbf{Y}_n(x_1, x_2, \cdots, x_n) := \sum_{k=0}^{n} \mathbf{B}_{n,k}(x_1, x_2, \cdots, x_{n-k+1})$$

with the partial Bell polynomials given by $\mathbf{B}_{0,0}(x_1)=1$ and

$$\begin{aligned}\mathbf{B}_{n,k}(x_1,x_2,\cdots,x_{n-k+1}):=&\frac{n!}{k!}\sum_{\sigma_k(n)}\binom{k}{j_1,j_2,\cdots,j_n}\prod_{i=1}^{n}\Big(\frac{x_i}{i!}\Big)^{j_i}\\ =&\frac{1}{k!}\sum_{\substack{i_1+i_2+\cdots+i_k=n\\(i_j\in\mathbb{N})}}\binom{n}{i_1,i_2,\cdots,i_k}\prod_{j=1}^{k}x_{i_j}.\end{aligned}$$

We have therefore the following alternative expression of $\Omega_n(y|m)$ by the complete Bell polynomials:

$$\begin{aligned}&\Omega_n(y|m)=\frac{1}{n!}\times\mathbf{Y}_n(\mathcal{Q}_1,\mathcal{Q}_2,\cdots,\mathcal{Q}_n),\\ \text{where}\qquad&\mathcal{Q}_k:=\mathcal{Q}_k(y|m)=(-1)^{k-1}(k-1)!\mathcal{P}_k(y|m).\end{aligned}\tag{4}$$

It may also be worth remarking that the last formula for Ω_n is effectively a sum over the permutations of $\{1,2,\cdots,n\}$. For example, there are 6 permutations of $\{1,2,3,4\}$ consisting of a single cycle, 1 permutation with four cycles of length 1, and $\binom{4}{2}=6$ permutations with cycle lengths (2,1,1). There are two different kinds of permutations of $\{1,2,3,4\}$ with two cycles; 3 with cycle lengths (2,2) and $2\binom{4}{3}=8$ permutations with cycle lengths (3,1). If we attach $(-1)^k$ to each of these coefficients, where k is the number of cycles, and divide by the total number $4!=24$ of permutations, then we get the formula for Ω_4 in terms of $\mathcal{P}_1,\mathcal{P}_2,\mathcal{P}_3,\mathcal{P}_4$.

The first five expressions are displayed below with the common parameters $(y|m)$ being suppressed:

$$\begin{aligned}\Omega_1=&\mathcal{P}_1,\\ \Omega_2=&\frac{1}{2}(\mathcal{P}_1^2-\mathcal{P}_2),\\ \Omega_3=&\frac{1}{6}(\mathcal{P}_1^3-3\mathcal{P}_1\mathcal{P}_2+2\mathcal{P}_3),\\ \Omega_4=&\frac{1}{24}(\mathcal{P}_1^4-6\mathcal{P}_1^2\mathcal{P}_2+8\mathcal{P}_1\mathcal{P}_3+3\mathcal{P}_2^2-6\mathcal{P}_4),\\ \Omega_5=&\frac{1}{120}(\mathcal{P}_1^5-10\mathcal{P}_1^3\mathcal{P}_2+20\mathcal{P}_1^2\mathcal{P}_3+15\mathcal{P}_1\mathcal{P}_2^2-30\mathcal{P}_1\mathcal{P}_4-20\mathcal{P}_2\mathcal{P}_3+24\mathcal{P}_5).\end{aligned}$$

2. Main theorem and proof

In classical analysis and enumerative combinatorics, the Lagrange expansion formula is fundamental.

Lemma 1 (Lagrange expansion formula [14]: see [9, §3.8] and [3,8])**.** *For a formal power series $\varphi(x)$ subject to $\varphi(0)\neq 0$, the functional equation $x=y/\varphi(y)$ determines y as an implicit function of x. Then for another formal power series $F(y)$ in the* *variable y, the following expansions hold for both composite series:*

$$F(y(x)) = F(0) + \sum_{n=1}^{\infty} \frac{x^n}{n}[y^{n-1}]\big\{F'(y)\varphi^n(y)\big\}, \tag{5}$$

$$\frac{F(y(x))}{1-\big(y\varphi'(y)/\varphi(y)\big)} = \sum_{n=0}^{\infty} x^n[y^n]\big\{F(y)\varphi^n(y)\big\}. \tag{6}$$

By taking $F(y) = (1+y)^\alpha$ and $\varphi(y) = (1+y)^\lambda$ in this lemma, we recover the binomial series due to Lambert [15,16] (see also [3,5,10], [11, §5.4] and [18, §5.4]):

$$\left.\begin{aligned} \sum_{k\geq 0} \frac{\alpha}{\alpha+k\lambda}\binom{\alpha+k\lambda}{k}x^k &= (1+y)^\alpha \\ \sum_{k\geq 0} \binom{\alpha+k\lambda}{k}x^k &= \frac{(1+y)^{\alpha+1}}{1+y-\lambda y} \end{aligned}\right\} \quad \text{where} \quad y := y_\lambda(x) = x(1+y)^\lambda.$$

Taking $\alpha = 1$ now gives the generating function

$$C_\lambda(x) := 1 + y_\lambda(x) = \sum_{n\geq 0}\binom{1+n\lambda}{n}\frac{x^n}{1+n\lambda}$$

of the extended Catalan numbers, where $y = x(1+y)^\lambda$. In particular, we have the generating function $C(x) = 1 + y_2(x)$ for the usual Catalan numbers.

By making use of (5), we can expand, under the specifications $F(y) = \log(1+y)$ and $\varphi(y)$ as before, the logarithm function

$$\log C_\lambda(x) = \sum_{m\geq 1}\frac{x^m}{m\lambda}\binom{m\lambda}{m}. \tag{7}$$

Furthermore, we can show the following main theorem.

Theorem 1. *Let $n \in \mathbb{N}$ and $\lambda \in \mathbb{N}$. It holds:*

$$\log^n C_\lambda(x) = n! \sum_{m\geq n}\frac{x^m}{m\lambda}\binom{m\lambda}{m}\Omega_{n-1}(m\lambda - m|m-1).$$

Proof. The formula in the theorem is clearly true for $n = 1$ in view of (7). Suppose that the formula is valid for n. Then we have to prove it for $n+1$. Because

$$\log^n C_\lambda(x) \times \log C_\lambda(x) = \log^{n+1} C_\lambda(x),$$

the formula to be shown is equivalent to

$$\sum_{k=n}^{m-1}\frac{\Omega_{n-1}(k\lambda-k|k-1)}{k\lambda}\binom{k\lambda}{k}\frac{\binom{m\lambda-k\lambda}{m-k}}{m\lambda-k\lambda} = \frac{n+1}{m\lambda}\binom{m\lambda}{m}\Omega_n(m\lambda-m|m-1). \tag{8}$$

Recall the Hagen–Rothe convolution formula (cf. [7] and [10])

$$\sum_{k=0}^{m}\frac{x}{x+k\lambda}\binom{x+k\lambda}{k}\frac{y}{y+m\lambda-k\lambda}\binom{y+m\lambda-k\lambda}{m-k}=\frac{x+y}{x+y+m\lambda}\binom{x+y+m\lambda}{m}.$$

Moving the $k=m$ term to the other side, dividing by y, and letting $y\to 0$, we get

$$\sum_{k=0}^{m-1}\frac{x\binom{x+k\lambda}{k}}{x+k\lambda}\frac{\binom{m\lambda-k\lambda}{m-k}}{m\lambda-k\lambda}=\lim_{y\to 0}\left\{\frac{(x+y)\binom{x+y+m\lambda}{m}}{y(x+y+m\lambda)}-\frac{x\binom{x+m\lambda}{m}}{y(x+m\lambda)}\right\}$$

which can be reformulated equivalently as the equality

$$\sum_{k=0}^{m-1}\frac{x\binom{x+k\lambda}{k}}{x+k\lambda}\frac{\binom{m\lambda-k\lambda}{m-k}}{m\lambda-k\lambda}=\left\{1+x\frac{\mathrm{d}}{\mathrm{d}x}\right\}\frac{\binom{x+m\lambda}{m}}{x+m\lambda}. \tag{9}$$

Recalling the generating function

$$\sum_{n=0}^{m-1}x^n\Omega_n(y|m-1)=\prod_{k=1}^{m-1}\left(1+\frac{x}{y+k}\right)=\frac{y+m}{x+y+m}\frac{\binom{x+y+m}{m}}{\binom{y+m}{m}}$$

we have

$$[x^n]\frac{\binom{x+y+m}{m}}{x+y+m}=\frac{\binom{y+m}{m}}{y+m}\Omega_n(y|m-1),$$

where $[x^n]\psi(x)$ stands for the coefficient of x^n in the formal power series $\psi(x)$. Then the coefficient of x^n from the sum on the left-hand side of (9) reads:

$$\begin{aligned}[x^n]\sum_{k=0}^{m-1}\frac{x\binom{x+k\lambda}{k}}{x+k\lambda}\frac{\binom{m\lambda-k\lambda}{m-k}}{m\lambda-k\lambda}&=\sum_{k=0}^{m-1}\frac{\binom{m\lambda-k\lambda}{m-k}}{m\lambda-k\lambda}[x^{n-1}]\frac{\binom{x+k\lambda}{k}}{x+k\lambda}\\&=\sum_{k=0}^{m-1}\frac{\Omega_{n-1}(k\lambda-k|k-1)}{k\lambda}\binom{k\lambda}{k}\frac{\binom{m\lambda-k\lambda}{m-k}}{m\lambda-k\lambda}.\end{aligned}$$

The coefficient of x^n from the right-hand side member of (9) can be extracted similarly

$$\begin{aligned}[x^n]\left\{1+x\frac{\mathrm{d}}{\mathrm{d}x}\right\}\frac{\binom{x+m\lambda}{m}}{x+m\lambda}=&[x^n]\frac{\binom{x+m\lambda}{m}}{x+m\lambda}+n[x^n]\frac{\binom{x+m\lambda}{m}}{x+m\lambda}\\=&\frac{n+1}{m\lambda}\binom{m\lambda}{m}\Omega_n(m\lambda-m|m-1).\end{aligned}$$

Therefore (8) is verified and Theorem 1 follows by induction. □

3. Examples as applications

By making use of the relation

$$\mathcal{P}_n(y|m)=H_{m+y}^{\langle n\rangle}-H_y^{\langle n\rangle}\quad\text{for}\quad y\in\mathbb{N}_0,$$

from Theorem 1 we can derive the following infinite series expressions.

Proposition 1. *Let $\lambda \in \mathbb{N}$. It holds:*

$$\log^2 C_\lambda(x) = 2\sum_{m\geq 2}\frac{x^m}{m\lambda}\binom{m\lambda}{m}\big\{H_{m\lambda-1}-H_{m\lambda-m}\big\},$$

$$\log^3 C_\lambda(x) = 3\sum_{m\geq 3}\frac{x^m}{m\lambda}\binom{m\lambda}{m}\Big\{\big(H_{m\lambda-1}-H_{m\lambda-m}\big)^2-\big(H^{\langle 2\rangle}_{m\lambda-1}-H^{\langle 2\rangle}_{m\lambda-m}\big)\Big\},$$

$$\log^4 C_\lambda(x) = 4\sum_{m\geq 4}\frac{x^m}{m\lambda}\binom{m\lambda}{m}\left\{\begin{array}{c}\big(H_{m\lambda-1}-H_{m\lambda-m}\big)^3+2\big(H^{\langle 3\rangle}_{m\lambda-1}-H^{\langle 3\rangle}_{m\lambda-m}\big)\\ -3\big(H_{m\lambda-1}-H_{m\lambda-m}\big)\big(H^{\langle 2\rangle}_{m\lambda-1}-H^{\langle 2\rangle}_{m\lambda-m}\big)\end{array}\right\}.$$

In particular, we recover, for $\lambda = 2$, Knuth's series (2) and two further series of cubic and quartic logarithms of the Catalan generating function in terms of generalized harmonic numbers.

Corollary 1.

$$\log^3 C(x) = 3\sum_{m\geq 3}\frac{x^m}{2m}\binom{2m}{m}\Big\{\big(H_{2m-1}-H_m\big)^2-\big(H^{\langle 2\rangle}_{2m-1}-H^{\langle 2\rangle}_{m}\big)\Big\},$$

$$\log^4 C(x) = 4\sum_{m\geq 4}\frac{x^m}{2m}\binom{2m}{m}\left\{\begin{array}{c}\big(H_{2m-1}-H_m\big)^3+2\big(H^{\langle 3\rangle}_{2m-1}-H^{\langle 3\rangle}_{m}\big)\\ -3\big(H_{2m-1}-H_m\big)\big(H^{\langle 2\rangle}_{2m-1}-H^{\langle 2\rangle}_{m}\big)\end{array}\right\}.$$

By applying the exponential law to Theorem 1, we get the next convolution formula.

Proposition 2. *For $\alpha,\gamma \in \mathbb{N}$ and $m \in \mathbb{N}$ it holds:*

$$\sum_{k=\alpha}^{m-\gamma}\binom{k\lambda}{k}\frac{\Omega_{\alpha-1}(k\lambda-k|k-1)}{k\lambda}\binom{m\lambda-k\lambda}{m-k}\frac{\Omega_{\gamma-1}((m-k)(\lambda-1)|m-k-1)}{(m-k)\lambda}$$
$$=\binom{\alpha+\gamma}{\alpha}\binom{m\lambda}{m}\frac{\Omega_{\alpha+\gamma-1}(m\lambda-m|m-1)}{m\lambda}.$$

When $\alpha = \gamma = 1$ and $\alpha = 2$, $\gamma = 1$, the above formula reduces to the following two further interesting ones.

Corollary 2. *Let $\lambda \in \mathbb{N}$ and $m \in \mathbb{N}$. It holds:*

$$\sum_{k=1}^{m-1}\binom{k\lambda}{k}\binom{m\lambda-k\lambda}{m-k}\frac{m}{k(m-k)} = 2\lambda\binom{m\lambda}{m}\big\{H_{m\lambda-1}-H_{m\lambda-m}\big\},$$

$$\sum_{k=2}^{m-1}\binom{k\lambda}{k}\binom{m\lambda-k\lambda}{m-k}\frac{H_{k\lambda-1}-H_{k\lambda-k}}{k(m-k)}$$
$$=\frac{3\lambda}{2m}\binom{m\lambda}{m}\Big\{\big(H_{m\lambda-1}-H_{m\lambda-m}\big)^2-\big(H^{\langle 2\rangle}_{m\lambda-1}-H^{\langle 2\rangle}_{m\lambda-m}\big)\Big\}.$$

In addition, by combining (7) with Theorem 1 we can derive the following multiple convolution identity.

Corollary 3. *Let $\lambda \in \mathbb{N}$ and $m, n \in \mathbb{N}$. Denote by $h_k(\lambda)$ the falling factorial*

$$h_k(\lambda) = (k\lambda - 1)(k\lambda - 2) \cdots (k\lambda - k + 1) \quad \textit{for} \quad k \geq 1.$$

Then there holds the summation formula

$$h_m(\lambda)\Omega_{n-1}(m\lambda - m|m-1) = \mathbf{B}_{m,n}\big(h_1(\lambda), h_2(\lambda), \cdots, h_{m-n+1}(\lambda)\big).$$

Analogously, there are the series for Abel coefficients [1] (see also [3,5] for example)

$$\left.\begin{aligned} \sum_{k\geq 0} \frac{\alpha}{\alpha + k\lambda} \frac{(\alpha + k\lambda)^k}{k!} x^k &= e^{z\alpha} \\ \sum_{k\geq 0} \frac{(\alpha + k\lambda)^k}{k!} x^k &= \frac{e^{z\alpha}}{1 - z\lambda} \end{aligned}\right\} \quad \text{where} \quad z := z_\lambda(x) = xe^{z\lambda}.$$

In this case, taking into account (5) we have the expressions

$$z_\lambda(x) = \sum_{m\geq 1} \frac{(m\lambda)^{m-1}}{m!} x^m \quad \text{and} \quad z_\lambda^n(x) = \sum_{m\geq n} \frac{n}{m} \frac{(m\lambda)^{m-n}}{(m-n)!} x^m.$$

They lead to the following multiple convolution formula for the number of forests of n labeled rooted trees on m vertices

$$\binom{m-1}{n-1} m^{m-n} = \mathbf{B}_{m,n}\big(1^0, 2^1, 3^2, \cdots, (m-n+1)^{m-n}\big)$$

as well as the convolution identity

$$\sum_{k=\alpha}^{m-\gamma} \frac{\alpha}{k} \frac{k^{k-\alpha}}{(k-\alpha)!} \frac{\gamma}{m-k} \frac{(m-k)^{m-k-\gamma}}{(m-k-\gamma)!} = \frac{\alpha + \gamma}{m} \frac{m^{m-\alpha-\gamma}}{(m-\alpha-\gamma)!},$$

which can also be derived from the Abel identity (see [7] and [9, §3.1]).

Acknowledgement

The author is grateful to an anonymous referee for the careful reading, critical comments and valuable suggestions that make the manuscript improved substantially in revision.

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