MATHEMATICAL COMMUNICATIONS
Math. Commun. **24**(2019), 91–105

Constructing formally self-dual codes from block λ-circulant matrices

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Received April 10, 2018; accepted September 9, 2018

Abstract. In this work, construction methods for formally self-dual codes are generalized in the form of block λ-circulant matrices. The constructions are applied over the rings $\mathbb{F}_2$, $R_1 = \mathbb{F}_2 + u\mathbb{F}_2$ and $S = \mathbb{F}_2[u]/(u^3 - 1)$. Using n-block λ-circulant matrices for suitable integers n and units λ, many binary FSD codes (as Gray images) with a higher minimum distance than best known self-dual codes of lengths 34, 40, 44, 54, 58, 70, 72 and 74 were obtained. In particular, ten new even FSD $[72, 36, 14]$ codes were constructed together with eight new near-extremal FSD even codes of length 44 and twenty-five new near-extremal FSD even codes of length 36.

AMS subject classifications: 94B05, 94B99.

Key words: formally self-dual codes, near-extremal codes, circulant codes

1. Introduction

Formally self-dual codes are those that have the same weight enumerators as their duals. Iso-dual codes are codes that are equivalent to their duals. Of course self-dual codes are both iso-dual and formally self-dual but there are formally self-dual codes that are not self-dual. Formally self-dual codes can have better minimum distances than self-dual codes of the same lengths, as will be showcased throughout the paper. Formally self-dual codes work in the exact same way for the Assmus-Mattson theorem as self-dual codes and they have been studied quite extensively in the literature. For shortness of the notation, throughout the paper, we will use the abbreviation FSD to describe formally self-dual codes.

Betsumiya and Harada studied binary optimal odd FSD codes in [2], whereas Dougherty et al. considered optimal FSD codes over $\mathbb{F}_5$ and $\mathbb{F}_7$ in [8]. In [3], a classification of FSD even codes of lengths up to 16 was done. In [10], a classification of extremal even FSD codes of lengths 20 and 22 was done. Extremal double circulant FSD even codes were classified in [13]. Betsumiya and Harada considered FSD codes related to Type II codes in [4]. Near extremal FSD even codes of lengths 24 and 32 and later FSD even codes of lengths divisible by 8 were studied in [14] and [20], respectively. In [6], a classification of extremal even FSD codes of length 30 was

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made. In [11], optimal subcodes of FSD codes of lengths 16-22 were analysed by using their distance profiles.

Parallel to our main methods, some recent studies have shown that good FSD codes can be obtained from FSD codes over certain rings. A family of rings of characteristic 2 was used quite effectively in [19] to obtain FSD codes with better minimum distances than the self-dual ones as well as some near-extremal FSD codes. In [1], constructions were applied to the rings $\mathbb{F}_3 + v\mathbb{F}_3$ and $\mathbb{F}_5 + v\mathbb{F}_5$ to find some good FSD codes over the fields $\mathbb{F}_3$ and $\mathbb{F}_5$ with better minimum distances than the self-dual codes.

Two main construction methods for FSD codes use double circulant and bordered double circulant matrices over the binary field as well as over appropriate rings. In this work, we will generalize these methods by introducing a general block-matrix construction as well as λ-circulant matrices. Our construction methods generalize all the existing methods of constructions and their effectiveness is demonstrated by the highly substantial number of good FSD codes (near-extremal or better than self-dual) that we have obtained. We will apply our methods to three main alphabets, all of characteristic 2, namely the binary field $\mathbb{F}_2$, the ring $R_1 = \mathbb{F}_2 + u\mathbb{F}_2$ and $S = \mathbb{F}_2[u]/(u^3 - 1)$. In the case of S, since the Gray map is not duality-preserving, we will prove the existence of the MacWilliams identities that are preserved under the Gray map, which help us find binary FSD codes as the images of FSD codes over S. Using n-block λ-circulant matrices for suitable integers n and units λ, we were able to obtain many binary FSD codes with a higher minimum distance than best known self-dual codes of lengths 34, 40, 44, 54, 58, 70, 72 and 74. Ten new even $[72, 36, 14]$ codes were also constructed. In addition to these, eight new near-extremal FSD even codes of length 44 and twenty-five new near-extremal FSD even codes of length 36 were obtained.

The rest of the paper is organized as follows: In Section 2, we give the necessary preliminaries about FSD codes as well as the rings over which the constructions are described. In particular, the MacWilliams identities are given for the Lee weight enumerators of codes over S, to demonstrate that the image of FSD codes are binary FSD codes. In Section 3, the main construction methods are described with the proofs. Section 4 includes numerical results, where many good FSD codes of different lengths constructed through our constructions are tabulated.

2. Preliminaries

In this paper, we will work mainly on the binary field $\mathbb{F}_2$ and two of its ring extensions, namely $\mathbb{F}_2 + u\mathbb{F}_2$, where $u^2 = 0$ and $S = \mathbb{F}_2[u]/(u^3 - 1)$. In order to understand the constructions better, we will recall some of the properties of these rings, leaving aside the binary field as an obvious case. We refer the reader to [16] for the definitions and notations on codes.

Let R be one of the rings; $\mathbb{F}_2$, R_1 or S and $\mathcal{C}$ a code of length m over R. The dual of $\mathcal{C}$ is defined as $\mathcal{C}^\perp = \{\overline{y} \in R^m \mid \overline{y} \cdot \overline{x} = 0 \text{ for all } \overline{x} \in \mathcal{C}\}$, where $\cdot$ denotes the Euclidean inner product $\overline{a} \cdot \overline{b} = \sum a_i b_i$. A code $\mathcal{C}$ is said to be self-orthogonal if $\mathcal{C} \subseteq \mathcal{C}^\perp$, self-dual if $\mathcal{C} = \mathcal{C}^\perp$ and iso-dual if $\mathcal{C}$ is permutation equivalent to $\mathcal{C}^\perp$. The code is said to be even if all the codewords have even Lee weight.

Definition 1. *A code $\mathcal{C}$ is called FSD if the code $\mathcal{C}$ and its dual $\mathcal{C}^{\perp}$ have the same weight enumerator. A binary FSD even code of length m is called extremal if $d = 2\lfloor m/4 \rfloor + 2$ and near-extremal if $d = 2\lfloor m/4 \rfloor$.*

The ring $\mathbb{F}_2 + u\mathbb{F}_2 \simeq \mathbb{F}_2[u]/(u^2)$ is a commutative ring of characteristic 2 and it contains the binary field as a subring. It has just one non-trivial ideal, given by (u). Type II, Type IV, self-dual codes and cyclic codes over $\mathbb{F}_2 + u\mathbb{F}_2$ have been studied extensively in [9]. Through its different extensions that appeared later in the literature, we now refer to this ring as R_1.

We recall that a linear code $\mathcal{C}$ of length m over the ring R_1 is an R_1-submodule of R_1^m.

The elements of R_1 are $0, 1, u, 1+u$ and their Lee weights are defined as $0, 1, 2, 1$, respectively.

A Gray map ϕ is defined as $\phi : R_1^m \longrightarrow \mathbb{F}_2^{2m}$

$$\phi\left(\overline{a} + \overline{b}u\right) = \left(\overline{b}, \overline{a} + \overline{b}\right), \tag{1}$$

where $\overline{a}, \overline{b}$ in $\mathbb{F}_2^m$. ϕ is a distance preserving, duality preserving isometry from (R_1^m, d_L) to $\left(\mathbb{F}_2^{2m}, d_H\right)$, where d_L and d_H denote the Lee and Hamming distance in R_1^m and $\mathbb{F}_2^{2m}$, respectively. This means if $\mathcal{C}$ is a linear code over R_1 with parameters $\left[m, 2^k, d\right]$, (here 2^k means the number of the codewords), then $\phi(\mathcal{C})$ is a binary linear code of parameters $[2m, k, d]$. Moreover, $\phi(\mathcal{C}^{\perp}) = \phi(\mathcal{C})^{\perp}$.

Since the Gray map is a duality-preserving map, self-dual codes over R_1 have self-dual binary images, but more importantly for our work we have the following theorem, as a consequence of the duality-preserving property of ϕ:

Theorem 1. *If $\mathcal{C}$ is an iso-dual code over R_1 of length m, then $\phi(\mathcal{C})$ is an iso-dual binary code of length $2m$. If $\mathcal{C}$ is an FSD code over R_1 of length m, then $\phi(\mathcal{C})$ is an FSD binary code of length $2m$.*

2.1. The ring $S = \mathbb{F}_2[u]/(u^3 - 1)$ and the MacWilliams Identity

The ring $\mathbb{F}_2[u]/(u^3 - 1)$, which will be denoted by S here, was used quite effectively in [17] to construct many new binary self-dual codes of length 68. Thus we recall some properties of the ring S from [17], leaving some of the details for the work in question.

We start by observing that the ring is structurally quite familiar, namely $S \cong \mathbb{F}_2 \times \mathbb{F}_4$. S is not local and its ideal structure is given by $I_0 \subset I_{u+u^2}, I_{1+u+u^2} \subset S$, where;

$$\begin{aligned} I_{1+u+u^2} &= \left(1 + u + u^2\right) = \left\{0, 1 + u + u^2\right\}, \\ I_{u+u^2} &= (1 + u) = \left\{0, u + u^2, 1 + u, 1 + u^2\right\}. \end{aligned}$$

The units in S are given by $\left\{1, u, u^2\right\}$ and the non-units are

$$\left\{u + u^2, 1 + u, 1 + u^2, 1 + u + u^2\right\}.$$

A linear code $\mathcal{C}$ of length m over S is an S-submodule of S^m and has a generating matrix that is permutation equivalent to

$$G=\begin{pmatrix} I_{k_1} & A & B & C \\ 0 & \left(u+u^2\right) I_{k_2} & 0 & \left(u+u^2\right) D \\ 0 & 0 & \left(1+u+u^2\right) I_{k_3} & \left(1+u+u^2\right) E \end{pmatrix}$$

We define a Gray map as follows:

$$\begin{aligned} \varphi \ &: \ S^m \rightarrow \mathbb{F}_2^{3m} \\ \overline{a}+\overline{b}u+\overline{c}u^2 &\mapsto \left(\overline{a},\overline{b},\overline{c}\right). \end{aligned}$$

Definition 2. *The Lee weight of an element $x=a+bu+cu^2\in S$ is the Hamming weight of its Gray image, i.e. $w_L\left(a+bu+cu^2\right)=w_H\left(a\right)+w_H\left(b\right)+w_H\left(c\right)$.*

Since the Gray map φ introduced above does not preserve orthogonality, we cannot immediately have a theorem such as Theorem 1. In order to have an analogous result, we need to show that the MacWilliams identities for the Lee weight enumerators of codes over S exist. We will quickly establish this by using Wood's results in [21].

Let us label the elements in $S=\{g_1,g_2,\ldots,g_8\}$ as $S=\{0,u^2,u,u+u^2,1,1+u^2,1+u,1+u+u^2\}$

Definition 3. *The complete weight enumerator of a linear code $\mathcal{C}$ over S is defined as*

$$cwe_{\mathcal{C}}(X_1,X_2,\ldots,X_8)=\sum_{\overline{c}\in\mathcal{C}} X_1^{n_{g_1}(\overline{c})}X_2^{n_{g_2}(\overline{c})}\cdots X_8^{n_{g_8}(\overline{c})},$$

where $n_{g_i}(\overline{c})$ is the number of appearances of g_i in the vector $\overline{c}$.

The complete weight enumerator gives us a lot of information about the code, such as the size of the code, the minimum weight of the code and the weight enumerator of the code for any weight function.

Now, since S is a Frobenius ring, the MacWilliams identities for the complete weight enumerator hold. To find the exact identities we define the following character on S :

Definition 4. *Define $\chi:S\rightarrow\mathbb{C}^\times$ by*

$$\chi(a+bu+cu^2)=(-1)^c.$$

It is easy to verify that χ is a non-trivial character when restricted to each nonzero ideal, hence it is a generating character for S.

Then we make up an 8×8 matrix T, by letting $T(i,j)=\chi(g_ig_j)$. The matrix T is given as follows:

$$T=\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 \\ 1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 \\ 1 & 1 & -1 & -1 & -1 & -1 & 1 & 1 \\ 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 \\ 1 & -1 & 1 & -1 & -1 & 1 & -1 & 1 \\ 1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 \\ 1 & -1 & -1 & 1 & -1 & 1 & 1 & -1 \end{bmatrix}.$$

The following theorem then follows from [21] (Theorem 8.1) quite easily:

Theorem 2. *Let $\mathcal{C}$ be a linear code over S of length m and suppose $\mathcal{C}^\perp$ is its dual. Then we have*

$$cwe_{\mathcal{C}^\perp}(X_1, X_2, \ldots, X_8) = \frac{1}{|\mathcal{C}|} cwe_{\mathcal{C}}(T \cdot (X_1, X_2, \ldots, X_8)^t),$$

where $()^t$ denotes the transpose.

Now, in S, we have $w_L(0) = 0$; $w_L(1) = w_L(u) = w_L(u^2) = 1$; $w_L(1+u) = w_L(1+u^2) = w_L(u+u^2) = 2$ and $w_L(1+u+u^2) = 3$. Thus, identifying the variables corresponding to the elements with the same weight we get the symmetrized weight enumerator, namely,

$$swe_{\mathcal{C}}(X, Y, Z, W) = cwe_{\mathcal{C}}(X, Y, Y, Z, Y, Z, Z, W). \tag{2}$$

Now, combining Theorem 2 and the definition of the symmetrized weight enumerator, we obtain the following theorem:

Theorem 3. *Let $\mathcal{C}$ be a linear code over S of length m and let $\mathcal{C}^\perp$ be its dual. Then we have*

$$swe_{\mathcal{C}^\perp}(X, Y, Z, W) =$$
$$\frac{1}{|\mathcal{C}|} swe_{\mathcal{C}}(X + 3Y + 3Z + W, X + Y - Z - W, X - Y - Z + W, X - 3Y + 3Z - W).$$

We are now ready to obtain the MacWilliams identity for the Lee weight enumerator. Define the Lee weight enumerator for codes over S as the homogeneous polynomial

$$Lee_{\mathcal{C}}(W, X) = \sum_{\overline{c} \in \mathcal{C}} W^{3n - w_L(\overline{c})} X^{w_L(\overline{c})}. \tag{3}$$

Considering the weights that the variables X, Y, Z, W of the symmetrized weight enumerator represent, we easily get the following theorem:

Theorem 4. *Let $\mathcal{C}$ be a linear code over S of length m. Then*

$$Lee_{\mathcal{C}}(W, X) = swe_{\mathcal{C}}(W^3, W^2X, WX^2, X^3).$$

Now combining Theorem 3 and Theorem 4 we obtain the following theorem:

Theorem 5. *Let $\mathcal{C}$ be a linear code over S of length m and let $\mathcal{C}^\perp$ be its dual. With $Lee_{\mathcal{C}}(W, X)$ denoting its Lee weight enumerator as given in (3), we have*

$$Lee_{\mathcal{C}^\perp}(W, X) = \frac{1}{|\mathcal{C}|} Lee_{\mathcal{C}}(W + X, W - X).$$

Note that this is exactly the same identity that is satisfied by the Hamming weight enumerators of binary codes. The existence of the identity gives us the following corollary that is of central importance for our work:

Corollary 1. *If $\mathcal{C}$ is an FSD code over S of length m, then $\varphi(\mathcal{C})$ is a binary FSD code of length $3m$. Moreover, the Lee weight enumerator of $\mathcal{C}$ is the same as the Hamming weight enumerator of $\varphi(\mathcal{C})$.*

3. Constructions

In this section, we introduce the construction methods for FSD codes over commutative Frobenius rings by block λ-circulant matrices.

Let R be a commutative Frobenius ring and $v=(v_1,v_2,\ldots,v_m)$ an element of the R-module R^m. The λ-cyclic shift of v is defined as $\sigma_\lambda(v)=(\lambda v_m,v_1,v_2,\ldots,v_{m-1})$, where $\lambda\in R$. An $m\times m$ square matrix A is called λ-circulant if every row is a λ-cyclic shift of the previous one, in other words, A is in the following form;

$$\begin{pmatrix} a_1 & a_2 & a_3 & \cdots & a_m \\ \lambda a_m & a_1 & a_2 & \cdots & a_{n-1} \\ \lambda a_{m-1} & \lambda a_m & a_1 & \cdots & a_{m-2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \lambda a_2 & \lambda a_3 & \lambda a_4 & \cdots & a_1 \end{pmatrix}.$$

If $\lambda=1$ the matrix is simply called circulant. Recently, in [19], Karadeniz et al. applied the double and bordered double circulant constructions for a family of commutative Frobenius rings of characteristic 2. The constructions were generalized to double λ-circulant codes in [1]. λ-circulant matrices share properties of circulant matrices such as they commute with each other and they are permutation equivalent to their transposes.

Theorem 6. *Let A_i be a λ-circulant matrix of size $m\times m$ for $1\le i\le n$ over R. Suppose that there is a weight function on R such that $wt(a)=wt(-a)$ for all $a\in R$. Then the code $\mathcal{C}$ generated by*

$$G=\left[I_{mn}\,\middle|M\right] \text{ where } M=\begin{bmatrix} A_1 & A_2 & \cdots & A_n \\ A_n & A_1 & \cdots & A_{n-1} \\ \vdots & \vdots & \ddots & \vdots \\ A_2 & \cdots & A_n & A_1 \end{bmatrix} \tag{4}$$

is an FSD code over R. Moreover, $\mathcal{C}$ is iso-dual if char$(R)=2$. The code is called an n-block λ-circulant code.

Proof. Let $\mathcal{C}$ be the code generated by G. Then the dual $\mathcal{C}^\perp$ of $\mathcal{C}$ is generated by

$$G'=\left[\,M^t\middle|-I_{mn}\right].$$

Let Π_n be the permutation

$$\Pi_n=(1,n)(2,n-1)\cdots(k-1,n-k+2)(k,n-k+1),$$

where $k=\lfloor n/2\rfloor$. It is easily observed that A_i and A_i^t are permutation equivalent since when Π_m is applied to rows and then to columns of A_i, we obtain A_i^t. So the matrix

$$M'=\begin{bmatrix} A_1^t & A_2^t & \cdots & A_n^t \\ A_n^t & A_1^t & \cdots & A_{n-1}^t \\ \vdots & \vdots & \ddots & \vdots \\ A_2^t & \cdots & A_n^t & A_1^t \end{bmatrix}$$

is obtained by applying Π_m to each block of M. Then M^t is obtained by applying Π_n to M' block-wise. Let $\mathcal{C}''$ be the code generated by $G'' = \left[\, M^T \middle| I_{mn} \,\right]$. The matrices G and G'' are permutation equivalent and so are the codes they generate. On the other hand, $\mathcal{C}^\perp$ and $\mathcal{C}''$ have the same weight enumerator since $wt(a) = wt(-a)$ for each $a \in R$. Hence $\mathcal{C}$ is formally self-dual. Moreover, $\mathcal{C}$ is isodual if R is a ring of characteristic 2 since $\mathcal{C}^\perp = \mathcal{C}''$. $\square$

Theorem 7. *Let M and R be as in (4); $a, b \in R$ and let $\boldsymbol{b}$ denote the all-b row of length mn. Then the code generated by*

$$\left[I_{mn+1} \middle| \begin{matrix} a & \boldsymbol{b} \\ \boldsymbol{b}^t & M \end{matrix} \right]$$

is an FSD code over R. The code is iso-dual when char $(R) = 2$ and it is called a bordered n-block λ-circulant code.

Proof. Let $\mathcal{C}$ be the code generated by G. Then the dual $\mathcal{C}^\perp$ of $\mathcal{C}$ is generated by

$$G' = \left[\begin{matrix} a & \boldsymbol{b} \\ \boldsymbol{b}^t & M^t \end{matrix} \middle| -I_{mn+1} \right]$$

Applying the same permutations as in the proof of Theorem 6, we see that C is equivalent to C'', which is generated by

$$G'' = \left[\begin{matrix} a & \boldsymbol{b} \\ \boldsymbol{b}^t & M^t \end{matrix} \middle| I_{mn+1} \right].$$

On the other hand, $\mathcal{C}^\perp$ and $\mathcal{C}''$ have the same weight enumerator since $wt(a) = wt(-a)$ for each $a \in R$. Hence $\mathcal{C}$ is formally self-dual. Moreover, $\mathcal{C}$ is isodual if R is a ring of characteristic 2 since $\mathcal{C}^\perp = \mathcal{C}''$. $\square$

The constructions given in theorems 6 and 7 can be generalized further by considering λ_0-circulating blocks of matrices. Such a code is called λ_0-n-block λ-circulant code. We illustrate this construction in the following example;

Example 1. *Let $R = R_1$ and $r_1 = (1+u, u)$, $r_2 = (1, 1+u)$, $r_3 = (1+u, 1+u)$ and $r_4 = (0, u)$ be the first rows of circulant matrices A_1, A_2, A_3 and A_4, respectively. Let $\mathcal{C}$ be the λ_0 block circulant code over R_1 generated by*

$$G = \left[I_8 \middle| \begin{matrix} A_1 & A_2 & A_3 & A_4 \\ \lambda_0 A_4 & A_1 & A_2 & A_3 \\ \lambda_0 A_3 & \lambda_0 A_4 & A_1 & A_2 \\ \lambda_0 A_2 & \lambda_0 A_3 & \lambda_0 A_4 & A_1 \end{matrix} \right],$$

where $\lambda_0 = 1+u$. Then $\mathcal{C}$ is an iso-dual code. The binary code $\varphi(\mathcal{C})$ is a binary even FSD code of parameters $[32, 16, 8]$, which is optimal. A partial weight distribution of the code is given by $1 + 364z^8 + 2048z^{10} + 6720z^{12} + \cdots$. Even FSD codes of these parameters have been studied extensively in [14].

More examples for the generalization of Theorem 7 are given in Tables 3,4 and 6.

4. Computational results

In this section, the methods from Section 3 are applied to the rings $\mathbb{F}_2$, R_1 and S. FSD codes with parameters better than self-dual codes are tabulated. New near-extremal FSD even codes are constructed for lengths 36, 38 and 44. For the rest of the paper $(.)^*$ is used to denote that the code is optimal as a linear code and $(.)^b$ is used to denote that the minimum distance of the code is the distance of the best known linear code with respect to the online database [12].

Remark 1. *In the tables that follow, n denotes the number of blocks, m denotes the length of each block, a and b denote the borders of the block circulant matrix (In the case of bordered double circulant constructions given in Theorem 7), A_i is a λ-circulant matrix for each $1 \leq i \leq n$ and λ_0 is as in Example 1. The order of the automorphism group of $\mathcal{C}$ is denoted by $|Aut(\mathcal{C})|$. In order to fit the upcoming tables we denote the element $1+u$ of R_1 as 3 and $1+u+u^2$ of S as 7. All computations were run on an average PC, using the magma algebra system ([5]).*

Example 2. *Let A_1 and A_2 be 2×2 $(1+u)$-circulant matrices over R_1 with first rows $r_1 = (1, 1+u)$ and $r_2 = (1+u, u)$. Then the bordered 2-block $(1+u)$-circulant code generated by*

$$G = \left[I_5 \left| \begin{array}{ccccc} u & 1 & 1 & 1 & 1 \\ 1 & 1 & 1+u & 1+u & u \\ 1 & 1 & 1 & u & 1+u \\ 1 & 1+u & u & 1 & 1+u \\ 1 & u & 1+u & 1 & 1 \end{array} \right. \right]$$

is an iso-dual code over R_1. The binary image of the code is an odd binary FSD $[20, 10, 6]^*$ code that is optimal as a linear code. A partial weight distribution of the code is $1+40z^6+160z^7+130z^8+\cdots$ and its automorphism group has order $2^8 \times 3 \times 5$.

Example 3. *Let $\mathcal{C}$ be the pure double circulant code with the first row*

$$r = (11000100101111110100111).$$

Then $\mathcal{C}$ is an iso-dual code by Theorem 6. The code $\mathcal{C}$ is an optimal linear $[46, 23, 11]^*$ code and a partial weight distribution of $\mathcal{C}$ is given by

$$1 + 3312z^{11} + 9660z^{12} + 121440z^{15} + 235290z^{16} + \cdots.$$

With respect to the online database of linear codes, [12], a binary linear $[46, 23, 11]^*$ code is constructed as a shortened puncturing of the unique Type II $[48, 24, 12]^*$. The code $\mathcal{C}$ is equivalent to the one kept in [12].

4.1. Binary even FSD $[72, 36, 14]$ codes

The existence of a Type II binary self-dual $[72, 36, 16]$ code is an open problem that has generated a lot of interest in the last four decades. Equivalent to the same problem is the existence of a binary Type I self-dual $[72, 36, 14]$ code. The best known binary self-dual codes of length 72 have minimum distance 12, see [7, 15].

The first examples of binary $[72, 36, 14]$ FSD even codes were constructed in [19]. In this section, we construct 10 binary FSD even $[72, 36, 14]$ codes, which are different than the codes given in [19]. The results are given in Table 1.

Example 4. *Let $\lambda = 1 + u$, $n = 1$ and let $\mathcal{C}$ be the double λ-circulant code with the first row $r_1 = (u, 0, u, u, 0, u, 0, 0, 0, u + 1, 0, 1, u + 1, u + 1, 1, 0, 1, 1,)$ over R_1. Then $\mathcal{C}$ is an iso-dual code. The Gray image $\varphi(\mathcal{C})$ of $\mathcal{C}$ is a binary FSD even code of parameters $[72, 36, 14]$. A partial weight enumerator of the code is given by* $1 + 9036z^{14} + 121959z^{16} \cdots$.

n	$r_1, \ldots, r_n$	Partial weight distribution
4	000110011, 111101100, 101111100, 100011011	$1 + 9144z^{14} + 120897z^{16} + \cdots$
4	000100101, 011101011, 101000000, 110110000	$1 + 8748z^{14} + 123525z^{16} + \cdots$
4	101110001, 000100010, 001010011, 101011101	$1 + 8748z^{14} + 123660z^{16} + \cdots$
4	000100101, 011101011, 000101111, 000100110	$1 + 9072z^{14} + 121383z^{16} + \cdots$
4	000100110, 001010101, 101000000, 110110000	$1 + 8640z^{14} + 124479z^{16} + \cdots$
3	111100111010, 110101101100, 111011001011	$1 + 8640z^{14} + 125433z^{16} + \cdots$
3	000111110111, 001101011000, 100110001101	$1 + 9288z^{14} + 120537z^{16} + \cdots$
1	100100010000010111111111000111010000	$1 + 8820z^{14} + 122841z^{16} + \cdots$
1	100100010000010111000011000100111011	$1 + 8748z^{14} + 123831z^{16} + \cdots$

Table 1: *Binary FSD $[72, 36, 14]$ codes as n-block circulant codes over $\mathbb{F}_2$*

4.2. New near extremal binary FSD even codes

Extremal binary FSD even codes do not exist for many lengths and the non-existence of near-extremal binary FSD even codes have been studied in [18]. In [13], possible weight enumerators of near-extremal binary FSD even codes of lengths 16 to 46 has been determined. In this section, we construct new binary near-extremal FSD even codes for lengths 36, 38 and 44, where possible weight enumerators are;

$$
\begin{aligned}
W_{36} &= 1 + (225 + \alpha)\, y^8 + (2016 - 6\alpha)\, y^{10} + \cdots, \\
W_{38} &= 1 + (171 + \alpha)\, y^8 + (1862 - 5\alpha)\, y^{10} + \cdots, \\
W_{44} &= 1 + (1320 + \alpha)\, z^{10} + (10461 - 8\alpha)\, z^{12} + \cdots.
\end{aligned}
$$

The existence of near-extremal binary FSD even codes is known for weight enumerators; α =0, 9, 13, 18, 27, 36, 45, 47, 63, 64, 72, −4, −9, −21, −27, −36, −38, −45, −54, −55 and −72 for W_{36}; α =0, 22, 31, 44, 66, 176, −19, −22, −44, −53, −66, −176 for W_{44}.

In this work, we were able to construct binary near-extremal FSD even codes with weight enumerators; α =3, 6, 12, 15, 21, 24, 30, 33, 39, 42, 48, 60, 96, 144, −3, −6, −12, −15, −18, −24, −30, −33, −42, −144 in W_{36}; α =10, 30, 50, 80, −10, −11, −30 and −70 in W_{44}.

Example 5. *For $n = 1$ and $\lambda = 1$, let $\mathcal{C}$ be the double circulant code over S of length 12 with the first row of A_1 as $r_1 = \left(1, u^2, 1+u, u+u^2, 1, 1+u^2\right)$. Then $\mathcal{C}$ is iso-dual by Theorem 6. The Gray image of $\mathcal{C}$ is a binary near-extremal FSD even code with weight enumerator $\alpha = 144$ in W_{36}. The weight enumerator of the code is different than the ones in [13].*

More new near extremal FSD even codes are constructed in Table 2.

λ	r_1	a, b	$\lvert Aut(\mathcal{C})\rvert$	α
1	$\left(u, 1+u^2, 0, 7, 1+u^2\right)$	$1+u^2, u$	30	3
1	$\left(1+u^2, 1+u^2, u, 0, 7\right)$	$1+u^2, 1$	15	6
u	$\left(0, 1+u^2, 1, u+u^2, u\right)$	$u+u^2, u$	3	12
u	$\left(1+u, 7, 0, 1, 0\right)$	$1+u, 1$	3	15
u	$\left(u, 0, u+u^2, u^2, 1+u\right)$	$1+u^2, 1$	3	21
u	$\left(1+u^2, u^2, u, u^2, 1\right)$	$u+u^2, 1$	3	24
u	$\left(1, 1, 0, u^2, 7\right)$	$1+u^2, u^2$	3	30
1	$\left(u^2, 0, 1+u^2, 1, 1+u\right)$	$1+u^2, 1$	15	33
u	$\left(1+u, 7, 7, 0, 1+u^2\right)$	$1+u, 1$	3	39
u	$\left(1+u^2, 7, 1, u^2, u^2\right)$	$1+u, u$	3	42
1	$\left(u, u^2, u, 1+u^2, u\right)$	$u+u^2, u$	15	48
u	$\left(7, u^2, 1+u^2, 7, 1\right)$	$1+u, u$	3	60
1	$\left(u+u^2, u+u^2, 1, 1, 1+u^2\right)$	$1+u, u^2$	60	96
u	$\left(7, u, u+u^2, 0, 0\right)$	$1+u^2, u$	3	-3
u	$\left(0, u^2, 1, u, 7\right)$	$1+u^2, 1$	3	-6
1	$\left(0, 7, 1+u^2, 0, 1\right)$	$u+u^2, u$	15	-12
u	$\left(1+u, u, 1+u, 0, u^2\right)$	$1+u^2, u$	3	-15
u	$\left(7, u, 1+u, 1, 7\right)$	$1+u, u^2$	3	-18
1	$\left(1+u^2, u, 1, 1+u^2, 0\right)$	$1+u, u$	15	-24
u	$\left(7, 0, u, u, 1\right)$	$1+u^2, u$	3	-30
u	$\left(1+u^2, u+u^2, 7, u, 0\right)$	$1+u, u$	3	-33
1	$\left(1+u, 1+u, u, u^2, 0\right)$	$1+u^2, u^2$	15	-42
1	$\left(1+u^2, 1+u^2, u, 1+u, u\right)$	$1+u^2, u^2$	1440	-144

Table 2: *24 new near extremal FSD even codes of length 36 over S by Theorem 7*

For a list of known near-extremal FSD even codes of length 38 we refer to [13]. In the following example we construct a code with a new weight enumerator by applying bordered 2-block circulant construction on $\mathbb{F}_2$.

Example 6. *Let $\mathcal{C}$ be the bordered 2-block circulant code with $r_1 = (111100111)$, $r_2 = (101011100)$ and $a = 1 = b$ over $\mathbb{F}_2$. The code $\mathcal{C}$ is an even code and it is FSD by Theorem 7. $\lvert Aut(\mathcal{C})\rvert = 2 \times 3^2$ and $\mathcal{C}$ has a weight enumerator with $\alpha = -9$ in W_{38}.*

Example 7. *Let $\mathcal{C}$ be the binary bordered 3-block circulant code with $r_1 = (1110101)$, $r_2 = (1100100)$, $r_3 = (0010100)$, $a = 0$ and $b = 1$. Then $\mathcal{C}$ is a near-extremal FSD* *even code with weight enumerator $\alpha = -11$ in W_{44}. The existence of an FSD code with this weight enumerator was unknown. The automorphism group of the code has order 3×7.*

We were able to construct seven near extremal binary FSD even codes of length 44 with new weight enumerators as Gray images of FSD codes over R_1, which are listed in Table 3.

n	λ_0	λ	$r_1, \ldots, r_n$	a, b	$\lvert Aut(\mathcal{C})\rvert$	α
2	3	1	$(u01u3), (03u01)$	$3, 3$	2×5	30
2	3	1	$(0330u), (33101)$	$3, 3$	2×5	50
5	1	1	$(13), (10), (uu), (3u)\,(33)$	$1, 1$	$2^2 5$	10
5	1	1	$(30), (11), (1u), (u3)\,(u1)$	$1, 1$	$2^2 5$	80
5	1	1	$(u0), (u3), (03), (11)\,(13)$	$1, 1$	$2^2 5$	-10
5	1	1	$(3u), (0u), (1u), (uu)\,(13)$	$1, 1$	$2^2 5$	-30
5	1	1	$(3u), (03), (33), (13)\,(u0)$	$1, 1$	$2^2 5$	-70

Table 3: *Optimal* $[44, 22, 10]$ *FSD even codes as images of bordered n-block circulant codes over R_1*

Remark 2. *Throughout the paper the binary FSD codes that have been listed have better minimum distances than the best known self-dual codes, except for the ones listed in Tables 2 and 10, which have the same distance as the corresponding extremal self-dual code.*

Theorem 6 is applied over $\mathbb{F}_2$ in Table 8. Its generalization is applied over R_1 and S in Table 4 and in Table 6, respectively. Theorem 7 is applied in Tables 5, 7, 9 and 10.

λ_0	λ	r_1	r_2	Partial weight distribution
1	1	(01113)	$(013u0)$	$1 + 280z^9 + 1052z^{10} + \cdots$
1	1	$(1u0uu)$	$(1330u)$	$1 + 320z^9 + 992z^{10} + \cdots$
1	1	$(u1u03)$	$(u303u)$	$1 + 340z^9 + 972z^{10} + \cdots$
3	1	$(1013u)$	$(3u303)$	$1 + 260z^9 + 1030z^{10} + \cdots$
3	1	$(11u01)$	(33100)	$1 + 280z^9 + 1010z^{10} + \cdots$
3	1	$(10u33)$	$(u1033)$	$1 + 300z^9 + 1020z^{10} + \cdots$
3	1	$(u3u00)$	$(110u1)$	$1 + 320z^9 + 1000z^{10} + \cdots$
3	1	$(u3311)$	(00013)	$1 + 320z^9 + 982z^{10} + \cdots$
3	1	$(0u31u)$	$(1111u)$	$1 + 360z^9 + 922z^{10} + \cdots$
3	3	(01133)	$(u31uu)$	$1 + 340z^9 + 982z^{10} + \cdots$
3	3	$(03u30)$	$(u1331)$	$1 + 240z^9 + 1166z^{10} + \cdots$
3	3	$(00u33)$	$(3u03u)$	$1 + 300z^9 + 1022z^{10} + \cdots$
3	3	$(1030u)$	(01311)	$1 + 320z^9 + 1012z^{10} + \cdots$

Table 4: *Binary* $[40, 20, 9]^b$ *FSD codes as Gray images of* 2*-block circulant codes over R_1*

n	m	$r_1,\ldots,r_n$	a,b	$\lvert Aut(\mathcal{C})\rvert$	Partial weight distribution
2	8	$11000001, 11010100$	$0,1$	2^6	$1+172z^8+352z^9+\cdots$
2	8	$10001010, 10110001$	$0,1$	2^7	$1+204z^8+384z^9+\cdots$
2	8	$10001101, 00011001$	$1,1$	2^6	$1+220z^8+384z^9+\cdots$
2	8	$10001011, 11101111$	$0,1$	2^6	$1+236z^8+416z^9+\cdots$
2	8	$10111110, 10000101$	$0,1$	2^7	$1+268z^8+448z^9+\cdots$
4	4	$1110, 1000, 0110, 0111$	$0,1$	$2^9 3$	$1+140z^8+320z^9+\cdots$
4	4	$1001, 1000, 1000, 0111$	$0,1$	$2^9 3$	$1+204z^8+512z^9+\cdots$
4	4	$1010, 0110, 0001, 1100$	$0,1$	$2^9 5^2 3$	$1+332z^8+640z^9+\cdots$
1	16	1000101110011100	$0,1$	2^4	$1+192z^8+298z^9+\cdots$
1	16	0111001100101101	$0,1$	2^4	$1+200z^8+380z^9+\cdots$
1	16	1100101001101001	$0,1$	2^4	$1+208z^8+282z^9+\cdots$
1	16	0001111011101111	$0,1$	2^5	$1+208z^8+388z^9+\cdots$
1	16	1110011000111110	$0,1$	2^4	$1+210z^8+284z^9+\cdots$
1	16	1000101110110000	$0,1$	2^4	$1+212z^8+392z^9+\cdots$
1	16	1100001111101001	$1,1$	2^7	$1+220z^8+384z^9+\cdots$
1	16	0110111010000110	$0,1$	2^4	$1+224z^8+266z^9+\cdots$
1	16	1010111011110010	$0,1$	2^4	$1+224z^8+282z^9+\cdots$
1	16	0101010000011001	$0,1$	2^4	$1+240z^8+266z^9+\cdots$
1	17	10101110111110110	pure	17	$1+153z^8+527z^9+\cdots$
1	17	01111110100011111	pure	17	$1+170z^8+527z^9+\cdots$
1	17	11101000001001011	pure	17	$1+187z^8+459z^9+\cdots$
1	17	11001111111100110	pure	2×17	$1+187z^8+493z^9+\cdots$
1	17	11011101000100100	pure	17	$1+204z^8+442z^9+\cdots$
1	17	11111010111011001	pure	17	$1+221z^8+408z^9+\cdots$
1	17	10011011011000010	pure	17	$1+255z^8+816z^9+\cdots$

Table 5: *Optimal binary linear* $[34, 17, 8]$ *codes as odd FSD codes over* $\mathbb{F}_2$

n	λ_0	λ	$r_1,\ldots,r_n$	Partial weight distribution
3	u^2	1	$(1+u,u,u), (u^2,u+u^2,0), (1+u,1,u+u^2)$	$1+729z^{11}+3096z^{12}+\cdots$
3	u^2	1	$(1+u,1+u^2,0), (1+u,0,1), (u^2,1,7)$	$1+864z^{11}+3123z^{12}+\cdots$
3	1	1	$(1,1,0), (1,0,1+u^2), (u^2,u+u^2,1+u^2)$	$1+1080z^{11}+5706z^{12}+\cdots$
3	1	1	$(0,7,1+u), (1+u,1+u^2,1+u^2), (u,1,u)$	$1+1404z^{11}+5031z^{12}+\cdots$
1	-	u	$(1+u,1+u^2,1+u^2,1,u+u^2,7,u+u^2,u,1)$	$1+621z^{11}+2970z^{12}+\cdots$
1	-	u	$(u+u^2,u^2,1,u+u^2,u,u+u^2,1,u+u^2,u+u^2)$	$1+648z^{11}+3051z^{12}+\cdots$
1	-	u	$(1,1,u,u,u,1+u^2,1+u+u^2,1+u^2,0)$	$1+729z^{11}+3051z^{12}+\cdots$
1	-	u	$(u^2,u+u^2,u+u^2,7,1+u,u,u+u^2,1+u,7,u)$	$1+864z^{11}+2916z^{12}+\cdots$
1	-	u	$(1+u,1+u^2,1+u^2,1,u+u^2,1+u,u,1,1)$	$1+918z^{11}+3132z^{12}+\cdots$
1	-	u^2	$(1,1+u^2,1,7,u+u^2,u,1,7,1+u)$	$1+891z^{11}+2889z^{12}+\cdots$

Table 6: *Binary FSD* $[54, 27, 11]^b$ *codes as Gray images of FSD codes over* S

r_1	r_2	r_3	r_4	Partial weight distribution
1111001	1011001	0111111	1000101	$1 + 3290z^{12} + 40565z^{14} + \cdots$
1011110	0000110	0011011	1111111	$1 + 3388z^{12} + 39879z^{14} + \cdots$
1001111	0111011	0011000	0101000	$1 + 3395z^{12} + 39830z^{14} + \cdots$
0000100	0100100	1000101	1001110	$1 + 3402z^{12} + 39783z^{14} + \cdots$
0001001	1101111	1000101	1110011	$1 + 3437z^{12} + 39536z^{14} + \cdots$
0111111	0100000	0111001	1001111	$1 + 3437z^{12} + 39482z^{14} + \cdots$
0011100	1010000	0011111	0011110	$1 + 3500z^{12} + 39097z^{14} + \cdots$
1001100	0110000	0000101	1001100	$1 + 3500z^{12} + 39095z^{14} + \cdots$
1000110	0011110	1011111	1001100	$1 + 3507z^{12} + 38990z^{14} + \cdots$
1111001	1010000	1010100	0011110	$1 + 3542z^{12} + 38803z^{14} + \cdots$
1001011	1010010	1000111	0111011	$1 + 3563z^{12} + 38600z^{14} + \cdots$
1110011	1001110	0000110	0000001	$1 + 3591z^{12} + 38402z^{14} + \cdots$
0101100	0101000	1111010	1001011	$1 + 3598z^{12} + 38411z^{14} + \cdots$
0011000	0100110	0110110	0111000	$1 + 3619z^{12} + 38208z^{14} + \cdots$
1100001	0101000	1010001	1000001	$1 + 3640z^{12} + 38061z^{14} + \cdots$
1101110	1000100	0101100	0101101	$1 + 3668z^{12} + 37921z^{14} + \cdots$
0101100	0010011	0111001	0011110	$1 + 3703z^{12} + 37674z^{14} + \cdots$
1110101	1100101	0110111	0111010	$1 + 3745z^{12} + 37382z^{14} + \cdots$
0001010	1000000	1000011	1101010	$1 + 3766z^{12} + 37177z^{14} + \cdots$
1010111	1010001	1010011	0110110	$1 + 3850z^{12} + 36589z^{14} + \cdots$

Table 7: *Binary* $[58, 29, 12]^b$ *FSD codes as bordered* 4*-block circulant codes over* $\mathbb{F}_2$ *with* $a = 1 = b$

r_1	r_2	r_3	r_4	r_5	Partial weight distribution
1000111	0101010	0100110	0100001	0111001	$1{+}910z^{13}{+}5880z^{14}{+}\cdots$
1110011	0001000	1000110	0100110	0000011	$1{+}1120z^{13}{+}5740z^{14}{+}\cdots$
1101101	0101000	0011010	0100010	1100110	$1{+}1155z^{13}{+}6405z^{14}{+}\cdots$
0110111	1001110	0001000	0001000	0110100	$1{+}1295z^{13}{+}5810z^{14}{+}\cdots$
0010101	1111000	1001100	1111001	1001001	$1{+}1295z^{13}{+}5635z^{14}{+}\cdots$
1100100	1110010	0001110	0110001	1010111	$1{+}1400z^{13}{+}6020z^{14}{+}\cdots$
0111100	0111101	1110100	1010101	1001100	$1{+}1435z^{13}{+}5565z^{14}{+}\cdots$
1010011	0010010	1001100	1100010	0010111	$1{+}1505z^{13}{+}5670z^{14}{+}\cdots$
1001111	0010110	1000000	0100110	0101110	$1{+}1505z^{13}{+}6125z^{14}{+}\cdots$
0010111	1001111	0011011	0100110	1111000	$1{+}1540z^{13}{+}5530z^{14}{+}\cdots$

Table 8: *Binary* $[70, 35, 13]$ *FSD codes as 5-block circulant codes over* $\mathbb{F}_2$

n	m	$r_1,\ldots,r_n$	Partial weight distribution
4	9	$101000100, 101111100, 010001001, 011100100$	$1+6681z^{14}+98703z^{16}+\cdots$
4	9	$100001110, 110100110, 000011001, 001000010$	$1+6714z^{14}+98538z^{16}+\cdots$
4	9	$100111101, 111010111, 101000000, 101100110$	$1+6723z^{14}+98673z^{16}+\cdots$
4	9	$111011010, 010010001, 111001000, 110100000$	$1+6876z^{14}+97980z^{16}+\cdots$
4	9	$000111101, 001011001, 011100111, 111011000$	$1+7272z^{14}+95568z^{16}+\cdots$
3	12	$001011000110, 101011001001, 000100000000$	$1+6078z^{14}+102177z^{16}+\cdots$
3	12	$000100001111, 100101010110, 111000101011$	$1+6222z^{14}+102177z^{16}+\cdots$
3	12	$111101000001, 110110100101, 010100100011$	$1+6270z^{14}+100497z^{16}+\cdots$
3	12	$001001100101, 000000100110, 010111110101$	$1+6618z^{14}+99189z^{16}+\cdots$
3	12	$000100011101, 010010001100, 101101010110$	$1+7374z^{14}+95553z^{16}+\cdots$
2	18	$101001111011011100, 010110010001101110$	$1+6390z^{14}+100446z^{16}+\cdots$
2	18	$101000100111110001, 010110011100111011$	$1+6552z^{14}+99906z^{16}+\cdots$
2	18	$100011101000011000, 011111010001010010$	$1+6732z^{14}+98538z^{16}+\cdots$
2	18	$010111001111100011, 011010100101111000$	$1+6930z^{14}+98088z^{16}+\cdots$
2	18	$001101101010011100, 110111011100101100$	$1+6957z^{14}+97035z^{16}+\cdots$

Table 9: *Bordered n-block circulant $[74,37,14]^b$ FSD codes over $\mathbb{F}_2$ with $a=1=b$*

n	m	$r_1,\ldots,r_n$	a,b	Partial weight distribution
2	9	$(u03033u00),(01100u131)$	$3,1$	$1+4518z^{14}+80364z^{16}+\cdots$
2	9	$(u10013303),(u00111uu0)$	$1,1$	$1+4554z^{14}+81072z^{16}+\cdots$
2	9	$(3u1011111),(u3u331010)$	$3,1$	$1+4740z^{14}+79836z^{16}+\cdots$
2	9	$(0u03013u3),(1uu33311u)$	$1,3$	$1+5202z^{14}+77610z^{16}+\cdots$
2	9	$(133uu0310),(3u310u101)$	$3,1$	$1+5208z^{14}+77916z^{16}+\cdots$
2	9	$(uuu003131),(0010uuuu1)$	$1,1$	$1+5262z^{14}+77232z^{16}+\cdots$
2	9	$(03u110100),(111113033)$	$3,1$	$1+5340z^{14}+77526z^{16}+\cdots$
3	6	$(11303u),(30u3u3),(330131)$	$3,1$	$1+4206z^{14}+82047z^{16}+\cdots$
3	6	$(uu0133),(u31u10),(00u0u0)$	$3,3$	$1+4716z^{14}+80295z^{16}+\cdots$
3	6	$(103uu3),(31u1u0),(1101u3)$	$1,1$	$1+4842z^{14}+78663z^{16}+\cdots$
3	6	$(331033),(31u010),(03u113)$	$3,3$	$1+4866z^{14}+79239z^{16}+\cdots$
3	6	$(111313),(u033u1),(0u311u)$	$1,1$	$1+4866z^{14}+79455z^{16}+\cdots$
3	6	$(111300),(1001u3),(u1303u)$	$1,3$	$1+5778z^{14}+75711z^{16}+\cdots$

Table 10: *Binary $[76,38,14]^b$ FSD codes as Gray images of bordered n-block circulant codes over R_1*

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