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Continue quadrilaterals*

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Abstract. In this article, we introduce the notion of continue quadrilateral, that is a quadrilateral whose sides are in geometric progression. We obtain an extension of the principal result referring to the growth of continue triangles. Precisely, we will see that the growth of a continue quadrilateral belongs to the interval $(1/\Phi_2, \Phi_2)$, where Φ_2 is the Silver mean. The main result is that in any circle a continue quadrilateral of growth μ can be inscribed for every μ belonging to the interval $(1/\Phi_2, \Phi_2)$. Our investigation is supported by dynamical software.

AMS subject classifications: 51M05, 51M15, 51N20

Key words: Cyclic quadrilaterals, geometric progressions, silver mean, continue quadrilaterals, continued quadrilaterals, growth

1. Introduction

A triangle $\mathcal{T} = (\mathbf{a}, \mathbf{b}, \mathbf{c})$ is said to be *continue* (see [15]) if its sides $\mathbf{a}, \mathbf{b}, \mathbf{c}$ satisfy the proportion: $\mathbf{a} : \mathbf{b} = \mathbf{b} : \mathbf{c}$. This implies that the lengths a, b, c of the sides of $\mathcal{T}$ follow a geometric progression law: a, $a\mu$, $a\mu^2$, for a suitable real number μ. Clearly, every equilateral triangle is continue, and every *Kepler triangle* (see Figure 1) is an example of a right continue triangle.

The family of continue triangles is very special and it is strictly connected with Φ: the celebrated "Golden Mean". It can be shown (see for example [8, Theorem 1], [15] p. 22, or [4, Theorem 2.1 and Remark 2.1]) the following result:

Theorem 1. *If $\mathcal{T} = (a, b, c)$ is a continue triangle, then a, b, c are in geometric progression of the mean lying in $(1/\Phi, \Phi)$. Conversely, for every geometric progression kr, kr^2, kr^3, where k is a positive real number and the mean r lies in $(1/\Phi, \Phi)$, there exist a continue triangle $\mathcal{T} = (kr, kr^2, kr^3)$.*

Using continue triangles, as in Figure 1, we can easily construct examples of pairs of triangles with 5 pieces pairwise congruent, but not congruent. Jones and Peterson, Pennisi, and Volio, researched in detail such pairs of triangles (see [8], [15] and [18]), and actually they are known in the literature as *pairs of almost congruent triangles.* Indeed, if $\mathcal{T} = (k, k\mu, k\mu^2)$ is a continue triangle, then $\mathcal{T}' = (k\mu, k\mu^2, k\mu^3)$

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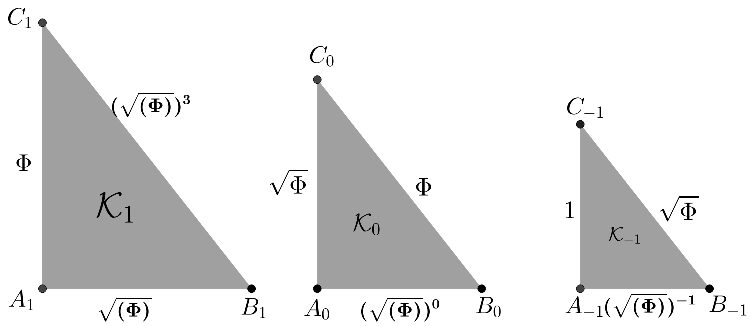

Figure 1: *Example of three 'consecutive' Kepler triangles*

is likewise a continue triangle sharing 5 pieces with $\mathcal{T}$, so that in particular $\mathcal{T}$ and $\mathcal{T}'$ are almost congruent.

It has recently been pointed out that "chains" of continue triangles can be used to describe logarithmic spirals (see [4]).

It is a natural question to investigate the corresponding notion for polygons and in particular for quadrilaterals.

In this article, we will introduce the notion of a 'continue quadrilateral' and investigate properties of such family of polygons.

First, we will study the link between continue quadrilaterals and almost congruent quadrilaterals. Then we will see the correspondent of the above Theorem 1 for continue quadrilaterals (see Theorem 3).

After that, we will investigate the relation between convex continue quadrilaterals and other relevant families of quadrilaterals such as orthodiagonal, tangential, equidiagonal or cyclic quadrilaterals. The main results refer to cyclic quadrilaterals: we will show that there are cyclic continue quadrilaterals of every possible 'growth' (Corollary 1) and finally, we will show that in any circle $\mathcal{C}$, for every arc $\overset{\frown}{AB} \in (0, 2\pi)$ there exists a unique continue cyclic quadrilateral $\mathcal{Q} = (A, B, C, D)$ inscribed in $\mathcal{C}$ (Theorem 5).

The article is suitable for a large audience of readers.

2. Definitions and preliminary considerations

In what follows, we will denote a convex quadrilateral by $\mathcal{Q} = (A, B, C, D)$, in which the vertices A, B, C, D are ordered counterclockwise (see Figure 2). The length of the sides of $\mathcal{Q}$

$$\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d}, \tag{1}$$

will be denoted by a, b, c, d. The inner angle at vertex A as well as its radian measure will be denoted by $B\hat{A}D$ or α; similarly the other angles.

Let $\mathcal{Q}$ be a convex quadrilateral. We will say that $\mathcal{Q} = (A, B, C, D)$ is *continue* if its sides $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$, $\mathbf{d}$ satisfy the proportion: $\mathbf{a} : \mathbf{b} = \mathbf{b} : \mathbf{c} = \mathbf{c} : \mathbf{d}$.

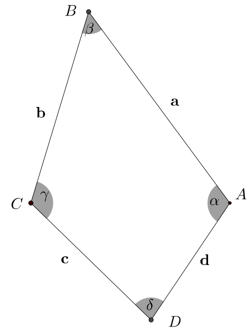

Figure 2: *Notations for a convex quadrilateral* $\mathcal{Q} = (A, B, C, D)$

Let $\mathcal{Q} = (A, B, C, D)$ be a continue quadrilateral. We can define the *growth* of $\mathcal{Q}$, as the factor $\mu_{\mathcal{Q}} := b/a$. Clearly, the growth of a continue quadrilateral $\mathcal{Q}$ is a positive real number, and it is exactly 1 if and only if $\mathcal{Q}$ is a rhombus.

Let $\mathcal{Q} = (A, B, C, D)$ be a continue quadrilateral. It follows from definitions that the measures of its sides are four consecutive numbers belonging to a geometric progression:

$$a, a\mu_{\mathcal{Q}}, a\mu_{\mathcal{Q}}^2, a\mu_{\mathcal{Q}}^3.$$

In particular, if $\mathcal{Q}$ is not a rhombus, then two dual cases may appear:

1) $\mu_{\mathcal{Q}} < 1$, that is, $a > b = a\,\mu_{\mathcal{Q}} > c = a\,{\mu_{\mathcal{Q}}}^2 > d = a\,{\mu_{\mathcal{Q}}}^3$. In this case, we refer to $\mathcal{Q}$ as a *decreasing continue quadrilateral*;

2) $\mu_{\mathcal{Q}} > 1$, that is, $a < b = a\,\mu_{\mathcal{Q}} < c = a\,{\mu_{\mathcal{Q}}}^2 < d = a\,{\mu_{\mathcal{Q}}}^3$. In this case, we refer to $\mathcal{Q}$ as an *increasing continue quadrilateral.*

Remark 1. *Note that if $\mathcal{Q} = (A, B, C, D)$ is a continue quadrilateral of growth $\mu_{\mathcal{Q}}$, then the quadrilateral $\mathcal{Q}' = (A', D', C', B')$, whose vertices are the symmetrical of A, B, C, D with respect to axis passing through A and C, has growth $\mu_{\mathcal{Q}'} = 1/\mu_{\mathcal{Q}}$ (see Figure 3).*

Remark 2. *Clearly, if $\mathcal{Q}$ is a continue quadrilateral, then any other quadrilateral $\mathcal{Q}'$ similar to $\mathcal{Q}$ is continue and its growth is either $\mu_{\mathcal{Q}}$ or $1/\mu_{\mathcal{Q}}$.*

Remark 3. *Applying a similarity of ratio $\mu \neq 1$ to a continue quadrilateral $\mathcal{Q} = (A, B, C, D)$, whose sides have lengths $a, a\mu, a\mu^2, a\mu^3$, we obtain a quadrilateral $\mathcal{Q}'$ whose sides have lengths $a\mu, a\mu^2, a\mu^3, a\mu^4$. Clearly, $\mathcal{Q}'$ is not congruent to $\mathcal{Q}$ but they have seven elements (four angles and three sides) pairwise congruent. This shows that, as in the case of triangles, the existence of a continue quadrilateral implies the existence of pairs of almost congruent quadrilaterals. However, contrary to what happens for the triangles (see [8]), the converse for quadrilaterals does not hold: in the following example, we show two almost congruent trapezoids that are neither continue nor similar.*

As observed in the introduction, Theorem 1 shows that the Golden Mean gives a "range" in which we can consider the growth of continue triangles.

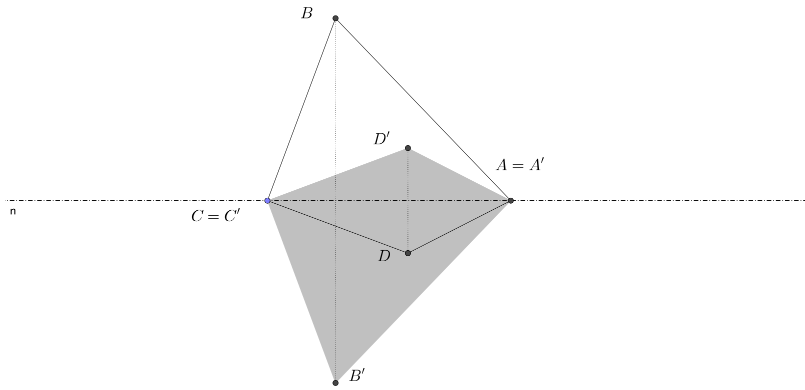

Figure 3: *Continue quadrilaterals with reciprocal growth* $\mu_{\mathcal{Q}'} = 1/\mu_{\mathcal{Q}}$

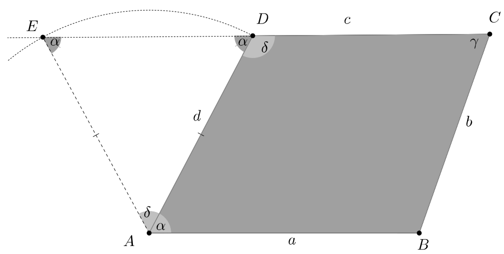

Figure 4: *The trapezoids* $Q = (A, B, C, D)$ *and* $Q' = (A, B, C, E)$ *are not congruent although they have 7 mutually congruent pieces: four angles and three sides*

Now we will see that the *Silver Mean* has a similar role for continue quadrilaterals. Recall that the Silver Mean, that is here denoted by Φ_2, is a real positive root of the polynomial (of Cauchy type) $p(x) = x^3 - x^2 - x - 1$. This is the characteristic polynomial of the 'tribonacci sequence' (see for example [2] or [7, Example 4.9]). Note that the Golden Mean Φ is less than $\Phi_2 = 1.83929\ldots$ (see [1], Table 1 on p. 619).

The following result gives the exact delimitations for the growth of a continue quadrilateral.

Theorem 2. *Let* $\mathcal{Q} = (A, B, C, D)$ *be a convex quadrilateral different from a rhombus. Then:*

i) If $\mathcal{Q}$ *is an increasing continue quadrilateral, then* $\mu_{\mathcal{Q}} \in (1, \Phi_2)$*. Conversely, for every positive real number* a *and every* $\mu \in (1, \Phi_2)$*, there is an increasing continue quadrilateral* $\mathcal{Q}$ *whose sides are* $a, a\mu, a\mu^2, a\mu^3$*.*

ii) If $\mathcal{Q}$ *is a decreasing continue quadrilateral, then* $\mu_{\mathcal{Q}} \in (1/\Phi_2, 1)$*. Conversely, for every positive real number* a *and every* $\mu \in (1/\Phi_2, 1)$*, there is a decreasing continue quadrilateral* $\mathcal{Q}$ *whose sides are* $a, a\mu, a\mu^2, a\mu^3$*.*

Proof. Suppose that $\mathcal{Q}$ is an increasing continue quadrilateral; then $\mu := \mu_{\mathcal{Q}} =$

$b/a > 1$, $a < b < c < d$ and

$$b = a\mu, \quad c = a\mu^2, \quad d = a\mu^3. \tag{2}$$

By elementary considerations we can note that the polynomial $p(x) = x^3 - x^2 - x - 1$ has only Φ_2 as real root and

$$p(\lambda) < 0 \text{ if and only if } \lambda < \Phi_2, \text{ for every real number } \lambda. \tag{3}$$

As $\mathcal{Q}$ is a quadrilateral, we have $a+b+c > d$, and it follows that $a + a\mu + a\mu^2 > a\mu^3$, so that $\mu^3 - \mu^2 - \mu - 1 < 0$. In particular, $\mu < \Phi_2$.

Conversely, if $a, a\mu, a\mu^2, a\mu^3$, where a is a positive real number and the mean μ lies in $(1, \Phi_2)$, we have $a\mu^3 - a\mu^2 - a\mu - a < 0$ by Equation (3), and this condition ensures the possibility to construct convex quadrilaterals whose longer side is $a\mu^3$. Condition (i) is proved.

Suppose now that $\mathcal{Q}$ is a decreasing continue quadrilateral. By Remark 1 the reflection with respect to the straight line AC gives an increasing quadrilateral $\mathcal{Q}'$ whose growth is $1/\mu_{\mathcal{Q}}$. The first part of the proof yields $1/\mu_{\mathcal{Q}} \in (1, \Phi_2)$, then μ_Q lies in $(1/\Phi_2, 1)$.

Conversely, suppose that μ lies in $(1/\Phi_2, 1)$, and let a be a positive real number. Note that $\mu' := 1/\mu$ lies in $(1, \Phi_2)$. It follows, by the first part of the proof, that there exists an increasing quadrilateral $\mathcal{Q}' = (A', B', C', D')$ whose sides are $a, a\mu', a\mu'^2, a\mu'^3$. The reflection with respect to the straight line $A'C'$ gives a decreasing quadrilateral $\mathcal{Q}$ whose growth is $1/\mu'_{\mathcal{Q}} = \mu$, by Remark 1. The proof is complete.

□

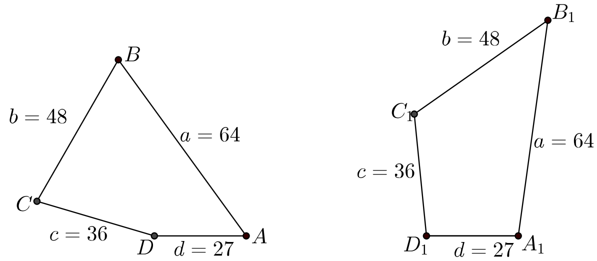

Figure 5: *Pairs of continue quadrilaterals that are not similar, and have the sides of the same length: they have the same growth 3/4*

We conclude this section by stating the corresponding 'characterization' for continue quadrilaterals of Theorem 1 (for triangles) that we have recalled in the introduction. It follows from the above Theorem 2.

Theorem 3. *Let $\mathcal{Q} = (A, B, C, D)$ be a continue convex quadrilateral; then a, b, c, d are in geometric progression of the mean lying in $(1/\Phi_2, \Phi_2)$. Conversely, for every geometric progression $a, a\mu, a\mu^2, a\mu^3$, where a is a positive number and the mean μ lies in $(1/\Phi_2, \Phi_2)$ there exists a continue quadrilateral $\mathcal{Q}$ whose sides are $a, a\mu, a\mu^2, a\mu^3$.*

3. Families of continue quadrilaterals

In this section we will study the relation between continue quadrilaterals and other types of relevant families of quadrilaterals such as orthodiagonal, tangential, equidiagonal and cyclic quadrilaterals.

Recall that a quadrilateral is *orthodiagonal* if it has orthogonal diagonals, and it is *tangential* if the lines on which its sides lie are all tangent to the same circle (as indicated by basic surveys on these topics, cf. §3.2 of [14]). Trivial examples of orthodiagonal or tangential quadrilaterals are rhombi. The following proposition shows that these notions collapse for continue quadrilaterals.

Proposition 1. *Let $\mathcal{Q} = (A, B, C, D)$ be a continue quadrilateral. Then the following statements are equivalent:*

i) $\mathcal{Q}$ is orthodiagonal;

ii) $\mathcal{Q}$ is tangential;

iii) $\mathcal{Q}$ is a rhombus.

Proof. If $\mathcal{Q}$ is an orthodiagonal quadrilateral, then by [9, Theorem 1] the following relation holds:

$$a^2 + c^2 = b^2 + d^2.$$

By hypothesis $b = a\mu$, $c = a\mu^2$ and $d = a\mu^3$. The roots of the polynomial $x^3 - x^2 + x - 1$ are $i, -i, 1$, so that we have the following equivalence:

$$a^2 + a^2\mu^4 = a^2\mu^2 + a^2\mu^6 \iff 1 = \mu^2.$$

It follows that $\mu = 1$ and $\mathcal{Q}$ is a rhombus. If $\mathcal{Q}$ is tangential, then

$$a + c = b + d.$$

Following the same argument above it can be proved that $\mu = 1$ and $\mathcal{Q}$ is a rhombus. □

Recall that a quadrilateral is *equidiagonal* if its diagonals are congruent (see [11] and reference therein for recent results on equidiagonal quadrilaterals). Clearly, every square is equidiagonal and continue. A natural question is: if we choose an arbitrary $\mu \in (1/\Phi_2, \Phi_2)$, are there any equidiagonal and continue quadrilaterals of growth μ? The answer is no. This can be easily seen as a consequence of the following nice characterization (see [11, Theorem 8]).

Theorem 4 (M. Josefsson)**.** *A convex quadrilateral (A, B, C, D) with consecutive sides a, b, c, d is equidiagonal if and only if $(a^2 - c^2)(b^2 - d^2) = 2abcd(cos(\hat{A} - \hat{C}) - cos(\hat{B} - \hat{D}))$.*

Therefore, if $\mathcal{Q} = (A, B, C, D)$ is a continue equidiagonal quadrilateral whose sides are $k, k\mu, k\mu^2, k\mu^3$, we have necessarily that:

$$k^2(1-\mu^4)k^2\mu^2(1-\mu^4) = 2k^4\mu^6((\cos(\hat{A}-\hat{C})-\cos(\hat{B}-\hat{D})), \text{ so that } \frac{(1-\mu^4)^2}{2\mu^4} < 2.$$

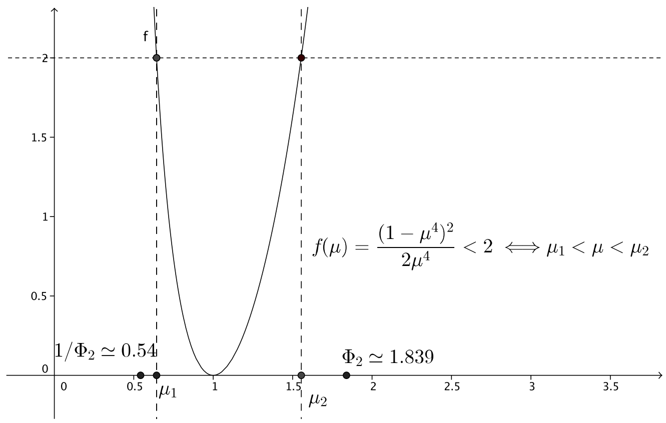

Figure 6: *Graph of the function* $f(\mu) = \frac{(1-\mu^4)^2}{2\mu^4}$

An easy check shows that for all values of μ close to $1/\Phi_2$ and Φ_2 the above inequality is not satisfied (see Figure 6). Therefore any continue quadrilateral whose growth is close to $1/\Phi_2$ and Φ_2 cannot be equidiagonal.

Recall that a quadrilateral is said to be *cyclic* if its vertices lie on a circle. It might seem that the restriction of being both a cyclic and continue quadrilateral is too strong and could reduce the study of such polygons to few cases. The next lemma shows that it is not so.

Lemma 1. *For every positive real number a and for every $\mu \in (1/\Phi_2, \Phi_2)$, there exists a cyclic continue quadrilateral $\mathcal{Q}_\mu$ whose sides have lengths $a, a\mu, a\mu^2, a\mu^3$.*

Proof. By Theorem 2 we may consider a continue quadrilateral whose sides have lengths $a, a\mu, a\mu^2, a\mu^3$; on the other hand, by [16, Theorem 1] 'for any quadrilateral with given edge lengths there is a cyclic quadrilateral with the same edge lengths'. Therefore, for every positive real number a and for every $\mu \in (1/\Phi_2, \Phi_2)$, there exists at least one cyclic continue quadrilateral $\mathcal{Q} = (A, B, C, D)$ inscribed in a circle $\mathcal{C}$, whose sides have lengths $a, a\mu, a\mu^2, a\mu^3$.

□

Corollary 1. *Let $\mathcal{C}$ be a circle and $A \in \mathcal{C}$. Then for every $\mu \in (1/\Phi_2, \Phi_2)$ there exists one cyclic continue quadrilateral $\mathcal{Q} = (A, B, C, D)$ of growth μ inscribed in $\mathcal{C}$.*

Proof. By Lemma 1 there exists a continue cyclic quadrilateral $\mathcal{Q}' = (A', B', C', D')$ inscribed in a circle $\mathcal{C}'$, whose sides have lengths $1, \mu, \mu^2, \mu^3$, respectively. In particular, $\mu_{\mathcal{Q}'} = \mu$. Put r and r' be the radius of $\mathcal{C}$ and $\mathcal{C}'$, respectively. Suppose that the two circles are centered in O, and consider the 'dilatation' of $\mathcal{C}'$ over $\mathcal{C}$ by r/r'. It turns out that the image of $\mathcal{Q}' = (A', B', C', D')$ is a cyclic continue quadrilateral inscribed in $\mathcal{C}$ with the same growth of $\mathcal{Q}'$. Among them, there is one, say $\mathcal{Q} = (A, B, C, D)$, that starts from A. □

The last question we are going to deal with is:

"Can any chord of an assigned circle $\mathcal{C}$ be a longer side of a cyclic continue quadrilateral inscribed in $\mathcal{C}$?"

We will see that the answer is positive, and in order to have the 'uniqueness', we will deal with this problem in terms of arcs of a circle.

The first step is to show that given a circle $\mathcal{C}$, every arc $\overset{\curvearrowright}{AB} \in (\pi/2, 2\pi)$ of $\mathcal{C}$ defines a chord that is the longer side AB of a decreasing continue cyclic quadrilateral $\mathcal{Q} = (A, B, C, D)$ inscribed in $\mathcal{C}$ (Lemma 2).

Clearly, the magnitude $\overset{\curvearrowright}{AB} = \pi/2$ deterimines a square $\mathcal{Q} = (A, B, C, D)$, thus the growth corresponding to this magnitude $\mu_{\mathcal{Q}}$ is 1 (see Figure 7 on the left). Moreover, easy considerations (see Figure 7 on the right) shows that when $\mathcal{Q} = (A, B, C, D)$ is a cyclic decreasing continue quadrilateral, then $\overset{\curvearrowright}{AB}$ must be greater than $\pi/2$.

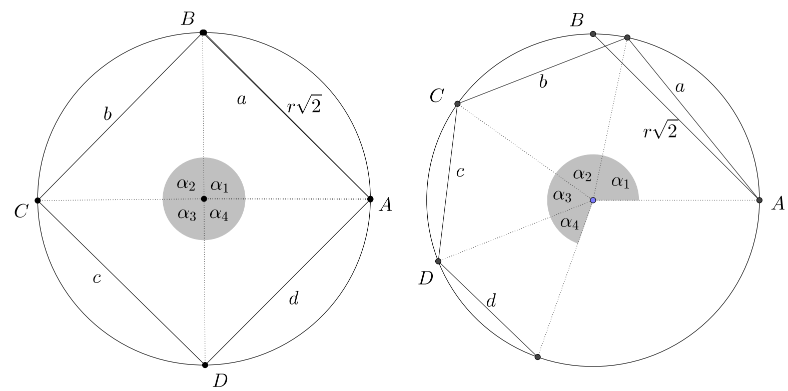

Figure 7: *A square has growth 1; In any cyclic decreasing continue quadrilateral $\mathcal{Q} = (A, B, C, D)$ the arc $\overset{\curvearrowright}{AB}$ cannot be less than $\pi/2$*

Lemma 2. *Let $\mathcal{C}$ be a circle with radius r, and fix $A \in \mathcal{C}$. Then for every arc $\overset{\curvearrowright}{AB} \in (\pi/2, 2\pi)$ there exists a unique decreasing continue cyclic quadrilateral $\mathcal{Q} = (A, B, C, D)$ inscribed in $\mathcal{C}$.*

Proof. For every arc $\overset{\curvearrowright}{AB}$ we put $\mathbf{a}$ be the chord AB. Assume first that $\overset{\curvearrowright}{AB} \in (\pi/2, \pi)$, so that

$$r\sqrt{2} < a < 2r, \text{ and hence } h^2 = \frac{a^2}{(2r)^2} > 1/2, \text{ where } \quad h := \frac{a}{2r}. \tag{4}$$

Now, let us consider the following function:

$$\begin{aligned} f_h : x \in [0,1] \to \\ f(x) &= (h\sqrt{1-h^2x^2} + hx\sqrt{1-h^2})(\sqrt{(1-h^2x^6)(1-h^2x^4)} - h^2x^5) \\ &\quad + (\sqrt{(1-h^2)(1-h^2x^2)} - h^2x)(hx^2\sqrt{(1-h^2x^6)} + hx^3\sqrt{(1-h^2x^4)}). \end{aligned}$$

In particular,

$$f_h(0) = h + \sqrt{1-h^2} \cdot 0 = h > 0. \tag{5}$$

On the other hand, by Equation 4 we have

$$f_h(1)=2h\sqrt{1-h^2}(1-2h^2)+(1-2h^2)\sqrt{1-h^2}\cdot 2h=4h\sqrt{1-h^2}(1-2h^2)<0. \quad (6)$$

Clearly, f_h is continue in $[0,1]$, so that there exist $\mu \in (0,1)$ such that $f(\mu)=0$.

Now, starting from **a**, let us consider four consecutive chords **a**, **b**, **c**, **d** on $\mathcal{C}$ (counterclockwise), whose lengths are $a, a\mu, a\mu^2, a\mu^3$, respectively, and put $\alpha_1,\alpha_2,\alpha_3,\alpha_4$, the magnitude of the respective arcs.

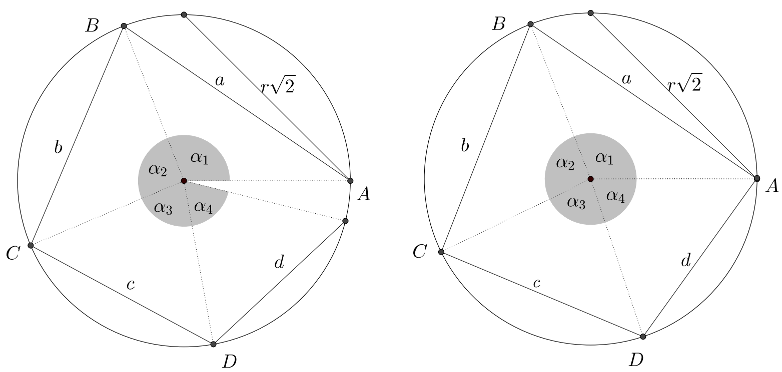

Figure 8: *Construction of continue quadrilaterals in a given circle* $\mathcal{C}$

In order to show that **a**, **b**, **c**, **d** are sides of an inscribed (cyclic) quadrilateral it is enough to show that $\alpha_1+\alpha_2+\alpha_3+\alpha_4=2\pi$ (see Figure 8).

Applying the theorem of sines and the fundamental trigonometric law we have:

$$\sin\frac{\alpha_1}{2}=h,\quad \sin\frac{\alpha_2}{2}=h\mu,\quad \sin\frac{\alpha_3}{2}=h\mu^2,\quad \sin\frac{\alpha_4}{2}=h\mu^3,$$
$$\cos\frac{\alpha_1}{2}=\sqrt{1-h^2},\ \cos\frac{\alpha_2}{2}=\sqrt{1-h^2\mu^2},\ \cos\frac{\alpha_3}{2}=\sqrt{1-h^2\mu^4},\ \cos\frac{\alpha_4}{2}=\sqrt{1-h^2\mu^6},$$

so that

$$\begin{aligned}
&\sin(\frac{\alpha_1}{2}+\frac{\alpha_2}{2}+\frac{\alpha_3}{2}+\frac{\alpha_4}{2})\\
&\quad=\sin(\frac{\alpha_1}{2}+\frac{\alpha_2}{2})\cos(\frac{\alpha_3}{2}+\frac{\alpha_4}{2})+\cos(\frac{\alpha_1}{2}+\frac{\alpha_2}{2})\sin(\frac{\alpha_3}{2}+\frac{\alpha_4}{2})\\
&\quad=(\sin\frac{\alpha_1}{2}\ \cos\frac{\alpha_2}{2}+\cos\frac{\alpha_1}{2}\ \sin\frac{\alpha_2}{2})(\cos\frac{\alpha_3}{2}\ \cos\frac{\alpha_4}{2}-\sin\frac{\alpha_3}{2}\ \sin\frac{\alpha_4}{2})\\
&\qquad+(\cos\frac{\alpha_1}{2}\ \cos\frac{\alpha_2}{2}-\sin\frac{\alpha_1}{2}\ \sin\frac{\alpha_2}{2})(\sin\frac{\alpha_3}{2}\ \cos\frac{\alpha_4}{2}+\cos\frac{\alpha_3}{2}\ \sin\frac{\alpha_4}{2})\\
&\quad=f(\mu)=0.
\end{aligned}$$

It follows that $\frac{\alpha_1}{2}+\frac{\alpha_2}{2}+\frac{\alpha_3}{2}+\frac{\alpha_4}{2}=m\pi$, for a suitable positive integer m, thus $\alpha_1+\alpha_2+\alpha_3+\alpha_4=2m\pi$. On the other hand, at least one chord is less than diameter of $\mathcal{C}$, for some $\alpha_i<\pi$. It follows that $\alpha_1+\alpha_2+\alpha_3+\alpha_4<4\pi$. It holds that $m = 1$ and $\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 = 2\pi$. This argument shows that the quadrilateral $\mathcal{Q} = (A, B, C, D)$ whose sides are $\mathbf{a}, \mathbf{a}\mu, \mathbf{a}\mu^{\mathbf{2}}, \mathbf{a}\mu^{\mathbf{3}}$ is a cyclic decreasing continue quadrilateral that contains the centre of $\mathcal{C}$.

In order to show that there exists a unique decreasing continue cyclic quadrilateral $\mathcal{Q} = (A, B, C, D)$ inscribed in $\mathcal{C}$, in which the larger sides are the chord corresponding to the arc $\overset{\frown}{AB} \in (\pi/2, \pi)$, let $\mathcal{Q}' = (A', B', C', D')$ be a decreasing continue cyclic quadrilateral inscribed in $\mathcal{C}$, with $A = A'$, $B = B'$. Suppose that $\mu_{\mathcal{Q}'} < \mu_{\mathcal{Q}}$. It follows that

$$a' = a, \quad a'\mu_{\mathcal{Q}'} < a\mu_{\mathcal{Q}}, \quad a'\mu_{\mathcal{Q}'}^2 < a\mu_{\mathcal{Q}}^2, \quad a'\mu_{\mathcal{Q}'}^3 < a\mu_{\mathcal{Q}}^3,$$

which give the corresponding relations on the arcs:

$$\overset{\frown}{A'B'} = \overset{\frown}{AB}, \quad \overset{\frown}{B'C'} < \overset{\frown}{BC}, \quad \overset{\frown}{C'D'} < \overset{\frown}{CD}, \quad \overset{\frown}{D'A'} < \overset{\frown}{DA}.$$

As $\mathcal{Q}'$ and $\mathcal{Q}$ are cyclic, we have

$$2\pi = \overset{\frown}{A'B'} + \overset{\frown}{B'C'} + \overset{\frown}{C'D'} + \overset{\frown}{D'A'} < 2\pi = \overset{\frown}{AB} + \overset{\frown}{BC} + \overset{\frown}{CD} + \overset{\frown}{DA}.$$

This is a contradiction. Similarly, it can be proved that $\mu_{\mathcal{Q}'}$ cannot be greater than $\mu_{\mathcal{Q}}$.

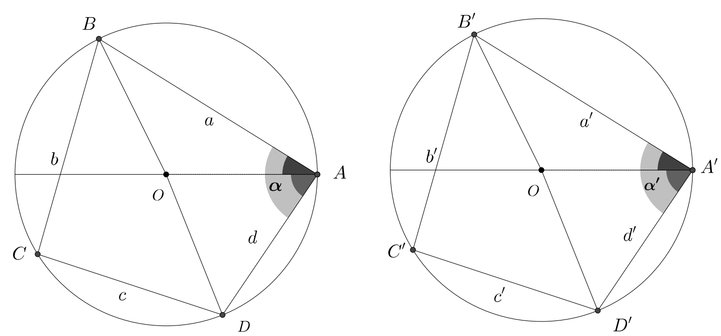

Figure 9: *If two quadrilaterals are inscribed in the same circle and have the sides in the same order and pairwise congruent, then they are congruent*

Therefore it must be $\mu_{\mathcal{Q}'} = \mu_{\mathcal{Q}}$, and $a' = a$, $b' = b$, $c' = c$, and $d' = d$.

A general argument (see Figure 9) shows that the triangles AOB and $A'O'B'$ are congruent, and in particular the angle $B\hat{A}O$ is equal to $B'\hat{A}'O'$. Similarly, it can be obtained that $O\hat{A}D = O'\hat{A}'D'$, so that $D\hat{A}B = D'\hat{A}'B'$. Therefore $\mathcal{Q}$ and $\mathcal{Q}'$ are congruent by the SAS criterion (see [12, Theorem 2.4]).

Assume now that $\overset{\frown}{AB} \in [\pi, 2\pi)$ (see Figure 10).

Put $h = \dfrac{a}{2r}$, and consider the function:

$$g_h : x \in [0,1] \to$$
$$g_h(x) = (h\sqrt{1 - h^2x^2} - hx\sqrt{1 - h^2})(\sqrt{(1 - h^2x^6)(1 - h^2x^4)} - h^2x^5)$$
$$- (\sqrt{(1 - h^2)(1 - h^2x^2)} + h^2x)(hx^2\sqrt{(1 - h^2x^6)} + hx^3\sqrt{(1 - h^2x^4)}).$$

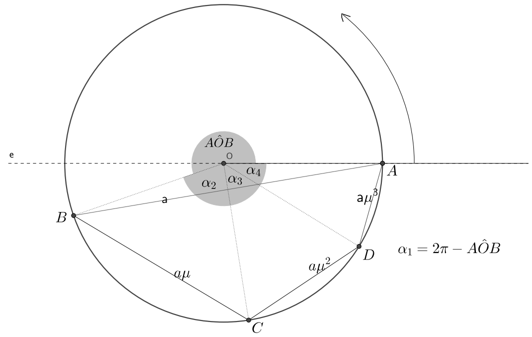

Figure 10: *Continue cyclic quadrilateral that does not contain the center*

Clearly, $g_h(0) > 0$ and $g_h(1) < 0$, so that there exists $\mu \in (0,1)$ such that $g_h(\mu) = 0$

Now, let us consider four consecutive chords $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$, $\mathbf{d}$ on $\mathcal{C}$, whose lengths are $a, a\mu, a\mu^2, a\mu^3$, respectively, and magnitudes of the respective (minor) arcs are $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ (see Figure 10).

A similar argument to we have just used in the first part of the proof shows that $\alpha_1 = \alpha_2 + \alpha_3 + \alpha_4$, so that $\mathcal{Q} = (A, B, C, D)$, whose sides are $\mathbf{a}, \mathbf{a}\mu, \mathbf{a}\mu^\mathbf{2}, \mathbf{a}\mu^\mathbf{3}$, is a decreasing continue quadrilateral which is also inscribed in $\mathcal{C}$.

To show the uniqueness we can argue as in the first case.

The proof is complete. $\square$

Remark 4. *Note that by Lemma 2, it is possible to determine a decreasing continue quadrilateral $\mathcal{Q} = (A, B, C, D)$ inscribed in a circle $\mathcal{C}$ of radius r, with $a > a\mu > a\mu^2 > a\mu^3$, even if $a < r\sqrt{2}$ (see Figure 11). Moreover, it is easy to see that for every chord DA of $\mathcal{C}$ with $r\sqrt{2} > DA > 0$ and $\overset{\frown}{DA} < \pi/2$, there exists a decreasing continue quadrilateral $\mathcal{Q} = (A, B, C, D)$ inscribed in a circle $\mathcal{C}$, in which the shorter side is DA (look at figures 11, 10, 8 and 7).*

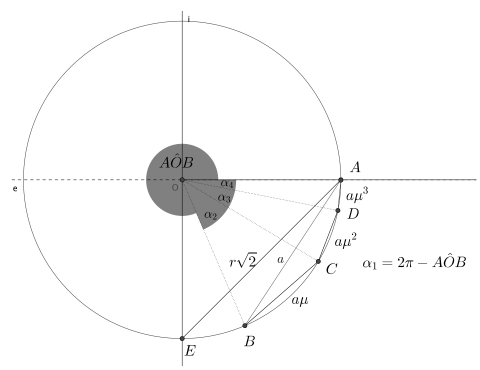

Figure 11: *Decreasing continue quadrilateral with all sides less than $r\sqrt{2}$*

Theorem 5. *Let $\mathcal{C}$ be a circle with radius r, and fix $A \in \mathcal{C}$. Then for every arc $\overset{\curvearrowright}{AB} \in (0, 2\pi)$ there exists only one continue cyclic quadrilateral $\mathcal{Q} = (A, B, C, D)$ inscribed in $\mathcal{C}$.*

Proof. By Lemma 2 it is enough to show that for every arc $\overset{\curvearrowright}{AB} \in (0, \pi/2)$ there exists a unique continue cyclic quadrilateral $\mathcal{Q} = (A, B, C, D)$ inscribed in $\mathcal{C}$. In order to find $\mathcal{Q}$, let us consider the chord $D'A$ corresponding to the arc $2\pi - \overset{\curvearrowright}{AB}$. By Remark 4 there exists a decreasing continue quadrilateral $\mathcal{Q}' = (A, B', C', D')$ inscribed in a circle $\mathcal{C}$, in which the shorter side is $D'A$ (see Figure 12).

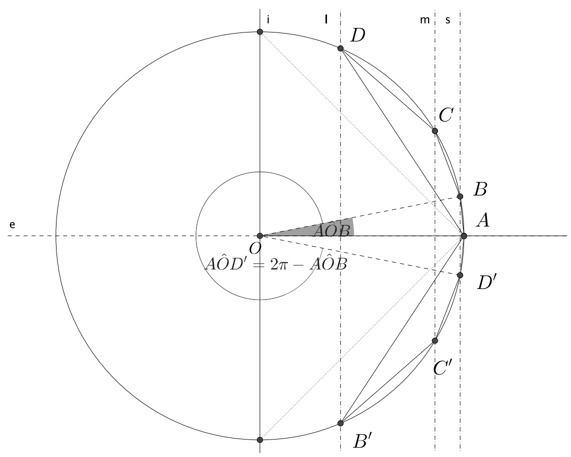

Figure 12: *Note that* $\mu_{\mathcal{Q}} = 1/\mu_{\mathcal{Q}'}$

It turns out that the symmetric of $\mathcal{Q}'$ gives an increasing quadrilateral $\mathcal{Q} = (A, B, C, D)$ inscribed in $\mathcal{C}$ such that $\overset{\curvearrowright}{AB} \in (0, \pi/2)$. The uniqueness follows trivially □

Remark 5. *Let $\overset{\curvearrowright}{AB} \in [\pi/2, 3/2\pi]$; then $\overset{\curvearrowright}{AB'} = 2\pi - \overset{\curvearrowright}{AB}$ also belongs to $[\pi/2, 3/2\pi]$. Clearly, the chords AB and AB' are equal, so that there are two different types of decreasing cyclic quadrilaterals, say $\mathcal{Q}$ and $\mathcal{Q}'$, inscribed in $\mathcal{C}$ with $a = a'$ (see Figure 13).*

We have seen (Corollary 1) that inside a circle $\mathcal{C}$ we can find cyclic continue quadrilaterals of any possible growth.

Remark 6. *Easy considerations show that a continue cyclic quadrilateral has a right angle if and only if it is a square.*

4. Conclusions and open questions

Geometric properties referring to quadrilaterals have been the object of many recent investigations and new results have been obtained (see [3] and reference therein). Experiment in classroom show that quadrilaterals could help us highlight and overcome

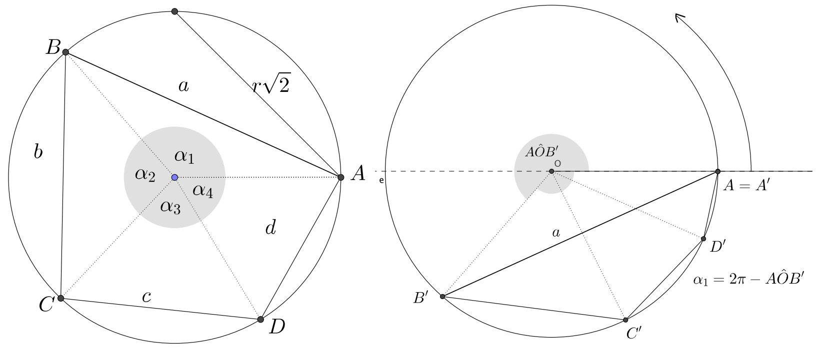

Figure 13: *Two different types of cyclic decreasing quadrilaterals $\mathcal{Q}$ and $\mathcal{Q}'$ with the same larger side. Their growths are $\mu_{\mathcal{Q}} \simeq 0.81 \neq \mu_{\mathcal{Q}'} \simeq 0.61$*

some misconceptions related to congruence of two polygons (see the last section of [12] and [13]).

We observe that in this article the use of dynamical software (Geogebra) has been useful, not only to show graphs of functions (see Figure 6) and all drawings of this article, but especially in order to conjecture the behavior of cyclic continue quadrilaterals. In fact, after proving a possible range of the growth of continue quadrilaterals (Theorem 2), starting from a circle $\mathcal{C} = (O, r)$, for every $a \in (0, 2r)$, one can consider the point $A \equiv (r, 0)$, and a variable (slide) parameter $\mu \in (1/\Phi_2, \Phi_2)$. It turns out that the chords whose lengths are $1, a\mu, a\mu^2, a\mu^3$ define a polygonal inscribed in $\mathcal{C}$ (see Figure 8 on the left). Increasing (or decreasing) μ, it can be seen that there are suitable values of μ such that the polygonal will be 'closed'; in other term it will be a quadrilateral (see Figure 8 on the right and Figure 11). This observation has been very useful in the article, in particular it allowed to conjecture the statement of Lemma 2 and Theorem 5. We think that the use of dynamical software should be encouraged at all levels of geometric exploration.

By a research point of view some questions arise. As we observed in the introduction, $\sqrt{\Phi}$ is the ratio of continue right triangles (Kepler triangles), and $\sqrt{\Phi}$ is the only positive root of the polynomial $x^4 - x^2 - 1$. Similarly, it is easy to see that $\sqrt[3]{\Phi_2}$ is the only positive root of the polynomial $x^9 - x^6 - x^3 - 1$, and that $\sqrt[3]{\Phi_2}$ is the ratio of continue quadrilaterals whose sides satisfy the relation $a^3 + b^3 + c^3 = d^3$.

Other properties related to elementary number theory could probably be connected to continue quadrilaterals.

Moreover, other euclidean geometric questions on continue quadrilaterals could be considered, for example:

1) What can we say about *orthic* or *pedal quadrilaterals* (as a general reference to these notions see [17] and [6] and references therein) of a continue quadrilateral?

2) The notion of continue polygons can be introduced and developed: In which terms can we extend the results obtained for quadrilaterals for general n-gons ($n > 4$)?

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