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Analytical Analysis of Distribution of Bending Stresses in Layers of Plywood with Numerical Verification

Analiza raspodjele naprezanja u slojevima furnirske ploče pri savijanju, s računskom provjerom

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ABSTRACT • *The study presents methods for accurate estimation of bending stresses in the 3-point flexural bending test of plywood, i.e. a wood-based laminate with an alternate crosswise ply configuration. The characteristic bending strength (MOR) and mean modulus of elasticity (MOE) of standard beech plywood was determined using European Standard bending tests EN 310. Correlations were determined between empirically determined bending moduli of the plywood and material moduli of the veneer layer. Calculations were conducted based on the classical plate theory for thin panels comprising the theory of elasticity including the Kirchhoff-Love hypothesis. Rigidity of individual layer was established theoretically in the axial configuration of transformed rigidity matrix values. Numerical laminate models were developed and simulation tests were conducted. Results of experimental and analytical studies were verified using the Finite Element Method (FEM). Analyses were performed in two plywood cross-band arrangement variants. An analysis of the distribution of stresses in individual layers of plywood used an analytical and numerical method assuming the plywood specimen to be a rhombic-anisotropic material.*
It was found that the bending load capacity of plywood depends on the configuration of individual layers (veneers). Values of stresses originating from bending do not only depend on the distance of the considered plywood layer from the middle layer but also on stiffness in the direction of operating stresses. Bending strength varies in individual directions of the plywood panel. Therefore, the distribution of stresses in individual layers differs from that resulting from the stress distribution for homogeneous isotropic materials. Results are presented in the form of tables, bitmaps, graphs and photographs. The tests were conducted based on the BFU-BU-18 standard beech plywood thickness of 18 mm.

Keywords: *plywood, stress distribution, veneers, layers, FEM, numerical analysis*

SAŽETAK • *U radu se opisuju metode točne procjene naprezanja pri testu savijanja furnirske ploče u tri točke, tj. pri savijanju laminata na bazi drva s naizmjenično okomito postavljenim slojevima. Karakteristični modul loma (MOR) i srednja vrijednost modula elastičnosti (MOE) standardne furnirske ploče od bukovine utvrđeni su*

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prema EN 310. Uočena je povezanost između empirijski određenih modula na savijanje furnirskih ploča i modula na savijanje slojeva furnira. Izračuni su provedeni na temelju klasične teorije za tanke ploče, koju čine teorija elastičnosti i Kirchhoff-Loveova hipoteza. Krutost pojedinog sloja utvrđena je teorijski u aksijalnoj konfiguraciji izmijenjenih vrijednosti matrice krutosti. Razvijeni su računski modeli laminata i provedena su simulacijska ispitivanja. Rezultati eksperimentalnih i analitičkih ispitivanja potvrđeni su metodom analize konačnih elemenata (FEM). Analize su provedene na dvije vrste furnirskih ploča s okomito usmjerenim slojevima furnira. Za analizu raspodjele naprezanja u pojedinim slojevima furnira furnirske ploče primijenjena je analitička i računska metoda, uz pretpostavku da je uzorak furnirske ploče rombo-anizotropni materijal. Utvrđeno je da kapacitet savijanja furnirske ploče ovisi o konfiguraciji pojedinih slojeva (furnira). Vrijednosti naprezanja koje proizlaze iz savijanja ne ovise samo o udaljenosti promatranog sloja od srednjeg sloja furnirske ploče, već i o krutosti u smjeru naprezanja. Čvrstoća na savijanje u pojedinim smjerovima furnirske ploče varira. Stoga se raspodjela naprezanja u pojedinim slojevima razlikuje od raspodjele naprezanja za homogene izotropne materijale. Rezultati su prikazani uz pomoć tablica, grafova i fotografija. Ispitivanja su provedena na standardnoj furnirskoj ploči od bukovine debljine 18 mm BFU-BU-18.

Ključne riječi: *furnirska ploča, raspodjela naprezanja, furniri, slojevi, FEM, računska analiza*

1 INTRODUCTION
1. UVOD

Plywood as a structural material exhibits properties comparable to those of composite materials. Thanks to its mechanical properties, plywood is a superior wood-based panel material. It is a building material consisting of veneers (thin wood layers) bonded with an adhesive. Depending on its type and properties, it has numerous applications in various sectors of industry, such as furniture making, joinery, construction engineering, e.g. in concrete forming, as well as boat building, aircraft and glider design, etc. Plywood is used in a variety of structural applications, such as floors, siding, formwork and engineered wood products prefabricated I-joists, box beams, and panel roofs (Stark *et al.,* 2010). Plywood is particularly useful in construction engineering as an engineered wood-based material with properties comparable to those of glulam. It is strengthened by cellulose fibers arranged along the tree stem or trunk and surrounded by a matric composed mainly of lignin. The macroscopic structure of plywood is created by gluing alternately arranged, thin cross-bands, i.e. veneers of various thickness. The cross-laminated veneer layers give plywood excellent strength characteristics, stiffness and dimensional stability (Youngquist, 1999). Physical and mechanical properties of plywood are dependent primarily on wood species, thickness, orientation of glued veneer sheets as well as the type of adhesive and the applied gluing method (Youngquist, 1999; Hráskỳ and Král, 2005; Stark *et al.,* 2010). Plywood is a major type of wood-based composite material for structural use. By identifying its properties and their modification focusing on performance characteristics in technological processes of plywood manufacture and thanks to its combination with other materials, the so-called modified plywood may be produced. Such an innovative structural material, e.g. with an aluminum core, has considerably extended applicability in various branches of industry. Additionally, such plywood is superior to other materials thanks to its increased strength and the simultaneously reduced effect of material defects. The application of plywood in structures as a load-transferring element occasionally requires an analysis of its rigidity and strength, particularly in critical areas of stress accumulation. It is crucial when the effect of stress accumulation determines the performance and load bearing capacity of the entire structure. This study investigates analytical and numerical methods to determine stress values in individual layers of plywood subjected to a three-point bending test. Analytical calculations were performed at the assumptions resulting from the classical theory of plates. The classical laminated plate theory (CLPT) is an extension of the classical plate theory to composites laminates (Reddy, 1997):

- Layers are permanently bound and their interface is indefinitely thin and prevents interlayer shear,
- Strains change edgewise in a continuous or constant manner depending on load (due to the varying layer thickness),
- A change in stresses at the interfaces is stepwise and edgewise of individual plies and it occurs in a continuous or constant manner,
- The Kirchhoff-Love theory referring to thin plate bending is binding: a straight line perpendicular to the mid-surface remains unchanged after the administration of a load acting in the mid-surface and the normal section to the mid-surface does not change in length ($\varepsilon_z = 0$),
- A composite as a whole forms a single layer, but with properties constituting a resultant of properties of individual layers,
- The assumption of small displacements is binding.

The proposed method of analytical calculations is relatively complex in comparison with other methods for the determination of strength parameters in wood and wood-based materials. Thanks to this approach, more accurate results can be obtained concerning the distribution of stresses in individual layers, particularly at their arbitrary arrangement in the laminated material (Kljak and Brezović, 2006). As most tests for mechanical properties are destructive in nature, they tend to be costly and time-consuming. It would, therefore, be worthwhile to examine current technology to adopt solutions that save time and are cost-effective. A recent approach to analyzing the mechanical behaviour of plywood is the Finite Element Method (FEM), which has been used to solve many special problems in timber design. However, this method is not commonly applied to analyze the mechanical properties of plywood specimens.

2 MATERIALS AND METHODS
2. MATERIJALI I METODE

2.1 Materials
2.1. Materijali

Experimental tests on plywood were performed in accordance with the PN-EN 310:1994 standard on small specimens cut from one sheet of commercial 18 mm thick beech plywood. A total of twelve specimens were collected in two variants differing in the arrangement of cross-bands, i.e. outer layers. In the first six specimens of variant I (0/90/0/90/0…), the direction of cross-band grain was parallel to specimens length, while in the other six specimens of variant II (90/0/90/0/90…), the arrangement was transverse. Geometrical dimensions of specimens are given in Table 1.

Tests were performed on a strength testing machine in a 3-point bending system with the computer measuring equipment recording values of loading force and displacement (Brancheriau et al., 2002). Bending forces were applied on specimens in the elastic-linear and plastic range until a complete sample failure. A total of 12 panels of plywood were used for each thickness in the cutting plan of (EN 310).

The dimensions, shape and load of the test specimens for the determination of bending strength of plywood specimens are presented in Figures 1.

Each plywood panel was cut into two groups of bending test specimens, 6 pieces parallel 0° to the grain direction and 6 pieces perpendicular 90° to the grain direction. The test specimen was rectangular with the width (b), 50 ± 1 mm, and the length 20 times the nominal thickness (t). The test specimens were conditioned to a constant mass in an atmosphere with a relative humidity of 65±5 % and a temperature of 20±2 °C. A cylindrical loading head with the diameter 30.0±0.5 mm was placed parallel to the supports. The load applied to

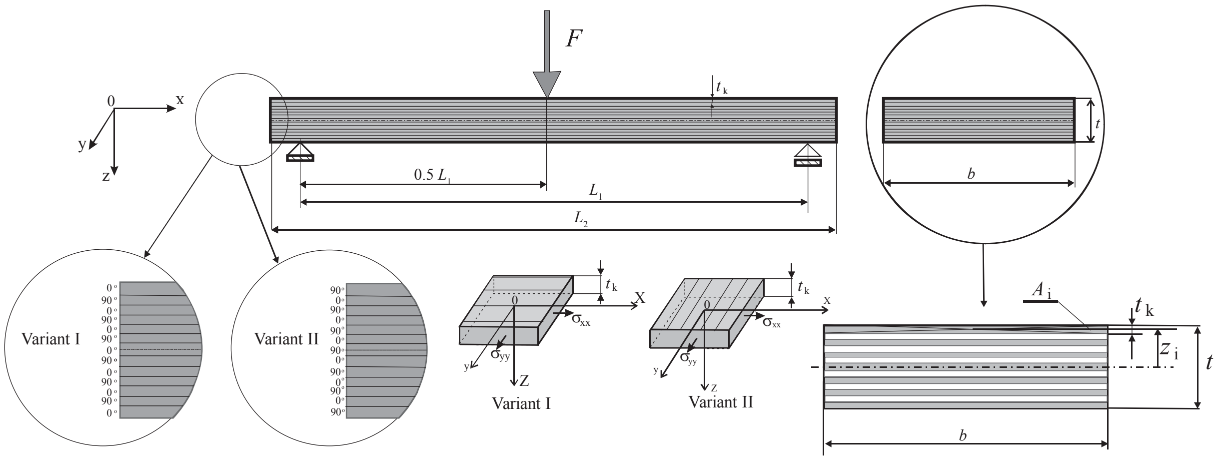

Figure 1 Dimensions and orientation of layers of plywood specimens
Slika 1. Dimenzije i smjer protezanja slojeva uzoraka furnirske ploče

Table 1 Dimensions of specimens
Tablica 1. Dimenzije uzoraka

Plywood / *Furnirska ploča*										
Number of specimen *Broj uzorka*	**Cross-sectional area of the layer** *Površina poprečnog presjeka sloja*	**Variant I** / *Varijanta I.* (0/90/0/90/0/90/0/90/0/90/0/90/0)				**Cross-sectional area of the layer** *Površina poprečnog presjeka sloja*	**Variant II** / *Varijanta II.* (90/0/90/0/90/0/90/0/90/0/90/0/90)			
		Length *Dužina*		Thickness *Debljina*	Width *Širina*		Length *Dužina*		Thickness *Debljina*	Width *Širina*
	A_i	L_1	L_2	t	b	A_i	L_1	L_2	t	b
	mm²	mm				mm²	mm			
1	901.45		410.09	17.95	50.22	894.33		410.09	17.78	50.30
2	894.13		409.89	17.79	50.26	895.01		409.89	17.79	50.31
3	893.48	360	410.42	17.77	50.28	889.92		410.42	17.77	50.08
4	889.63		410.07	17.75	50.12	892.65	360	410.07	17.75	50.29
5	891.58		410.31	17.75	50.23	892.29		410.31	17.75	50.27
6	887.09		410.20	17.71	50.09	892.58		410.20	17.71	50.40
Mean *Srednja vrijednost*	892.89		410.17	17.79	50.20	892.80		410.16	17.76	50.26

Table 2 Bending properties of plywood (Variant I)
Tablica 2. Savojna svojstva furnirske ploče (varijanta I.)

Specimen No. / *Broj uzorka*	**Plywood Variant I** / *Furnirska ploča, varijanta I.* (0/90/0/90/0/90/0/90/0/90/0/90/0)					
	Maximum Load *Najveće opterećenje*	**Modulus of elasticity** *Modul elastičnosti*	**Modulus of rapture** *Modul loma*	**Increment of load** *Povećanje opterećenja*	**Increment of deflection** *Povećanje progiba*	
					Experimental *Eksperimentalno*	**Numeric** *Računski*
	F_{max}	MOE	MOR	F_2-F_1	a_2-a_1	f_{FEM}
	N	N/mm²	N/mm²	N	mm	
1	2482	9455	82.13	744	3.16	3.20
2	2266	8913	76.93	679	3.14	3.12
3	2448	9337	83.26	734	3.25	3.16
4	2164	9002	74.00	649	3.00	2.79
5	2420	9510	82.57	726	3.17	3.12
6	2587	9596	88.92	776	3.39	3.33
Mean *Srednja vrijednost*	2394	9302	81.42	718	3.18	3.12

Table 3 Bending properties of plywood (Variant II)
Tablica 3. Savojna svojstva furnirske ploče (varijanta II.)

Specimen No. / *Broj uzorka*	**Plywood Variant II** / *Furnirska ploča, varijanta II.* (0/90/0/90/0/90/0/90/0/90/0/90/0)					
	Maximum Load *Najveće opterećenje*	**Modulus of elasticity** *Modul elastičnosti*	**Modulus of rapture** *Modul loma*	**Increment of load** *Povećanje opterećenja*	**Increment of deflection** *Povećanje progiba*	
					Experimental *Eksperimentalno*	**Numeric** *Računski*
	F_{max}	MOE	MOR	F_2-F_1	a_2-a_1	f_{FEM}
	N	N/mm²	N/mm²	N	mm	
1	1946	6119	66.08	583	3.88	3.89
2	1893	6044	64.20	568	3.87	3.79
3	1918	6104	65.49	575	3.91	3.83
4	1908	5931	65.03	572	4.00	3.79
5	1864	6119	63.55	559	3.79	3.73
6	1823	5997	62.27	547	3.80	3.62
Mean *Srednja vrijednost*	1892	6052	64.44	567	3.87	3.78

the test specimen was adjusted so that the maximum load was reached within 60 ±30 s throughout the test.

The modulus of rigidity (MOR) of each test pieces is calculated as follows:

$$\mathrm{MOR} = \frac{3 \cdot F_{max} \cdot L_1}{2 \cdot b \cdot t^2} \quad (\mathrm{N/mm^2}) \qquad (1)$$

The modulus of elasticity (MOE) of each test pieces is calculated as follows:

$$MOE = \frac{L_1^3 \cdot (F_2 - F_1)}{4 \cdot b \cdot t^3 \cdot (a_2 - a_1)} \quad (\mathrm{N/mm^2}) \qquad (2)$$

Where:
F_{max} –maximum load (N),
L_1 – distance (mm) between the centre of two supports (mm),
b – width (mm) of test pieces,
t – thickness (mm) of test pieces as depicted in (Figure 1).
(F_2 - F_1) – increment of load on straight line portion of the load–deflection curve, where F_1 was approximately 10 % and F_2 was approximately 40 % of the maximum load F_{max},
(a_2 - a_1) – increment of deflection corresponding to (F_2 - F_1) in the load–deflection curve.

The results of MOE, MOR and deflection in all tested directions of plywood specimens are shown in Tables 2 and 3.

3 ANALYTICAL CALCULATIONS
3. ANALITIČKI IZRAČUNI

Strength analysis in the macroscopic scale of wood-based composite was conducted taking into consideration the theory of thin plate bending (Bodig and Jayne, 1982; Goodman and Bodig, 1970; German, 1996). Mechanical properties of plywood as a whole are mainly dependent on the properties of its individual layers (Kljak and Brezović, 2007; Makowski, 2014). These parameters have been successfully described by the rhomboid-anisotropic material model in the range of linear-elastic values (Goodman and Bodig, 1970; Lekhnitskii, 1982; Reddy, 1997). Layers in a standard plywood are rotated at an angle of 90°, which corresponds to the so-called symmetric laminate with an alternate layers arrangement. Each layer contains two main material axes: longitudinal and transverse. In the conducted theoretical and numerical analyses, the following material data were used (Keylwerth, 1951; Goodman and Bodig, 1970; Neuhaus, 1994) corresponding to mean values of beech wood (*Fagus sylvatica* L.): density ρ=690 kg/m^3, 12 % moisture content, coefficient of elasticity $E_L=E_{11}$=14000 N/mm^2, $E_T=E_{22}$=1160 N/mm^2, $G_{12}=G_{LT}$=1080 N/mm^2, $\nu_{LT}=\nu_{12}$=0.52 (major), $\nu_{TL}=\nu_{21}$=0.043 (minor), where E_L,E_T, and E_R are Young's moduli in the longitudinal (L), tangential (T) and radial (R) directions of the veneer. Based on the above-mentioned data using the indirect method, a correlation was described between Young's modulus of layers and the modulus of the laminate. In analytical calculations in the plane perpendicular to the panel, the former modulus was estimated at bending parallel to cross-band grain and the latter - at bending perpendicular to grain. It was assumed that flexural rigidity of the layer system depended on the flexural rigidity of its individual layers, which is consistent with the assumptions of the Kirchhoff-Love theory. Applying denotations from Figure 1, the equation of rigidity for the laminated material may be written as follows (Bodig and Jayne, 1982; Wang and Chang, 1978; Lemaitre and Chaboche, 1990; Tsai and Hahn, 1980).

$$E_{gx} I_y = \sum_{i=1}^{n} E_{xi} I_{iy}^{(o)} \quad (3)$$

Where:
E_{gx} – modulus of elasticity at laminate bending,
E_{xi}– modulus of elasticity at laminate layer bending,
I_y – moment of inertia for cross-section in relation to neutral axis y,
n – number of layers with an identical orientation of veneers,
$I_{iy}^{(o)}$ – moment of inertia for cross-section of the i-th layer in relation to the neutral axis y,
$I_{iy}^{(o)} = I_{iy}^{'} + A_i (z_i^{'})^2$- acc. with Steiner's theorem (Figure1), $I_{iy}^{'} = \frac{b \cdot t_k^3}{12}$

Taking into consideration the crosswise arrangement and geometrical symmetry of laminate layers the above dependence may be expressed as follows:

$$E_{gx} I_y = 2\sum_{i=1}^{n} E_{xi}^{0} \cdot I_{iy}^{0} + 2\sum_{i=1}^{n} E_{xi}^{90} \cdot I_{iy}^{90} + E_x^m \cdot I_{my} \quad (4)$$

Where: m – denotes the direction of grain arrangement in the core.

Based on the above-mentioned dependencies, the theoretical bending moduli were estimated:
- for the laminate with a 0° angle of cross-band layer orientation:

$$E_{gx}^{0} = \frac{2}{I_y}\left\{\sum_{i=1}^{n} E_{xi}^{0} \cdot I_{iy}^{0} + \sum_{i=1}^{n} E_{xi}^{90} \cdot I_{iy}^{90} + \frac{1}{2} E_x^m \cdot I_{my}\right\} = \\ = \frac{2}{I_y}\left\{E_{xi}^{0} \sum_{i=1}^{n} I_{iy}^{0} + E_{xi}^{90} \sum_{i=1}^{n} I_{iy}^{90} + \frac{1}{2} E_x^m \cdot I_{my}\right\},$$

Where: $E_{\mathrm{xi}}^{0} = 14000\ \frac{\mathrm{N}}{\mathrm{mm}^2}$; $E_{\mathrm{xi}}^{90} = 1160\ \frac{\mathrm{N}}{\mathrm{mm}^2}$;
$I_{\mathrm{y}} = \frac{b \cdot t^3}{12} = 24300\ \mathrm{mm}^4$; $I_{\mathrm{m}} = \frac{b \cdot t_{\mathrm{k}}^3}{12} = 11.43\ \mathrm{mm}^4$;
$E_{\mathrm{x}}^{m} = E_{\mathrm{xi}}^{0}$

$$E_{\mathrm{gx}}^{0} = 9076.9 \frac{N}{\mathrm{mm}^2}$$

- for the laminate with a 90° angle of cross-band layer orientation:

$$E_x^m = E_{xi}^{90^o}$$

$$E_{gx}^{90} = \frac{2}{I_y}\left\{\sum_{i=1}^{n} E_{xi}^{90} \cdot I_{iy}^{90} + \sum_{i=1}^{n} E_{xi}^{0} \cdot I_{iy}^{0} + \frac{1}{2} E_x^m \cdot I_{my}\right\} = \\ = \frac{2}{I_y}\left\{E_{xi}^{90} \sum_{i=1}^{n} I_{iy}^{90} + E_{xi}^{0} \sum_{i=1}^{n} I_{iy}^{0} + \frac{1}{2} E_x^m \cdot I_{my}\right\},$$

$$E_{\mathrm{gx}}^{90} = 6108.6 \frac{N}{\mathrm{mm}^2}$$

Flexural moduli, calculated using this method, differ slightly from the results of laboratory experiments: $E_{gx}{}^{0}$=9076.9 N/mm^2, $E_{gx}{}^{90}$=*6108.6* N/mm^2. The values in parentheses refer to empirical results. Scatter of comparative results is very limited, for the longitudinal modulus amounting to 2.42 % and for transverse modulus - to 0.9 %, respectively. This shows that the adopted methodology of determining plywood parameters is appropriate. As a result, for these material data, it was decided to establish analytically the distribution of stresses in plywood layers subjected to bending tests. By limiting the study to linear relationships in terms of applicability of Hooke's law, the following dependencies between stresses and strains are obtained (Bodig and Jayne, 1982; Reddy, 1997; Tsai and Hahn, 1980; Stefańczyk *et al.,* 2005):

$$\{\sigma\} = [\mathbf{Q}][\mathbf{T}]\{\varepsilon\} = [\mathbf{Q}^*]\{\varepsilon\};$$
$$\begin{Bmatrix}\sigma_{\mathrm{xx}}\\ \sigma_{\mathrm{yy}}\\ \tau_{\mathrm{xy}}\end{Bmatrix} = \begin{bmatrix}\mathrm{Q}_{11}^* & \mathrm{Q}_{12}^* & \mathrm{Q}_{16}^*\\ \mathrm{Q}_{12}^* & \mathrm{Q}_{22}^* & \mathrm{Q}_{26}^*\\ \mathrm{Q}_{16}^* & \mathrm{Q}_{26}^* & \mathrm{Q}_{66}^*\end{bmatrix}\begin{Bmatrix}\varepsilon_{\mathrm{xx}}\\ \varepsilon_{\mathrm{yy}}\\ \gamma_{\mathrm{xy}}\end{Bmatrix} \quad (5)$$

The reduced matrix of rigidity [Q*] for any given layer k in the axial configuration, expressed in terms of engineering constants, takes the form:

$$[\mathbf{Q}^*]^k = [\mathbf{Q}]^k = \begin{bmatrix} mE_{11} & m\nu_{12}E_{22} & 0\\ m\nu_{21}E_{11} & mE_{22} & 0\\ 0 & 0 & G_{12}\end{bmatrix}^k = \begin{bmatrix}\mathrm{Q}_{11} & \mathrm{Q}_{12} & 0\\ \mathrm{Q}_{21} & \mathrm{Q}_{22} & 0\\ 0 & 0 & \mathrm{S}_{66}\end{bmatrix}^k \quad (6)$$

Where: $m=(1-\nu_{12}\nu_{21})^{-1}$ assuming that $\nu_{12}/\mathrm{E}_{11}=\nu_{21}/\mathrm{E}_{22}$, E_{11} - the so-called longitudinal Young's modulus, E_{22} - the so-called transverse Young's modulus, G_{12} – shear modulus, ν_{12} – the so-called major Poisson's ratio, ν_{21} – the so-called minor Poisson's ratio.

Elementary component layers of the laminated material have various reduced rigidity matrices $[Q^*]^k$. The constitutive equation of any k-th layer is written as:

$$\begin{Bmatrix}\sigma_{\mathrm{xx}}\\ \sigma_{\mathrm{yy}}\\ \tau_{\mathrm{xy}}\end{Bmatrix}^k = [\mathbf{Q}^*]^k \begin{Bmatrix}\varepsilon_{\mathrm{xx}}\\ \varepsilon_{\mathrm{yy}}\\ \gamma_{\mathrm{xy}}\end{Bmatrix}^k \quad (7)$$

From Eq. 5, values of stresses found in any laminate layer may be calculated having an a priori knowledge of the value of its strain. Strains were calculated using geometrical relationships linking strain and curvature of the laminate mid-surface:

$$\varepsilon_{xx} = \varepsilon_{xx}^0 - zk_x$$
$$\varepsilon_{yy} = \varepsilon_{yy}^0 - zk_y \quad (8)$$
$$\gamma_{xy} = \gamma_{xy}^0 - zk_{xy}$$

Thus the relationships from Eq. 7 may be written as:

$$\begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \tau_{xy} \end{Bmatrix}^k = \left[\mathbf{Q}^*\right]^k \begin{Bmatrix} \varepsilon_{xx}^0 - zk_x^0 \\ \varepsilon_{yy}^0 - zk_y^0 \\ \varepsilon_{xy}^0 - zk_{xy}^0 \end{Bmatrix}^k \quad (9)$$

By replacing stresses with external forces acting on the laminate (external forces N and bending moments M), the above equation takes the form:

$$\begin{Bmatrix} N \\ M \end{Bmatrix} = \begin{bmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{B} & \mathbf{D} \end{bmatrix} \begin{Bmatrix} \varepsilon^0 \\ k^0 \end{Bmatrix} \quad (10)$$

Where: **A** – longitudinal rigidity matrix,
B – coupling rigidity matrix,
D – bending rigidity matrix.

At bending of symmetric laminates (an odd number of plies), the coupling matrix in the absence of external forces N=0 flexing the composite is reduced to zero [**B**]=0. The above equation takes the form:

$$\{M\} = [\mathbf{D}]\{k^0\} = \begin{bmatrix} D_{11} & D_{21} & 0 \\ D_{12} & D_{22} & 0 \\ 0 & 0 & D_{66} \end{bmatrix} \begin{Bmatrix} k_x^0 \\ k_Y^0 \\ k_{xY}^0 \end{Bmatrix} \quad (11)$$

By transforming the equation in terms of the bending curvature of the laminate in the neutral surface the following is obtained:

$$\{k^0\} = [\mathbf{D}]^{-1}\{M\} \quad \text{or}$$

$$\begin{Bmatrix} k_x^0 \\ k_y^0 \\ k_{xy}^0 \end{Bmatrix} = \begin{bmatrix} D_{11} & D_{21} & 0 \\ D_{12} & D_{22} & 0 \\ 0 & 0 & D_{66} \end{bmatrix}^{-1} \{M\} \quad (12)$$

Where: bending moment for cross-section width is expressed as follows:

$$\{M\} = \begin{Bmatrix} M'/b \\ 0 \\ 0 \end{Bmatrix} \quad (13)$$

Where $M'=0.5\cdot F\cdot 0.5\cdot L_1$ – maximum resultant moment of bending of plywood specimens in the linear-elastic range ($F=F_2-F_1$). Components of the bending rigidity matrix depending on the type of laminated material, i.e. regular and symmetric with a cross-wise ply arrangement, are expressed as follows (German, 1996):

$$D_{11} = \frac{Q_{11}t^3}{12}\frac{1}{n^3}\left[P(4P^2-3) + \frac{E_{22}}{E_{11}}R(4R^2-3)\right]$$

$$D_{22} = \frac{Q_{22}t^3}{12}\frac{1}{n^3}\left[P(4P^2-3) + \frac{E_{11}}{E_{22}}R(4R^2-3)\right] \quad (14)$$

$$D_{12} = \frac{Q_{12}t^3}{12},\ D_{66} = \frac{Q_{66}t^3}{12},\ D_{16} = D_{26} = 0$$

Where: t - total laminate thickness, n - number of layers 13, P - number of layers oriented at $\alpha=0^o$, R - number of layers oriented at $\alpha = 90^o$.

3.1 Results of analytical calculations
3.1. Rezultati analitičkih izračuna

Reduced rigidity matrices for the 0° layer and the 90° layer based on (Eq.6) may be determined as:

$$[\mathbf{Q}]^k = [\mathbf{Q}^*]^{0^o} = \begin{bmatrix} 14320 & 617 & 0 \\ 617 & 1186 & 0 \\ 0 & 0 & 1080 \end{bmatrix} \frac{\text{N}}{\text{mm}^2}$$

$$[\mathbf{Q}]^k = [\mathbf{Q}^*]^{90^o} = \begin{bmatrix} 1186 & 617 & 0 \\ 617 & 14320 & 0 \\ 0 & 0 & 1080 \end{bmatrix} \frac{\text{N}}{\text{mm}^2}$$

Bending moment for variant I based on equation (Eq.13), using data contained in Table 2 and Figure 1, is as follows:

$$\{M\} = -\begin{Bmatrix} \dfrac{0.5\cdot 0.718\ \text{kN}\cdot 180\,\text{mm}}{50\,\text{mm}} \\ 0 \\ 0 \end{Bmatrix} = -1.292\ \text{kN mm/mm}$$

Taking into consideration the above results and after algebraic calculations, the elements of the bending rigidity matrix $[\mathbf{D}]_0$ for the 0° core layer at P=7, R=6, are as follows:

$$[\mathbf{D}]_0 = \begin{bmatrix} D_{11} & D_{21} & 0 \\ D_{12} & D_{22} & 0 \\ 0 & 0 & D_{66} \end{bmatrix} = \begin{bmatrix} 4501.663 & 299.862 & 0 \\ 299.862 & 3033.187 & 0 \\ 0 & 0 & 524.880 \end{bmatrix} \text{kNmm}$$

while elements of the inverse bending rigidity matrix are:

$$[\mathbf{D}]_0^{-1} = \begin{bmatrix} 2.2361 & -0.2211 & 0 \\ -0.2211 & 3.3187 & 0 \\ 0 & 0 & 19.0519 \end{bmatrix} 10^{-4} \frac{1}{\text{kNmm}}$$

Estimated curvatures of the bending neutral layer, according to formula (Eq.12) for plywood variant I, are as follows:

$$\begin{Bmatrix} k_x^0 \\ k_y^0 \\ k_{xy}^0 \end{Bmatrix} = \begin{bmatrix} D_{11} & D_{21} & 0 \\ D_{12} & D_{22} & 0 \\ 0 & 0 & D_{66} \end{bmatrix}^{-1} \begin{Bmatrix} M_x \\ 0 \\ 0 \end{Bmatrix} = \begin{Bmatrix} -2.8899 \\ 0.2857 \\ 0 \end{Bmatrix} 10^{-4} \frac{1}{\text{mm}}$$

For plywood variant II, calculations for its parameters are as follows:
The bending moment is:

$$\{M\} = -\begin{Bmatrix} \dfrac{0.5\cdot 0.567\,\text{kN}\cdot 180\,\text{mm}}{50\,\text{mm}} \\ 0 \\ 0 \end{Bmatrix} = -1.021\ \text{kNmm/mm}$$

Curvature values:

$$\begin{Bmatrix} k_x^0 \\ k_y^0 \\ k_{xy}^0 \end{Bmatrix} = \begin{bmatrix} D_{11} & D_{21} & 0 \\ D_{12} & D_{22} & 0 \\ 0 & 0 & D_{66} \end{bmatrix}^{-1} \begin{Bmatrix} M_x \\ 0 \\ 0 \end{Bmatrix} = \begin{Bmatrix} -3.3871 \\ 0.2256 \\ 0 \end{Bmatrix} 10^{-4} \frac{1}{\text{mm}}$$

Knowing values of these curvatures referring to the neutral surface, values of true strain in individual layers may be estimated. Assuming the linear character of strain distribution, their values are dependent on the distance from the laminate middle layer. In the analysis, the distribution of shear stresses was disregarded as

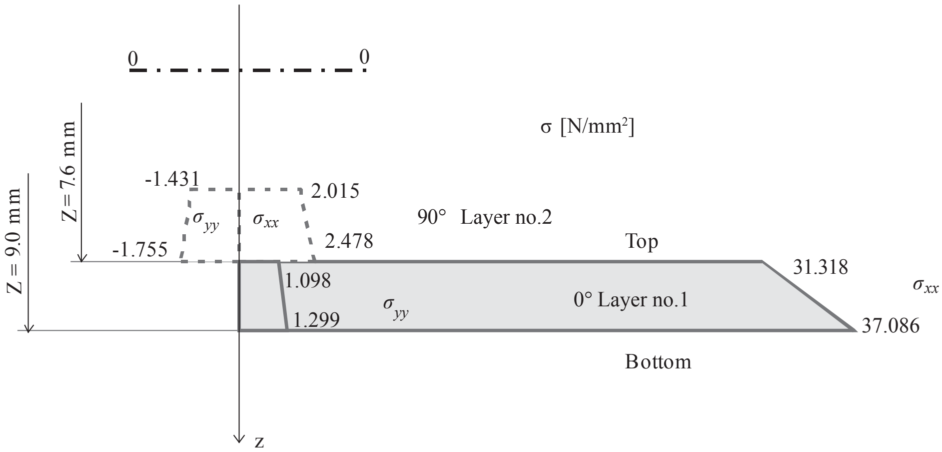

Figure 2 Distribution of stresses in the bottom layer No. 1 of variant I
Slika 2. Raspodjela naprezanja u donjem sloju (sloju br. 1.) za varijantu I.

small in relation to the t/L_1 ratio. A detailed algorithm to calculate stresses using the analytical method for an arbitrarily selected layer No.1 from variants I and II is as follows:
Calculations for variant I

$$[\sigma]_z=[\mathbf{Q}^*]\{\varepsilon\}$$

The bottom surface of layer 1
Data: $z = 9$ mm, $\varepsilon_x^0 = \varepsilon_y^0 = 0$
$\varepsilon_{xx} = \varepsilon_x^0 - z\,k_x = 2.6009\ 10^{-3}$
$\varepsilon_{yy} = \varepsilon_y^0 - z\,k_y = -\ 0.2571\ 10^{-3}$

$$\begin{Bmatrix}\sigma_{xx}\\ \sigma_{yy}\\ \tau_{xy}\end{Bmatrix}^9 = \left[\mathbf{Q}^*\right]_{0^o}\begin{Bmatrix}\varepsilon_{xx}^0 - zk_x^0\\ \varepsilon_{yy}^0 - zk_y^0\\ \varepsilon_{xy}^0 - zk_{xy}^0\end{Bmatrix} = \begin{Bmatrix}37.086\\ 1.299\\ 0\end{Bmatrix}\frac{\mathrm{N}}{\mathrm{mm}^2}$$

The upper surface of layer 1
Data: $z = 7.6$ mm, $\varepsilon_x^0 = \varepsilon_y^0 = 0$
$\varepsilon_{xx} = \varepsilon_x^0 - z\,k_x = 2.1963\ 10^{-3}$
$\varepsilon_{yy} = \varepsilon_y^0 - z\,k_y = -0.2172\ 10^{-3}$

$$\begin{Bmatrix}\sigma_{xx}\\ \sigma_{yy}\\ \tau_{xy}\end{Bmatrix}^{7.6} = \left[\mathbf{Q}^*\right]_{0^o}\begin{Bmatrix}\varepsilon_{xx}^0 - zk_x^0\\ \varepsilon_{yy}^0 - zk_y^0\\ \varepsilon_{xy}^0 - zk_{xy}^0\end{Bmatrix} = \begin{Bmatrix}31.318\\ 1.098\\ 0\end{Bmatrix}\frac{\mathrm{N}}{\mathrm{mm}^2}$$

Calculations for variant II
The bottom surface of layer 1
Data: $z = 9$ mm, $\varepsilon_x^0 = \varepsilon_y^0 = 0$
$\varepsilon_{xx} = 3.0484\ 10^{-3}$
$\varepsilon_{yy} = -0.2031\ 10^{-3}$

$$\begin{Bmatrix}\sigma_{xx}\\ \sigma_{yy}\\ \tau_{xy}\end{Bmatrix}^9 = \begin{Bmatrix}3.490\\ -1.027\\ 0\end{Bmatrix}\frac{\mathrm{N}}{\mathrm{mm}^2}$$

The upper surface of layer 1
Data: $z = 7.6$ mm, $\varepsilon_x^0 = \varepsilon_y^0 = 0$
$\varepsilon_{xx} = 2.574\ 10^{-3}$
$\varepsilon_{yy} = -0.1715\ 10^{-3}$

$$\begin{Bmatrix}\sigma_{xx}\\ \sigma_{yy}\\ \tau_{xy}\end{Bmatrix}^{7.6} = \begin{Bmatrix}2.947\\ -0.867\\ 0\end{Bmatrix}\frac{\mathrm{N}}{\mathrm{mm}^2}$$

The other results of calculations for normal stresses for both plywood variants in individual layers are analogous, while the results for overall calculations are presented in graphs in Figures 7 and 8.

4 NUMERICAL CALCULATIONS
4. MATEMATIČKI IZRAČUNI

Numerical analysis of the two plywood model variants was based on the Finite Element Method (FEM) using a professional Algor ® Version 13.08 WIN programme. The concepts for the fundamentals and application of FEA are described in detail in literature (Zienkiewicz and Taylor, 1988). Components of laminate elements are modelled using finite elements, which disregard edgewise changes in mechanical parameters of the layers. In the simulation, the material parameters along with load values and restrain conditions were assumed to be identical as in analytical calculations. When creating numerical models, an orthotropic material model was assumed, which was subjected to physical digitization using 8-node Linear Brick Elements as spatial finite elements (Smardzewski, 1998; Spyrakos, 1994; Tanakut *at al.,* 2014;). The numerical model was composed of 1485 nodes with a total number of 1040 finite elements. Results of the numerical analysis were analyzed using the program presented in Figures 3-4 and 5-6. Images of displacement and stress distribution in the form of bitmaps are described as maximum in the middle of the specimen span. In this model, a lack of displacement between the model components was assumed.

5 RESULTS AND DISCUSSION
5. REZULTATI I RASPRAVA

Experimental tests and analytical calculations including the numerical simulation in the 3-point bending test provided data on strength properties of a wood-based composite material (Tables 2 and 3 and Figures 3 and 5). Results for both plywood variants present markedly different stress distributions in its individual layers, resulting in varied bending strength properties. This was confirmed in empirical, analytical and numerical studies. It may be observed that stress

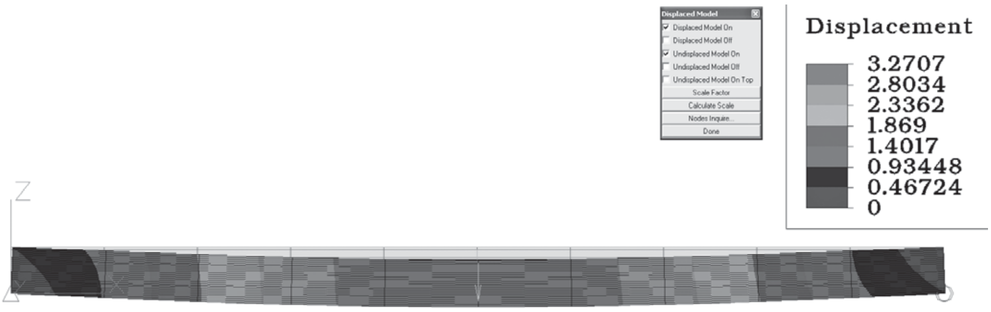

Figure 3 Values of vertical displacement (mm) in the model of plywood variant I
Slika 3. Vrijednosti vertikalnog pomaka (mm) u modelu za furnirsku ploču varijante I.

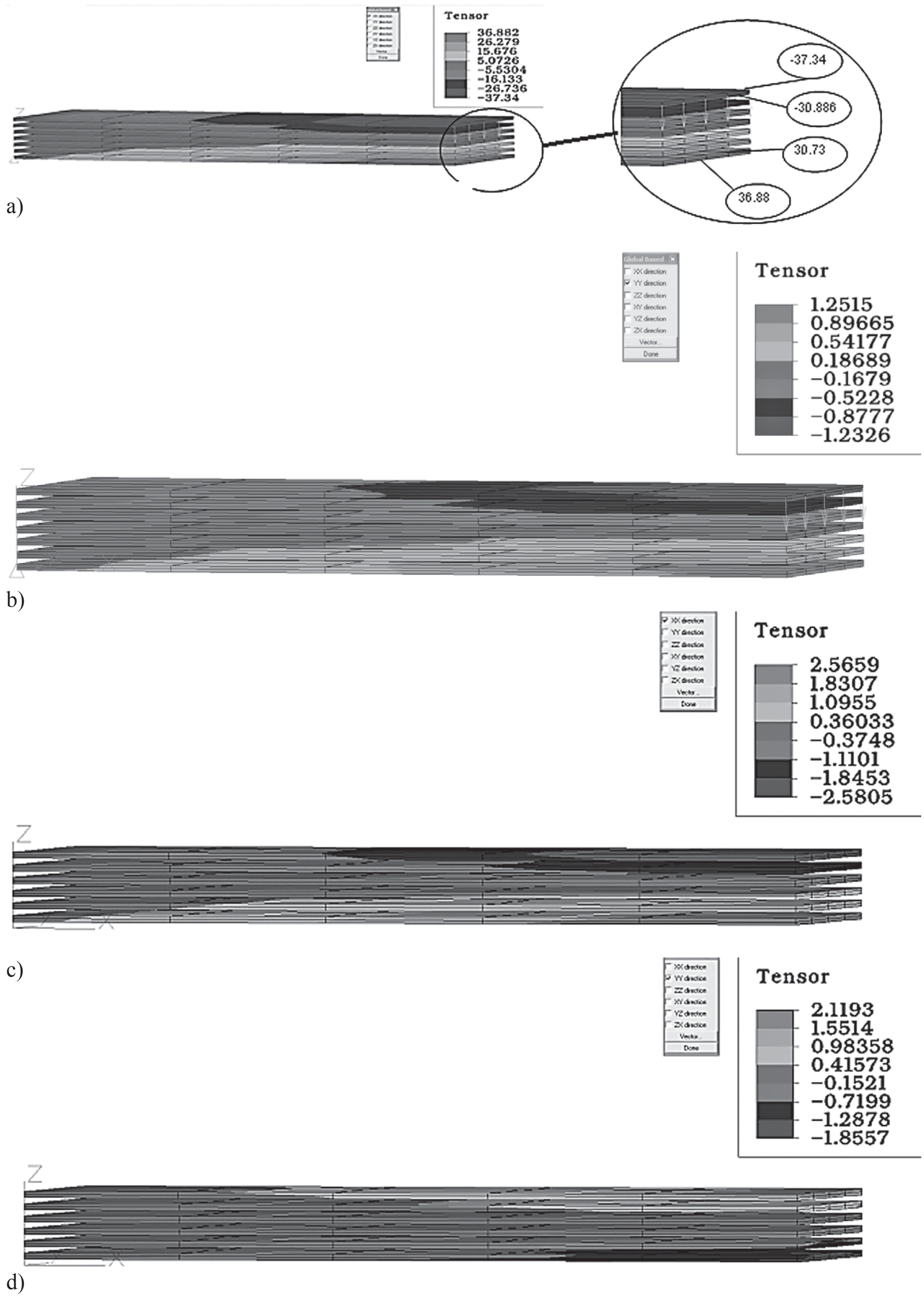

Figure 4 Stress distribution in layers at cross-section of plywood FEM model variant I: a), b) normal stresses σ_{xx} N/mm^2 and σ_{yy} N/mm^2 in plywood layers oriented at $\alpha=0^o$, c), d) normal stresses σ_{xx} N/mm^2 and σ_{yy} N/mm^2 in plywood layers oriented at $\alpha=90^o$
Slika 4. Raspodjela naprezanja u slojevima na poprečnom presjeku furnirske ploče, FEM model varijante I.: a), b) normalna naprezanja σ_{xx} N/mm^2 i σ_{yy} N/mm^2 u slojevima furnirske ploče orijentiranima pri $\alpha = 0°$, c), d) normalna naprezanja σ_{xx} N/mm^2 i σ_{yy} N/mm^2 u slojevima furnirske ploče orijentiranima pri $\alpha = 90°$

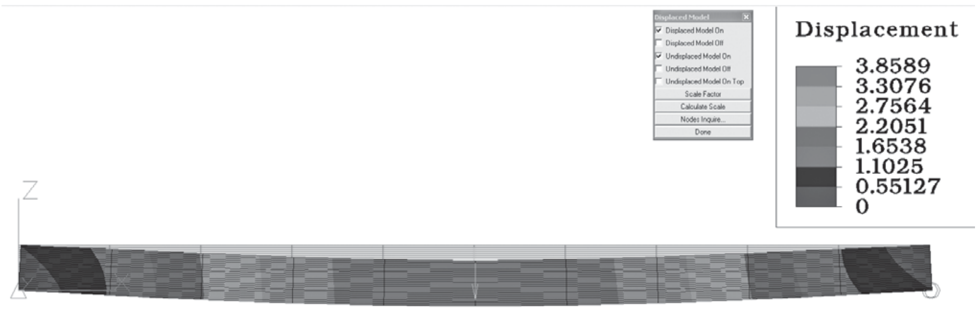

Figure 5 Values of vertical displacement (mm) in plywood model variant II
Slika 5. Vrijednosti vertikalnog pomaka (mm) u modelu za furnirsku ploču varijante II.

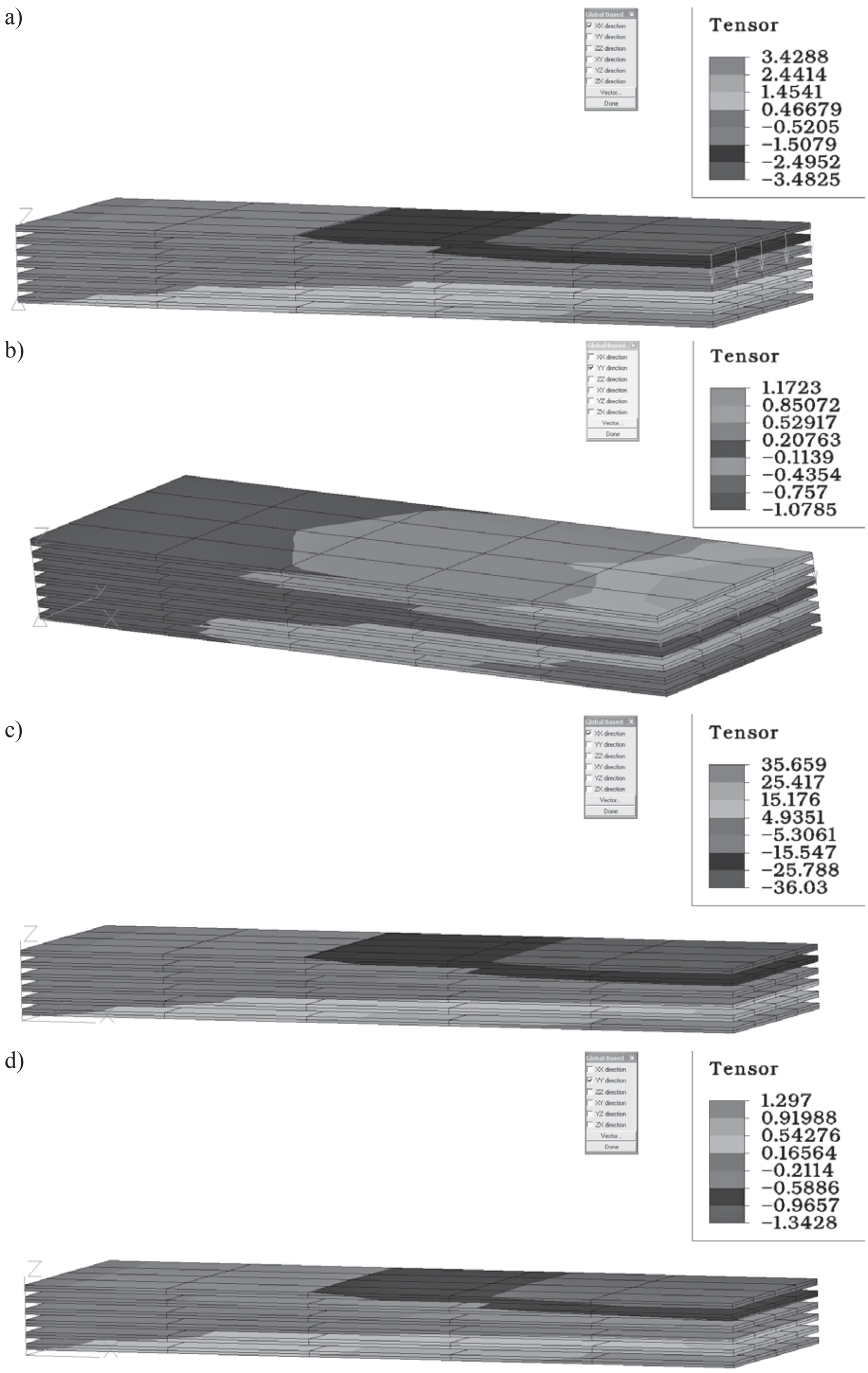

Figure 6 Stress distribution in layers at cross-section of plywood FEM model variant II: a), b) normal stresses σ_{xx} N/mm^2 and σ_{yy} N/mm^2 in plywood layers oriented at α=90^o, c), d) normal stresses σ_{xx} N/mm^2 and σ_{yy} N/mm^2 in plywood layers oriented at α=0^o
Slika 6. Raspodjela naprezanja u slojevima na poprečnom presjeku furnirske ploče, FEM model varijante II.: a), b) normalna naprezanja σ_{xx} N/mm^2 i σ_{yy} N/mm^2 u slojevima furnirske ploče orijentiranima pri α = 90°, c), d) normalna naprezanja σ_{xx} N/mm^2 i σ_{yy} N/mm^2 u slojevima furnirske ploče orijentiranima pri α = 0°

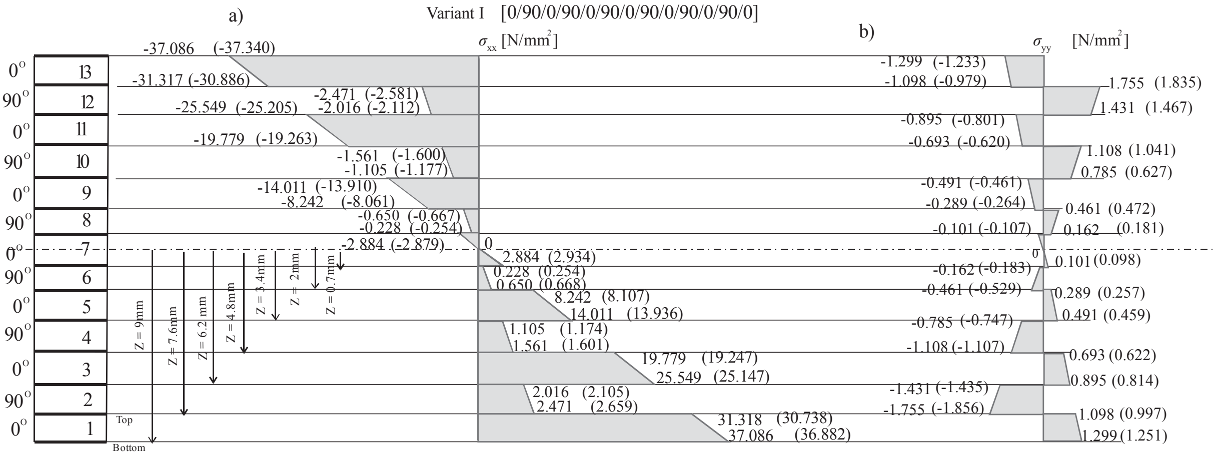

Figure 7 Graphic presentation of edgewise normal stresses in the linear range for plywood variant I: a) σ_{xx} N/mm^2, b) σ_{yy} N/mm^2 (Values given refer to numerical calculations-FEM)
Slika 7. Grafički prikaz normalnih naprezanja na rubovima u linearnom rasponu za furnirsku ploču varijante I.: a) σ_{xx} N/mm^2, b) σ_{yy} N/mm^2 (navedene se vrijednosti odnose na matematičke izračune – FEM)

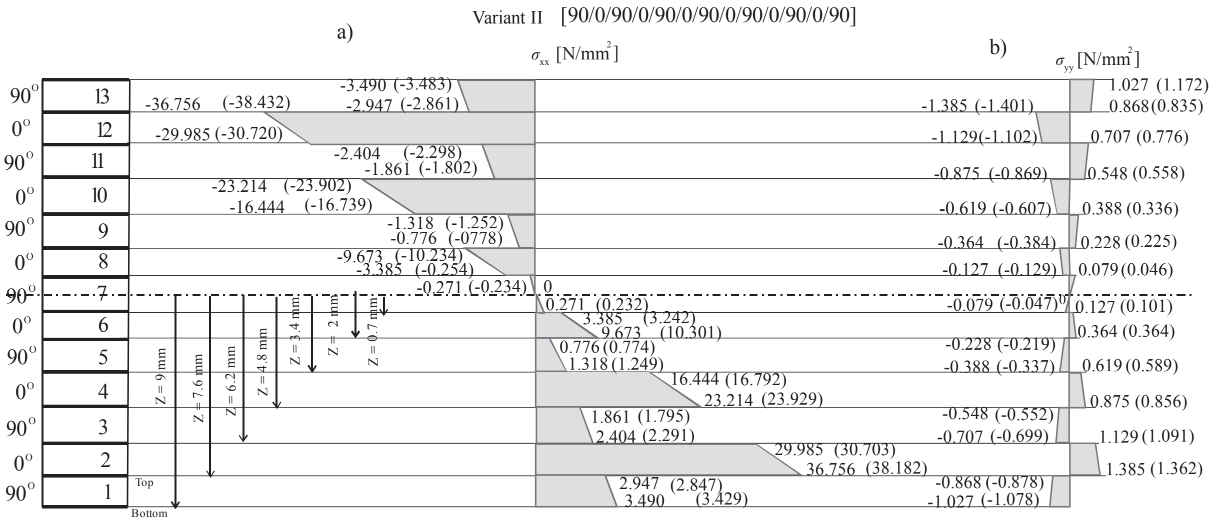

Figure 8 Graphic presentation of edgewise normal stresses in the linear range for plywood variant II: a) σ_{xx} N/mm^2, b) σ_{yy} N/mm^2 (Values given refer to numerical calculations-FEM)
Slika 8. Grafički prikaz normalnih naprezanja na rubovima u linearnom rasponu za furnirsku ploču varijante II.: a) σ_{xx} N/mm^2, b) σ_{yy} N/mm^2 (navedene se vrijednosti odnose na matematičke izračune – FEM)

distribution within the range of elastic deformation is linear in layers with an identical grain orientation. The edgewise distribution of normal stresses σ_{xx}, σ_{yy} is linear; however, it varies between veneers depending on their mechanical properties. Estimated values are almost identical. Slight discrepancies were observed in the distribution of normal stresses in the middle cross-section of plywood estimated using analytical and numerical methods. Depending on the values of deformation and the modulus of elasticity for individual veneers, stresses change stepwise. It is particularly evident in individual plywood layers (Figures 2, 4 and 6). Plywood models with more veneers in their structure, that run parallel to the span of the loaded panel, have greater values of bending strength MOR and modulus of elasticity in bending MOE (Table 2). The differences in the values of bending strength in different directions of the plywood panel are a result of the orientation of wood fibres in the plywood structure in relation to the direction of the action of bending moment. This means that the orientation of veneers in the plywood structure has a direct impact on plywood bending strength. The bending strength of wood is greater in the direction of wood fibres. The image of experimental failure of selected samples for both plywood variants in the bending test is presented in photographs (Figure 9). The character of failure is primarily dependent on the applied veneer orientation, particularly that of outer layers. The produced damage indicates that short-term strength is exceeded in lower plies. Lengthwise fibre effort in cross-bands (α=0° variety I, Figure 9a) exceeded tensile strength with fragmentary loosening of bottom veneers. In turn, in veneers of variant II (crossband angle α=90°, Figure 9b), lignin in the veneer was damaged as a result of their delamination. Moreover, fibres in adjacent lower veneers were partially broken and a comparable interlayer failure was observed.

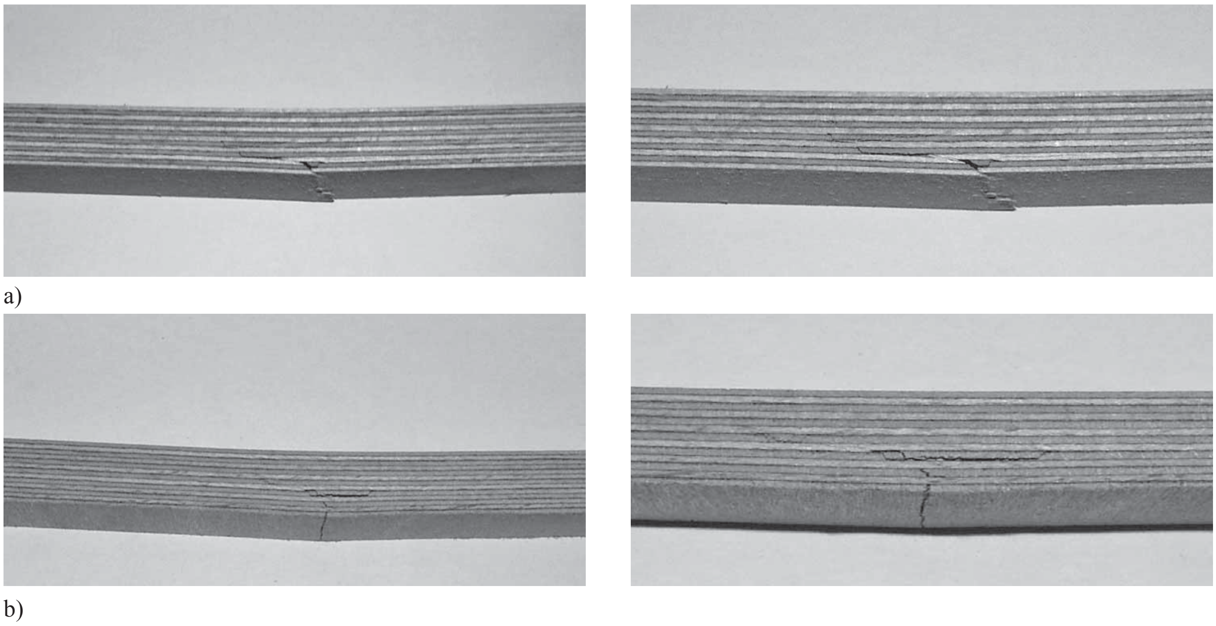

Figure 9 Failure variants of test specimens for the determination of bending properties of plywood: a) variant I, b) variant II
Slika 9. Varijante loma ispitivanih uzoraka pri određivanju savojnih svojstava furnirske ploče: a) varijanta I., b) varijanta II.

6 CONCLUSIONS
6. ZAKLJUČAK

Based on the analyses, it may be stated that the adoption of the thin plate bending theory including Kirchhoff's hypothesis in analytical calculations confirms its applicability in such structures. For this reason, it may be assumed that the method adopted to determine the distribution of normal stresses in the multi-layer wood-based laminated material, including orthotropic material properties, is appropriate for practical analyses of multi-layer systems. This confirms the thesis included in the research hypothesis that the load-bearing capacity of plywood is dependent on the configuration of its individual layers, which was confirmed when evaluating plywood displacement in both empirical and numerical analyses. This is particularly related to a detailed analysis of the distribution of stresses and deformations in a multi-layer wood-based material subjected to elastic bending. The stresses σx, σy are different because of the values of the deformation are different on directions x and y, respectively, and different stresses in the elastic range are the result of diversified elasticity modules of veneers in the direction parallel and perpendicular to the grain.

In view of the above data, these methods need to be considered when designing plywood structures, particularly in terms of loading direction. Unfortunately, the computational process is more complex in comparison to other methods. Nevertheless, it provides more detailed results, particularly at the non-axial (arbitrary) layer configuration, when shear stresses need to be considered. Such an analysis is significantly supplemented when using FEM. The consistency of numerical and experimental results shows that costly failure tests may be replaced by non-destructive computer methods. However, for the results to be fully consistent with the actual values, it is necessary to improve the model and adopt correct boundary conditions.

From the mechanical point of view, an arbitrary configuration of layer orientation, next to the type of used material, in the analysis of laminated composites is a factor determining how properties of individual layers influence properties of the entire composite material.

7 REFERENCES
7. LITERATURA

1. Bodig, J.; Jayne, B. A., 1982: Mechanics of wood and wood composites. Van Nostran Company Inc., New York.
2. Brancheriau, L.; Bailleres, H.; Guitard, D., 2002: Comparison between modulus of elasticity values calculated using 3 and 4point bending tests on wooden samples. Wood Science and Technology, 36: 367-383. https://doi.org/10.1007/s00226-002-0147-3.
3. German, J., 1996: Foundations of mechanics of fibrous composite materials. Publishing Cracow University of Technology, Cracow.
4. Goodman, J. R.; Bodig, J., 1970: Orthotropic elastic properties of wood. Journal of the Structural Division, 96 (11): 2301-2319.
5. Hrázský, J.; Král, P., 2005: Assessing the bending strength and modulus of elasticity in bending of exterior foiled plywoods in relation to their construction. Journal of Forest Science, 51 (2): 77-94.
6. Keylwerth, R., 1951: Die anisotrope Elastizität des Holzes und der Lagerhölzer. VDI – Forschungsheft 430, Ausgabe 13, Band 17.
7. Kljak, J.; Brezović, M.; Jambreković, V., 2006: Plywood stress optimization using the finite element method. Wood Research, 51 (1): 1-10.
8. Kljak, J.; Brezovic, M., 2007: Relationship between bending and tensile stress distribution in veneer plywood. Forest Products Journal, 57 (12): 65.
9. Lekhnitskii, S. G., 1982: Theory of elasticity of an anisotropic body. Mir Publishers, Moscow.
10. Lemaitre, J.; Chaboche, J. L., 1990: Mechanics of solid materials. Cambridge University Press: Cambridge, UK.

11. Makowski, A., 2014: Assessment of strength parameters of beech plywood in terms of failure criteria, Ann. WULS – SGGW, Forestry and Wood Technology, 88: 143-146.
12. Neuhaus, H., 1994: Lehrbuch des Ingenieurholzbaus. Teubner, Stuttgart.
13. Reddy, J. N., 1997: Mechanics of Laminated Composite Plates and Shells – Theory and Analysis. 2nd ed. CRC Press, Boca Raton.
14. Tanakut, N.; Tanakut, A. L.; Zor, M., 2014: Finite element analysis of wood materials. Drvna industrija, 65 (2): 159-171. https://doi.org/10.5552/drind.2014.1254.
15. Smardzewski, J., 1998: Numerical analysis of furniture constructions. Wood Science and Technology, 32: 273-286. http://doi.org/10.1007/BF00702895.
16. Spyrakos, C. C., 1994: Finite element analysis in engineering practice. Algor Inc. Publishing Division Pittsburgh. PA. USA.
17. Stark, N. M.; Cai, Z.; Carll, C., 2010: Wood handbook – Wood as engineering material, Chapter 11: Wood-based composite materials. USDA, Forest Product Laboratory, General Technical Report, 190: 11-28.
18. Stefańczyk, B., 2005: General construction technology, vol. 1 – Construction materials and products. Arkady, Warsaw.
19. Tsai, S. W.; Wu, E. M., 1971: A general theory of strength for anisotropic materials. Journal of Composite Materials, 5: 58-80.
20. Tsai, S. W.; Hahn, H., 1980: T. Introduction to composite materials. Technomic Publishing Company Westport, Connecticut, U.S.
21. Youngquist, J. A., 1999: Wood-based composites and panel products. Chapter 10 in USDA, Forest Service, Wood as an engineering material. General Technical report FPL-GTR-113. Wood Handbook. USDA, Forest Service, Forest Products Laboratory.
22. Zienkiewicz, O. C.; Taylor, R. L. 1988: The finite element method: vol. 1 – Basic formulation and linear problems. McGraw-Hill, London.
23. Zienkiewicz, O. C.; Taylor, R. L., 1989: The finite element method: vol. 2 – Solid and fluid mechanics, dynamics and non-linearity. McGraw Hill, London.
24. Wang, S. Y.; Chang, T., 1978: Studies on the flexural properties of laminated beams with different wood species (1) Bending modulus of elasticity. Q. J. Chinese Forest, 11: 43-52.
25. *** EN 310, 1994: Wood-based panels – determination of modulus of elasticity in bending and of bending strength. European Committee for standardization, Brussel.

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