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# "Almost" universality of the Lerch zeta-function*

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**Abstract.** The Lerch zeta-function $L(\lambda, \alpha, s)$ with a transcendental parameter $\alpha$, or with rational parameters $\alpha$ and $\lambda$ is universal, i.e., a wide class of analytic functions is approximated by shifts $L(\lambda, \alpha, s + i\tau)$, $\tau \in \mathbb{R}$. The case of an algebraic irrational $\alpha$ is an open problem. In the paper, it is proved that for all parameters $\alpha$, $0 < \alpha < 1$, and $\lambda$, $0 < \lambda \leqslant 1$, including an algebraic irrational $\alpha$, there exists a closed non-empty set of analytic functions $F_{\alpha,\lambda}$ such that every function $f \in F_{\alpha,\lambda}$ can be approximated by shifts $L(\lambda, \alpha, s + i\tau)$.

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## 1. Introduction

Let $s = \sigma + it$ be a complex variable, $\lambda \in \mathbb{R}$ and $\alpha$, $0 < \alpha \leqslant 1$, fixed parameters. The Lerch zeta-function $L(\lambda, \alpha, s)$ was introduced independently by Lerch [7] and Lipschitz [8], and is defined, for $\sigma > 1$, by the Dirichlet series

$$L(\lambda, \alpha, s) = \sum_{m=0}^{\infty} \frac{e^{2\pi i \lambda m}}{(m + \alpha)^s}.$$

For $\lambda \in \mathbb{Z}$, the function $L(\lambda, \alpha, s)$ reduces to the Hurwitz zeta-function

$$\zeta(s, \alpha) = \sum_{m=1}^{\infty} \frac{1}{(m + \alpha)^s}, \quad \sigma > 1,$$

which can be analytically continued to the whole complex plane, except for a simple pole at the point $s = 1$ with residue 1. Moreover, $L(k, 1, s)$, $k \in \mathbb{Z}$, coincides with the Riemann zeta-function

$$\zeta(s) = \sum_{m=1}^{\infty} \frac{1}{m^s}, \quad \sigma > 1,$$

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and

$$L\left(k, \frac{1}{2}, s\right) = (2^s - 1)\,\zeta(s).$$

For $\lambda \notin \mathbb{Z}$, the function is analytically continued to an entire function. If the parameter $\lambda$ is rational, then the function $L(\lambda, \alpha, s)$ becomes a periodic Hurwitz zeta-function because of the periodicity of $\mathrm{e}^{2\pi i\lambda m}$. Thus, the Lerch zeta-function is a generalization of some classical zeta-functions. The function $L(\lambda, \alpha, s)$ is not so important as, say, $\zeta(s)$; however, it is an interesting analytic object depending on two parameters, and occupies a proper place in analytic number theory.

After a pioneer Voronin's work [9], it is known that some zeta and $L$-functions are universal in the sense that their shifts approximate a wide class of analytic functions. There exists a conjecture that the Lerch zeta-function is also universal in the Voronin sense; however, this conjecture is proved only for some classes of parameters $\alpha$ and $\lambda$. The simplest case is of transcendental $\alpha$ because of the linear independence over the field of rational numbers $\mathbb{Q}$ of the set $L(\alpha) \overset{def}{=} \{\log(m+\alpha) : m \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}\}$. The universality of the function $L(\lambda, \alpha, s)$ with transcendental $\alpha$ was obtained in [4] and has the following form. Let $D = \left\{s \in \mathbb{C} : \frac{1}{2} < \sigma < 1\right\}$, $\mathcal{K}$ be the class of compact subsets of the strip $D$ with connected complements, and let $H(K)$ with $K \in \mathcal{K}$ denote the class of continuous functions on $K$ that are analytic in the interior of $K$. Moreover, let meas$A$ stand for the Lebesgue measure of a measurable set $A \subset \mathbb{R}$. Then [4] we have

**Theorem 1.** *Suppose that the parameter $\alpha$ is transcendental. Let $K \in \mathcal{K}$ and $f(s) \in H(K)$. Then, for every $\varepsilon > 0$ and $\lambda \in \mathbb{R}$,*

$$\liminf_{T\to\infty} \frac{1}{T}\mathrm{meas}\left\{\tau \in [0,T] : \sup_{s\in K} |L(\lambda, \alpha, s + i\tau) - f(s)| < \varepsilon\right\} > 0.$$

The inequality of the theorem shows that the set of shifts $L(\lambda, \alpha, s+i\tau)$ approximating a given function $f(s) \in H(K)$ with accuracy $\varepsilon$ has a positive lower density. Hence, the set of those shifts is infinite. Theorem 1 can also be found in [5].

Obviously, in view of the periodicity of $\mathrm{e}^{2\pi i\lambda m}$, it suffices to assume that $0 < \lambda \leqslant 1$.

The next universality result for $L(\lambda, \alpha, s)$ was obtained for rational $\alpha$ and $\lambda$. If $\lambda = \frac{r}{q}$, $0 < r \leqslant q$, $(r, q) = 1$, then the sequence $\left\{\mathrm{e}^{2\pi i(r/q)m} : m \in \mathbb{N}_0\right\}$ is periodic with a period $q$, and the function $L(\lambda, \alpha, s)$ becomes a periodic Hurwitz zeta-function, whose universality with rational $\alpha$ was considered in [6]. Thus, we have the following statement

**Theorem 2.** *Suppose that $\alpha = \frac{a}{b}$, $a, b \in \mathbb{N}$, $a < b$, $(a,b) = 1$, $\alpha \neq \frac{1}{2}$, $\lambda = \frac{r}{q}$, $r, q \in \mathbb{N}$, $r < q$, $(r,q) = 1$, and $(bl + a, bq) = 1$ for all $l = 0, 1, \ldots, q-1$. Let $K \in \mathcal{K}$ and $f(s) \in H(K)$. Then, for every $\varepsilon > 0$,*

$$\liminf_{T\to\infty} \frac{1}{T}\mathrm{meas}\left\{\tau \in [0,T] : \sup_{s\in K} \left|L\left(\frac{r}{q}, \frac{a}{b}, s + i\tau\right) - f(s)\right| < \varepsilon\right\} > 0.$$

The case of an algebraic irrational $\alpha$ is an open difficult problem.

In this note, we propose a certain "approximation" to universality of the function $L(\lambda, \alpha, s)$ for all parameters $\alpha$ and $\lambda$. We recall that $D = \left\{s \in \mathbb{C} : \frac{1}{2} < \sigma < 1\right\}$, and denote by $H(D)$ the space of analytic functions on $D$ endowed with the topology of uniform convergence on compacta.

**Theorem 3.** *Suppose that the parameters $\lambda$, $0 < \lambda \leqslant 1$, and $\alpha$, $0 < \alpha < 1$, are arbitrary numbers. Then there exists a non-empty closed set $F_{\alpha,\lambda} \subset H(D)$ such that, for every compact subset $K \subset D$, $f(s) \in F_{\alpha,\lambda}$ and $\varepsilon > 0$,*

$$\liminf_{T\to\infty} \frac{1}{T}\text{meas}\left\{\tau \in [0,T] : \sup_{s\in K} |L(\lambda, \alpha, s + i\tau) - f(s)| < \varepsilon\right\} > 0.$$

A lower density of shifts $L(\lambda, \alpha, s + i\tau)$ can be replaced by density, but not for all $\varepsilon > 0$. We have

**Theorem 4.** *Suppose that the parameters $\lambda$, $0 < \lambda \leqslant 1$, and $\alpha$, $0 < \alpha < 1$, are arbitrary numbers. Then there exists a non-empty closed set $F_{\alpha,\lambda} \subset H(D)$ such that, for every compact subset $K \subset D$ and $f(s) \in F_{\alpha,\lambda}$, the limit*

$$\lim_{T\to\infty} \frac{1}{T}\text{meas}\left\{\tau \in [0,T] : \sup_{s\in K} |L(\lambda, \alpha, s + i\tau) - f(s)| < \varepsilon\right\} > 0$$

*exists for all but at most countably many $\varepsilon > 0$.*

The above theorems remain valid for some compositions $\Phi(L(\lambda, \alpha, s))$, where $\Phi : H(D) \to H(D)$ is a certain operator.

**Theorem 5.** *Suppose that the parameters $\lambda$, $0 < \lambda \leqslant 1$, and $\alpha$, $0 < \alpha < 1$, are arbitrary numbers. There exists a non-empty closed set $F_{\alpha,\lambda} \subset H(D)$ such that if $\Phi : H(D) \to H(D)$ is a continuous operator such that, for every open set $G \subset H(D)$, the set $\left(\Phi^{-1}G\right) \cap F_{\alpha,\lambda}$ is not empty; then, for every compact subset $K \subset D$, $f(s) \in \Phi\left(F_{\alpha,\lambda}\right)$ and $\varepsilon > 0$,*

$$\liminf_{T\to\infty} \frac{1}{T}\text{meas}\left\{\tau \in [0,T] : \sup_{s\in K} |\Phi\left(L(\lambda, \alpha, s + i\tau)\right) - f(s)| < \varepsilon\right\} > 0.$$

It is not difficult to see that the condition $\left(\Phi^{-1}G\right) \cap F_{\alpha,\lambda} \neq \varnothing$ for every open set $G \subset H(D)$ can be replaced by the following: for every polynomial $p = p(s)$, the set $\left(\Phi^{-1}\{p\}\right) \cap F_{\alpha,\lambda}$ is not empty.

We call Theorems 3 – 5 "almost" universality theorems for the Lerch zeta-function.

## 2. Preparatory results

In this section, we present some measure theory results including the weak convergence in the space $H(D)$. Denote by $\mathcal{B}(X)$ the Borel $\sigma$-field of the space $X$, and for $A \in \mathcal{B}(H(D))$ define

$$P_{T,\alpha,\lambda}(A) = \frac{1}{T}\text{meas}\left\{\tau \in [0,T] : L(\lambda, \alpha, s + i\tau) \in A\right\}.$$

The main result of this section is the following theorem.

**Theorem 6.** *Suppose that the parameters $\alpha$, $0<\alpha<1$, and $\lambda$, $0<\lambda\leqslant 1$, are arbitrary numbers. Then, on $(H(D),\mathcal{B}(H(D)))$, there exists a probability measure $P_{\alpha,\lambda}$, such that $P_{T,\alpha,\lambda}$ converges weakly to $P_{\alpha,\lambda}$ as $T\to\infty$.*

We divide the proof of Theorem 6 into lemmas. The first is a limit theorem on the infinite-dimensional torus. Let $\gamma=\{s\in\mathbb{C}:|s|=1\}$ and

$$\Omega=\prod_{m=0}^{\infty}\gamma_m,$$

where $\gamma_m=\gamma$ for all $m\in\mathbb{N}_0$. By the Tikhonov theorem, the torus $\Omega$, with the product topology and poinwise multiplication, is a compact topological Abelian group. Denote by $\omega(m)$, $m\in\mathbb{N}_0$, the $m$th component of the element $\omega\in\Omega$. The characters $\chi$ of the group $\Omega$ are of the form

$$\chi(\omega)=\prod_{m=0}^{\infty}\omega^{k_m}(m),\quad \omega\in\Omega,$$

where only a finite number of integers $k_m$ is distinct from zero. Therefore, the Fourier transform $g(\underline{k})$, $\underline{k}=\{k_m\in\mathbb{Z}:m\in\mathbb{N}\}$, of a measure $P$ on $(\Omega,\mathcal{B}(\Omega))$ is given by the formula

$$g(\underline{k})=\int_{\Omega}\left(\prod_{m=0}^{\infty}\omega^{k_m}(m)\right)\mathrm{d}P, \tag{1}$$

where only a finite number of integers $k_m$ is distinct from zero. It is well known that the measure $P$ is uniquely determined by its Fourier transform $g(\underline{k})$. Moreover, the convergence of Fourier transforms implies weak convergence for the corresponding probability measures.

Let, for brevity,

$$\underline{k}_{0,\alpha}=\left\{k_m\in\mathbb{Z}:\sum_{m=0}^{\infty}{}'k_m\log(m+\alpha)=0\right\}$$

and

$$\hat{\underline{k}}_{0,\alpha}=\left\{k_m\in\mathbb{Z}:\sum_{m=0}^{\infty}{}'k_m\log(m+\alpha)\neq 0\right\},$$

where the sign " $'$ " shows that only a finite number of integers $k_m$ is distinct from zero. For $A\in(\Omega,\mathcal{B}(\Omega))$, define

$$Q_{T,\alpha}(A)=\frac{1}{T}\mathrm{meas}\left\{\tau\in[0,T]:\left((m+\alpha)^{-i\tau}:m\in\mathbb{N}_0\right)\in A\right\}.$$

**Lemma 1.** *On $(\Omega,\mathcal{B}(\Omega))$, there exists a probability measure $Q_\alpha$ such that $Q_{T,\alpha}$ converges weakly to $Q_\alpha$ as $T\to\infty$. Moreover, the Fourier transform of the measure $Q_\alpha$ is*

$$g_\alpha(\underline{k})=\begin{cases}1 \text{ if } & \underline{k}=\underline{k}_{0,\alpha},\\ 0 \text{ if } & \underline{k}=\hat{\underline{k}}_{0,\alpha}.\end{cases} \tag{2}$$

**Proof**. We apply the Fourier transform method. In view of (1), the Fourier transform $g_{T,\alpha}(\underline{k})$ of $Q_{T,\alpha}$, is of the form

$$g_{T,\alpha}(\underline{k}) = \frac{1}{T}\int_0^T \left(\prod_{m=0}^{\infty}(m+\alpha)^{-i\tau k_m}\right) \mathrm{d}\tau = \frac{1}{T}\int_0^T \exp\left\{-i\tau \sum_{m=0}^{\infty}{}' k_m \log(m+\alpha)\right\} \mathrm{d}\tau.$$

Hence,

$$g_{T,\alpha}(\underline{k}_{0,\alpha}) = 1,$$

and

$$g_{T,\alpha}(\hat{\underline{k}}_{0,\alpha}) = \frac{1-\exp\left\{-iT\sum_{m=0}^{\prime\infty} k_m \log(m+\alpha)\right\}}{iT\sum_{m=0}^{\prime\infty} k_m \log(m+\alpha)}.$$

Therefore,

$$\lim_{T\to\infty} g_{T,\alpha}(\underline{k}) = g_\alpha(\underline{k}),$$

where $g_\alpha(\underline{k})$ is given by (2). Since the function $g_\alpha(\underline{k})$ is continuous in the discrete topology, this and the continuity theorem for probability measures on compact groups, see, for example, Theorem 1.4.2 from [3], prove the lemma. □

Lemma 1 allows to prove a limit theorem for absolutely convergent Dirichlet series related to the function $L(\lambda,\alpha,s)$. Let $\sigma_0 > \frac{1}{2}$ be a fixed number,

$$v_n(m,\alpha) = \exp\left\{-\left(\frac{m+\alpha}{n+\alpha}\right)^{\sigma_0}\right\}, \quad m\in\mathbb{N}_0,\ n\in\mathbb{N},$$

and

$$L_n(\lambda,\alpha,s) = \sum_{m=0}^{\infty}\frac{\mathrm{e}^{2\pi i\lambda m}v_n(m,\alpha)}{(m+\alpha)^s}.$$

Then it is known [5] that the latter Dirichlet series is absolutely convergent for $\sigma > \frac{1}{2}$. The next lemma deals with weak convergence for

$$P_{T,n,\alpha,\lambda}(A) \overset{def}{=} \frac{1}{T}\mathrm{meas}\left\{\tau\in[0,T] : L_n(\lambda,\alpha,s+i\tau)\in A\right\}, \quad A\in\mathcal{B}(H(D)).$$

**Lemma 2.** *On $(H(D),\mathcal{B}(H(D)))$, there exists a probability measure $\hat{P}_{n,\alpha,\lambda}$ such that $P_{T,n,\alpha,\lambda}$ converges weakly to $\hat{P}_{n,\alpha,\lambda}$ as $T\to\infty$.*

**Proof**. Define the function $u_{n,\alpha,\lambda} : \Omega \to H(D)$ by the formula

$$u_{n,\alpha,\lambda}(\omega) = \sum_{m=0}^{\infty}\frac{\mathrm{e}^{2\pi i\lambda m}v_n(m,\alpha)\omega(m)}{(m+\alpha)^s}, \quad \omega\in\Omega.$$

Since the latter series is absolutely convergent for $\sigma > \frac{1}{2}$, the function $u_{n,\alpha,\lambda}$ is continuous; thus, it is $(\mathcal{B}(\Omega),\mathcal{B}(H(D)))$-measurable. Therefore, the measure $Q_\alpha$

($Q_\alpha$ is the limit measure in Lemma 1) induces on $(H(D),\mathcal{B}(H(D)))$ the unique probability measure $\hat{P}_{n,\alpha,\lambda} \overset{def}{=} Q_\alpha u_{n,\alpha,\lambda}^{-1}$, where, for $A\in\mathcal{B}(H(D))$,

$$Q_\alpha u_{n,\alpha,\lambda}^{-1}(A) = Q_\alpha\left(u_{n,\alpha,\lambda}^{-1}A\right).$$

By the definitions of $P_{T,n,\alpha,\lambda}$, $Q_{T,\alpha}$ and $u_{n,\alpha,\lambda}$, we have that, for $A\in\mathcal{B}(H(D))$,

$$P_{T,n,\alpha,\lambda}(A) = \frac{1}{T}\text{meas}\left\{\tau\in[0,T] : \left((m+\alpha)^{-i\tau} : m\in\mathbb{N}_0\right)\in u_{n,\alpha,\lambda}^{-1}A\right\}.$$

Thus, we see that $P_{T,n,\alpha,\lambda} = Q_{T,\alpha}u_{n,\alpha,\lambda}^{-1}$. This together with Lemma 1, the continuity of $u_{n,\alpha,\lambda}$ and Theorem 5.1 of [1] shows that $P_{T,n,\alpha,\lambda}$ converges weakly to the measure $\hat{P}_{n,\alpha,\lambda} \overset{def}{=} Q_\alpha u_{n,\alpha,\lambda}^{-1}$. □

The next step of the proof of Theorem 6 consists of the approximation in the mean of the function $L(\lambda,\alpha,s)$ by $L_n(\lambda,\alpha,s)$. For this, we recall a metric in $H(D)$ that induces the topology of uniform convergence on compacta. It is well known, see, for example, [2], that there exists a sequence of compact subsets $\{K_l : l\in\mathbb{N}\}\subset D$ such that

$$D=\bigcup_{l=1}^{\infty}K_l,$$

$K_l\subset K_{l+1}$ for all $l\in\mathbb{N}$, and if $K\subset D$ is a compact set, then $K$ lies in $K_l$ for some $l$. For $g_1,g_2\in H(D)$, define

$$\rho(g_1,g_2)=\sum_{l=1}^{\infty}2^{-l}\frac{\sup_{s\in K_l}|g_1(s)-g_2(s)|}{1+\sup_{s\in K_l}|g_1(s)-g_2(s)|}.$$

Then $\rho$ is the desired metric in the space $H(D)$.

**Lemma 3.** *Suppose that* $\lambda$, $0<\lambda\leqslant 1$, *and* $\alpha$, $0<\alpha<1$, *are arbitrary numbers. Then*

$$\lim_{n\to\infty}\limsup_{T\to\infty}\frac{1}{T}\int_0^T\rho\left(L(\lambda,\alpha,s+i\tau),L_n(\lambda,\alpha,s+i\tau)\right)\mathrm{d}\tau=0.$$

**Proof**. Let $\sigma_0$ be the same as in the definition of $v_n(m,\alpha)$. Then, for $\sigma>\frac{1}{2}$, we have the integral representation [5]

$$L_n(\lambda,\alpha,s)=\frac{1}{2\pi i}\int_{\sigma_0-i\infty}^{\sigma_0+i\infty}L(\lambda,\alpha,s+z)l_n(z,\alpha)\frac{\mathrm{d}z}{z}, \tag{3}$$

where

$$l_n(s,\alpha)=\frac{s}{\sigma_0}\Gamma\left(\frac{s}{\sigma_0}\right)(n+\alpha)^s,$$

and $\Gamma(s)$ denotes the Euler gamma-function. Suppose that $K\subset D$ is a fixed compact set, and $\varepsilon>0$ is such that $\frac{1}{2}+2\varepsilon\leqslant \mathrm{Re}w\leqslant 1-\varepsilon$ for any point $w\in K$. Let $\theta>0$. Then, by (3) and the residue theorem,

$$L_n(\lambda,\alpha,s)-L(\lambda,\alpha,s)=\frac{1}{2\pi i}\int_{-\theta-i\infty}^{-\theta+i\infty}L(\lambda,\alpha,s+z)l_n(z,\alpha)\frac{\mathrm{d}z}{z}+R_n(s),$$

where

$$R_n(s) = \begin{cases} 0 & \text{if} \quad 0 < \lambda < 1, \\ \frac{l_n(1-s,\alpha)}{1-s} & \text{if} \quad \lambda = 1. \end{cases}$$

Denote the points of $K$ by $s = \sigma + iv$, and taking

$$\theta = \sigma - \varepsilon - \frac{1}{2}, \qquad \sigma_0 = \frac{1}{2} + \varepsilon,$$

we find that

$$\begin{aligned} &|L_n(\lambda, \alpha, s + i\tau) - L(\lambda, \alpha, s + i\tau)| \\ &\quad \leqslant \frac{1}{2\pi} \int_{-\infty}^{\infty} |L(\lambda, \alpha, s + i\tau - \theta + it)| \frac{|l_n(-\theta + it, \alpha)|}{|-\theta + it|} \,\mathrm{d}t + |R_n(s + i\tau)| \,. \end{aligned}$$

Now, in the integral we replace $t + v$ by $t$. This gives the inequality

$$\begin{aligned} &|L_n(\lambda, \alpha, s + i\tau) - L(\lambda, \alpha, s + i\tau)| \\ &\quad \leqslant \frac{1}{2\pi} \int_{-\infty}^{\infty} \left| L\left(\lambda, \alpha, \frac{1}{2} + \varepsilon + i(t + \tau)\right) \right| \frac{|l_n\left(1/2 + \varepsilon - s + it, \alpha\right)|}{|1/2 + \varepsilon - s + it|} \,\mathrm{d}t \\ &\quad + |R_n(s + i\tau)| \,. \end{aligned}$$

Hence,

$$\frac{1}{T} \int_0^T \sup_{s \in K} |L(\lambda, \alpha, s + i\tau) - L_n(\lambda, \alpha, s + i\tau)| \,\mathrm{d}\tau \leqslant I_1 + I_2, \tag{4}$$

where

$$\begin{aligned} I_1 = \frac{1}{2\pi} \int_{-\infty}^{\infty} &\left( \frac{1}{T} \int_0^T \left| L\left(\lambda, \alpha, \frac{1}{2} + \varepsilon + i(t + \tau)\right) \right| \mathrm{d}\tau \right) \\ &\times \sup_{s \in K} \frac{|l_n\left(1/2 + \varepsilon - s + it, \alpha\right)|}{|1/2 + \varepsilon - s + it|} \,\mathrm{d}t \end{aligned}$$

and

$$I_2 = \frac{1}{T} \int_0^T \sup_{s \in K} |R_n(s + i\tau)| \,\mathrm{d}\tau.$$

Using the definition of $l_n(s, \alpha)$ and applying the Stirling formula for the gamma-function, we obtain the bound

$$\begin{aligned} \frac{|l_n\left(1/2 + \varepsilon - s + it, \alpha\right)|}{|1/2 + \varepsilon - s + it|} &= \frac{(n + \alpha)^{1/2 + \varepsilon - \sigma}}{\sigma_0} \left| \Gamma\left( \frac{1/2 + \varepsilon - \sigma}{\sigma_0} + \frac{i(t - v)}{\sigma_0} \right) \right| \\ &\ll \frac{(n + \alpha)^{-\varepsilon}}{\sigma_0} \left( 1 + \frac{|t - v|}{\sigma_0} \right)^{(1/2 + \varepsilon - \sigma)/\sigma_0 - 1/2} \exp\left\{ -\frac{\pi}{2\sigma_0} |t - v| \right\}. \end{aligned} \tag{5}$$

Let $v_0 = v_0(K) = \sup_{s\in K} |\mathrm{Im} s| + 1$. Then we have $|t-v| \geqslant |t| - |v| \geqslant |t| - v_0$. Therefore, in view of (5),

$$\begin{aligned} &\frac{|l_n(1/2+\varepsilon-s+it,\alpha)|}{|1/2+\varepsilon-s+it|} \qquad (6)\\ &\quad \ll \frac{(n+\alpha)^{-\varepsilon}}{\sigma_0} \exp\left\{\frac{\pi v_0}{2\sigma_0}\right\} \exp\left\{-\frac{\pi|t|}{2\sigma_0}\right\}\\ &\quad \ll_{\sigma_0,K} (n+\alpha)^{-\varepsilon} \exp\left\{-\frac{\pi|t|}{2\sigma_0}\right\}. \qquad (7) \end{aligned}$$

By the estimate

$$\int_0^T |L(\lambda,\alpha,\sigma+it)|^2\,\mathrm{d}t \ll T$$

for $\sigma > \frac{1}{2}$, we find that

$$\begin{aligned} &\frac{1}{T}\int_0^T \left|L\left(\lambda,\alpha,\frac{1}{2}+\varepsilon+i(t+\tau)\right)\right|\,\mathrm{d}\tau\\ &\quad \ll \left(\frac{1}{T}\int_0^T \left|L\left(\lambda,\alpha,\frac{1}{2}+\varepsilon+i(t+\tau)\right)\right|^2\,\mathrm{d}\tau\right)^{1/2} \ll (1+|t|). \end{aligned}$$

This and (6) show that

$$I_1 \ll_{\sigma_0,K} (n+\alpha)^{-\varepsilon} \int_{-\infty}^{\infty} (1+|t|) \exp\left\{-\frac{\pi|t|}{2\sigma_0}\right\}\,\mathrm{d}t \ll_{\sigma_0,K} (n+\alpha)^{-\varepsilon}. \qquad (8)$$

Similarly to the above, we arrive at the estimate

$$|R_n(s)| \leqslant \frac{|l_n(1-s+it,\alpha)|}{|1-s+it|} \ll_{\sigma_0,K} (n+\alpha)^{1-\sigma} \exp\left\{-\frac{\pi|t|}{2\sigma_0}\right\}.$$

Therefore,

$$I_2 \ll_{\sigma_0,K} (n+\alpha)^{1-\sigma} \frac{1}{T}\int_0^T \exp\left\{-\frac{\pi\tau}{2\sigma_0}\right\}\,\mathrm{d}\tau \ll_{\sigma_0,K} \frac{(n+\alpha)^{1/2-2\varepsilon}}{T}.$$

This estimate, (8) and (4) give

$$\frac{1}{T}\int_0^T \sup_{s\in K} |L(\lambda,\alpha,s+i\tau) - L_n(\lambda,\alpha,s+i\tau)|\,\mathrm{d}\tau \ll_{\sigma_0,K} (n+\alpha)^{-\varepsilon} + \frac{(n+\alpha)^{1-\sigma}}{T}.$$

Hence,

$$\lim_{n\to\infty} \limsup_{T\to\infty} \frac{1}{T}\int_0^T \sup_{s\in K} |(L(\lambda,\alpha,s+i\tau) - L_n(\lambda,\alpha,s+i\tau))|\,\mathrm{d}\tau = 0,$$

and the definition of the metric $\rho$ proves the lemma. □

Now, we recall two notions used in the theory of weak convergence of probability measures. Let $\{P\}$ be a family of probability measures on the space $(X, \mathcal{B}(X))$. The family $\{P\}$ is called relatively compact if every sequence $\{P_n\} \subset \{P\}$ contains a subsequence $\{P_{n_r}\}$ weakly convergent to a certain probability measure $P$ on $(X, \mathcal{B}(X))$, and the family $\{P\}$ is called tight if, for every $\varepsilon > 0$, there exists a compact set $K = K(\varepsilon) \subset X$ such that

$$P(K) > 1 - \varepsilon$$

for all $P \in \{P\}$. These two notions are connected by the Prokhorov theorem [1]: If the family is tight, then it is relatively compact. Hence, it is important to have an information on the tightness of a given family of probability measures. The next lemma consider the relative compactness of the sequence $\{\hat{P}_{n,\alpha,\lambda} : n \in \mathbb{N}\}$, where $\hat{P}_{n,\alpha,\lambda}$ is a limit measure in Lemma 2.

**Lemma 4.** *The sequence $\{\hat{P}_{n,\alpha,\lambda}\}$ is relatively compact.*

**Proof**. As it follows from the Prokhorov theorem, it is sufficient to prove that $\{\hat{P}_{n,\alpha,\lambda}\}$ is tight. It is convenient, in place of weak convergence, to use the convergence in distribution. Thus, let $\xi$ be a random variable defined on a certain probability space with measure $\mu$ and uniformly distributed on $[0, 1]$. Denote by $\hat{X}_{n,\alpha,\lambda} = \hat{X}_{n,\alpha,\lambda}(s)$ the $H(D)$-valued random element having the distribution $\hat{P}_{n,\alpha,\lambda}$, and define the $H(D)$-valued random element as

$$X_{T,n,\alpha,\lambda} = X_{T,n,\alpha,\lambda}(s) = L_n(\lambda, \alpha, s + iT\xi).$$

Using the above random elements, the conclusion of Lemma 2 can be rewritten in the form

$$X_{T,n,\alpha,\lambda} \xrightarrow[T\to\infty]{\mathcal{D}} \hat{X}_{n,\alpha,\lambda}. \tag{9}$$

The series for $L_n(\lambda, \alpha, s)$ is absolutely convergent for $\sigma > \frac{1}{2}$; therefore

$$\begin{aligned}\lim_{T\to\infty} \frac{1}{T} \int_0^T |L_n(\lambda, \alpha, \sigma + it)|^2 \, \mathrm{d}t &= \sum_{m=0}^{\infty} \frac{\left|\mathrm{e}^{2\pi i \lambda m}\right|^2 v_n^2(m, \alpha)}{(m+\alpha)^{2\sigma}} \\ &\leqslant \sum_{m=0}^{\infty} \frac{1}{(m+\alpha)^{2\sigma}} \leqslant C_{\sigma,\alpha} < \infty \end{aligned} \tag{10}$$

for $\sigma > \frac{1}{2}$. Let $K_l$ be a compact set from the definition of the metric $\rho$. Then an application of the Cauchy integral formula and (10) show that

$$\sup_{n\in\mathbb{N}} \limsup_{T\to\infty} \frac{1}{T} \int_0^T \sup_{s\in K_l} |L_n(\lambda, \alpha, s + i\tau)| \, \mathrm{d}\tau \leqslant R_{l,\alpha} < \infty. \tag{11}$$

Let $\varepsilon > 0$ be an arbitrary fixed number, and $M_{l,\alpha} = M_{l,\alpha}(\varepsilon) = R_{l,\alpha} 2^l \varepsilon^{-1}$. Then

(11) implies

$$\begin{aligned}
&\limsup_{T\to\infty}\mu\left(\sup_{s\in K_l}|X_{T,n,\alpha,\lambda}(s)|>M_{l,\alpha}\right)\\
&\qquad=\limsup_{T\to\infty}\frac{1}{T}\text{meas}\left\{\tau\in[0,T]:\sup_{s\in K_l}|L_n(\lambda,\alpha,s+i\tau)|>M_{l,\alpha}\right\}\\
&\qquad\leqslant\sup_{n\in\mathbb{N}}\limsup_{T\to\infty}\frac{1}{TM_{l,\alpha}}\int_0^T\sup_{s\in K_l}|L_n(\lambda,\alpha,s+i\tau)|\,\mathrm{d}\tau\leqslant\frac{\varepsilon}{2^l}
\end{aligned}\tag{12}$$

for all $l\in\mathbb{N}$. Define the set

$$H_\varepsilon=\left\{g\in H(D):\sup_{s\in K_l}|g(s)|\leqslant M_{l,\alpha},\ l\in\mathbb{N}\right\}.$$

Then the set $H_\varepsilon$ is compact in the space $H(D)$, and, in view of (9) and (12),

$$\mu\left(\hat{X}_{n,\alpha,\lambda}\in H_\varepsilon\right)\geqslant 1-\varepsilon$$

for all $n\in\mathbb{N}$, or in other words,

$$\hat{P}_{n,\alpha,\lambda}(H_\varepsilon)\geqslant 1-\varepsilon$$

for all $n\in\mathbb{N}$, i.e., the sequence $\{\hat{P}_{n,\alpha,\lambda}\}$ is tight. □

*Proof of Theorem 6.* By Lemma 4, there exists a subsequence $\{\hat{P}_{n_r,\alpha,\lambda}\}$ weakly convergent to a certain probability measure $P_{\alpha,\lambda}$ on $(H(D),\mathcal{B}(H(D)))$ as $r\to\infty$, i.e.,

$$\hat{X}_{n_r,\alpha,\lambda}\xrightarrow[r\to\infty]{\mathcal{D}}P_{\alpha,\lambda}.\tag{13}$$

Define one more $H(D)$-valued random element

$$X_{T,\alpha,\lambda}=X_{T,\alpha,\lambda}(s)=L(\lambda,\alpha,s+iT\xi).$$

Then Lemma 3 implies, for every $\varepsilon>0$,

$$\begin{aligned}
&\lim_{n\to\infty}\limsup_{T\to\infty}\mu\left(\rho\left(X_{T,\alpha,\lambda},X_{T,n,\alpha,\lambda}\right)\geqslant\varepsilon\right)\\
&\qquad=\lim_{n\to\infty}\limsup_{T\to\infty}\frac{1}{T}\text{meas}\left\{\tau\in[0,T]:\rho\left(L(\lambda,\alpha,s+i\tau),L_n(\lambda,\alpha,s+i\tau)\right)\geqslant\varepsilon\right\}\\
&\qquad\leqslant\lim_{n\to\infty}\limsup_{T\to\infty}\frac{1}{T\varepsilon}\int_0^T\rho\left(L(\lambda,\alpha,s+i\tau),L_n(\lambda,\alpha,s+i\tau)\right)\mathrm{d}\tau=0.
\end{aligned}$$

The latter equality, (9) and (13) show that all conditions of Theorem 4.2 from [1] are satisfied. Therefore,

$$X_{T,\alpha,\lambda}\xrightarrow[T\to\infty]{\mathcal{D}}P_{\alpha,\lambda},$$

i.e., $P_{T,\alpha,\lambda}$ converges weakly to the measure $P_{\alpha,\lambda}$ as $T\to\infty$. The theorem is proved. □

## 3. Proof of approximation theorems

*Proof of Theorem 3.* Denote by $F_{\alpha,\lambda}$ the support of the limit measure $P_{\alpha,\lambda}$ in Theorem 6, i.e., $F_{\alpha,\lambda}$ is a minimal closed subset of $H(D)$ such that $P_{\alpha,\lambda}(F_{\alpha,\lambda}) = 1$. Clearly, $F_{\alpha,\lambda} \neq \varnothing$. Moreover, the set $F_{\alpha,\lambda}$ consists of elements $g \in H(D)$ such that for every open neighbourhood $G$ of $g$, the inequality $P_{\alpha,\lambda}(G) > 0$ is valid.

For $f \in F_{\alpha,\lambda}$, we set

$$G_\varepsilon = \left\{ g \in H(D) : \sup_{s \in K} |g(s) - f(s)| < \varepsilon \right\}.$$

Then $G_\varepsilon$ is an open neighbourhood of an element $f$. Therefore, $P_{\alpha,\lambda}(G_\varepsilon) > 0$. Hence, using the equivalent of weak convergence of probability measures in terms of open sets [1, Theorem 2.1] and Theorem 6, we obtain the inequality

$$\liminf_{T\to\infty} P_{T,\alpha,\lambda}(G_\varepsilon) \geqslant P_{\alpha,\lambda}(G_\varepsilon) > 0.$$

Thus, the definitions of $P_{T,\alpha,\lambda}$ and $G_\varepsilon$ complete the proof. □

*Proof of Theorem 4.* Denote by $\partial A$ the boundary of a set $A \in \mathcal{B}(H(D))$. The set $A \in \mathcal{B}(H(D))$ is called a continuity set of the measure $P_{\alpha,\lambda}$ if $P_{\alpha,\lambda}(\partial A) = 0$. Let the set $G_\varepsilon$ be the same as in the proof of Theorem 3. Then $\partial G_\varepsilon$ lies in the set $\{g \in H(D) : \sup_{s\in K} |g(s) - f(s)| = \varepsilon\}$, therefore, $\partial G_{\varepsilon_1} \cap \partial G_{\varepsilon_2} = \varnothing$ for different positive $\varepsilon_1$ and $\varepsilon_2$. This shows that the set $G_\varepsilon$ is a continuity set of the measure $P_{\alpha,\lambda}$ for all but at most countably many $\varepsilon > 0$. This remark together with the equivalent of weak convergence of probability measures in terms of continuity sets [1, Theorem 2.1] and Theorem 6 give the inequality

$$\lim_{T\to\infty} P_{T,\alpha,\lambda}(G_\varepsilon) = P_{\alpha,\lambda}(G_\varepsilon) > 0$$

for all but at most countably many $\varepsilon > 0$. Combining this with definitions $P_{T,\alpha,\lambda}$ and $G_\varepsilon$ proves the theorem. □

*Proof of Theorem 5.* For $A \in \mathcal{B}(H(D))$, define

$$P_{T,\Phi,\alpha,\lambda}(A) = \frac{1}{T} \text{meas} \left\{ \tau \in [0, T] : \Phi(L(\lambda, \alpha, s + i\tau)) \in A \right\}.$$

Since the operator $\Phi$ is continuous, Theorem 6 and Theorem 5.1 of [1] show that $P_{T,\Phi,\alpha,\lambda}$ converges weakly to $P_{\Phi,\alpha,\lambda} = P_{\alpha,\lambda}\Phi^{-1}$. It remains to find the support of the measure $P_{\alpha,\lambda}\Phi^{-1}$.

Let $g$ be an arbitrary element of the set $\Phi(F_{\alpha,\lambda})$, and $G$ its any open neighbourhood. Then the set $\Phi^{-1}G$ is open, and by the hypothesis $(\Phi^{-1}G) \cap F_{\alpha,\lambda} \neq \varnothing$, it is an open neighbourhood of a certain element of the set $F_{\alpha,\lambda}$. Therefore, by the definition of $F_{\alpha,\lambda}$, the inequality $P_{\alpha,\lambda}(\Phi^{-1}G) > 0$ is valid. Hence,

$$P_{\alpha,\lambda}\Phi^{-1}(G) = P_{\alpha,\lambda}(\Phi^{-1}G) > 0.$$

This shows that the support of the measure $P_{\alpha,\lambda}\Phi^{-1}$ contains the set $\Phi(F_{\alpha,\lambda})$. Since the support is a closed set, the support of the measure $P_{\alpha,\lambda}\Phi^{-1}$ contains the closure of the set $\Phi(F_{\alpha,\lambda})$.

Further proof repeat that of Theorem 3. For $f \in \Phi(F_{\alpha,\lambda})$ define the set $G_\varepsilon$. Then $G_\varepsilon$ is an open neighbourhood of an element of the support of the measure $P_{\alpha,\lambda}\Phi^{-1}$, thus, $P_{\alpha,\lambda}\Phi^{-1}(G_\varepsilon) > 0$. Since $P_{T,\Phi,\alpha,\lambda}$ converges weakly to $P_{\alpha,\lambda}\Phi^{-1}$ as $T \to \infty$, using the equivalent of weak convergence in terms of open sets, gives the inequality

$$\liminf_{T\to\infty} P_{T,\Phi,\alpha,\lambda}(G_\varepsilon) \geqslant P_{\alpha,\lambda}\Phi^{-1}(G_\varepsilon) > 0,$$

and, by the definitions of $P_{T,\Phi,\alpha,\lambda}$ and $G_\varepsilon$,

$$\liminf_{T\to\infty} \frac{1}{T}\text{meas}\left\{\tau \in [0,T] : \sup_{s\in K} |\Phi(L(\lambda,\alpha,s+i\tau)) - f(s)| < \varepsilon\right\} > 0.$$

□

Obviously, Theorem 5 has a modification of the type of Theorem 4.

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## References


[1] P. Billingsley, *Convergence of Probability Measures*, Willey, New York, 1968.

[2] J. B. Conway, *Functions of One Complex Variable*, Springer-Verlag, Berlin, 1978.

[3] H. Heyer, *Probability Measures on Locally Compact Groups*, Springer-Verlag, Berlin, 1977.

[4] A. Laurinčikas, *The universality of the Lerch zeta-function*, Liet. Matem. Rink. **37**(3)(1997), 367–375 (in Russian); Lith. Math. J. **37**(3)(1997), 275–280.

[5] A. Laurinčikas, R. Garunkštis, *The Lerch Zeta-Function*, Kluwer Academic Publishers, Dordrecht, Boston, London, 2002.

[6] A. Laurinčikas, R. Macaitienė, D. Mochov, D. Šiaučiūnas, *Universality of the periodic Hurwitz zeta-function with rational parameter,* Sib. Math. J. **59**(2018), 894–900.

[7] M. Lerch, *Note sur la fonction* $K(w,x,s) = \sum_{n\geqslant 0} \exp\{2\pi inx\}(n+w)^{-s}$, Acta Math. **11**(1887), 19–24.

[8] R. Lipschitz, *Untersuchung einer aus vier Elementen gebildeten Reihe*, J. Reine Angew. Math. **105**(1889), 127–156.

[9] S. M. Voronin, *Theorem on the "universality" of the Riemann zeta-function*, Izv. Akad. Nauk SSSR, Ser. Matem. **39**(1975), 475–486 (in Russian); Math. USSR Izv. **9**(1975), 443–453.