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Original scientific paper

Computer-Based Validation of $3n+1$ Hypothesis for Numbers 3^n-1

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Abstract: The formulation of the 3^n-1 problem is simple but no one has found the solution yet. This paper transforms the original problem into its equivalent so that it becomes more suitable for computer validation. A new algorithm is proposed and implemented. The hypothesis is tested and proven to be valid for numbers 3^n-1, conclusive with number $3^{32768}-1$.

Keywords: Collatz's problem; transformation; total stopping time; trajectories; 3^n-1

1 INTRODUCTION

There are certain problems in mathematics whose formulation is simple and understandable to many, yet no one has found the solution to them. One of such problems is the $3n+1$ problem. It was circulated by word of mouth only and as a result there were no publications related to the problem before the beginning of the 1970s. There are some sources [4, 5] saying that this problem has been referred to by different lecturers in university lectures from 1929 on. However, today we generally attribute this problem to LotharCollatz, which is why the problem $3n+1$ is widely known as the Collatz problem, and the related function is referred to as the Collatz's function. It is defined as follows:

$$C(n)=\begin{cases}\dfrac{n}{2}, & n-\text{even},\\ 3n+1, & n-\text{odd}.\end{cases} \tag{1}$$

The following hypothesis is related to this function.

Hypothesis 1: For each natural number n there is a natural number m such that $C^m(n)=1$.

Let's mention that $C^m(n)$ stands for the consecutive composition of function $C(n)$ and it is recursively defined with $C^0(n) = n$, $C^{m+1}(n) = C(C^m(n))$, for $m \geq 0$.

The formulation of the $3n+1$ problem reads: is the above hypothesis true? In other words, if we apply the Collatz's function consecutively to a random natural number, will the result always be one? We consider the problem to be solved either if the above hypothesis proves to be valid or if we find a natural number for which the hypothesis is not valid. It is possible to formulate the $3n+1$ problem as a game. Namely, Alice writes down a natural number. Bob wins if the number she wrote is one. Otherwise, Bob checks if the last number she wrote is an even number and if it is, he divides it by two; but if it is an odd number, he multiplies it with three and adds one. Bob writes down the number he gets and repeats the procedure. The question is: will Bob win the game every time?

Example: If Alice writes down number 7, then we have the following sequence of numbers:

7, 22, 11, 34, 17, 52, 26, 13, 40, 20, 10, 5, 16, 8, 4, 2, 1.

Namely, Bob made the following calculation 3×7 + 1 = 22, and then 22/2 = 11, etc. Finally, Bob wrote number one and thus won the game. The number of iterations which Bob made in this case is 16, i.e. $C^{16}(7) = 1$. Also, from the quoted sequence we can conclude that $C^{12}(17) = 1$, while $C^4(16) = 1$. The maximum value we got through such computations is 52.

Despite the fact that the formulation of the problem $3n+1$ in terms of the game is plausible to everyone who knows basic mathematical operations, as far as we know the problem has not been solved. Up to this date there have been a couple of hundreds of publications which either directly or indirectly addressed the $3n+1$ problem from various points of view (recent papers [10, 11, 12, 13]).

When it comes to more recent results, there is a particularly interesting paper [10] in which a new hypothesis is formulated, and from which exact upper limits of stopping can be obtained. However, although this hypothesis has been confirmed on all the calculations carried out, it remains without a proof.

There are even a couple of papers [2, 3] claiming to have solved the 3n+1 problem, but each of them proved to have had an error in the proof. One of the aspects related to the 3n+1 problem is the development of efficient algorithms which would check the correctness of Hypothesis 1 for the given number n (see [6, 7]), as well as all other characteristics related to the $3n+1$ problem [9].

This paper focuses on this aspect itself. Namely, an efficient algorithm is developed and then used for testing Hypothesis 1 on big numbers of a special format. Anyway, the main objective of this paper is to discover all characteristics which can help mathematicians prove Hypothesis 1 or find a counterexample.

After the introduction, this paper is organized into four sections. The second section gives a short overview of some of the results related to the $3n+1$ problem. The third section outlines the proposed transformations necessary for the construction of an algorithm and for the description of the algorithm itself. The fourth section gives an overview of experimental results and some statistical data generated by testing. The fifth section is the conclusion.

2 OVERVIEW OF RELATED WORK

The most frequent reformulation of the $3n+1$ problem iterates a different function, which is called the $3n+1$ function, shown as:

$$T(n)=\begin{cases}\dfrac{n}{2}, & n-\text{even},\\ \dfrac{3n+1}{2}, & n-\text{odd}.\end{cases} \tag{2}$$

From the point of view of iterations, these two functions are in a simple relation. The iteration of function $T(n)$ skips some steps in the iterations of function $C(n)$. The skipping happens when n is an odd number. Namely, the following happens:

$$T(n)=\begin{cases}C(n), & n-\text{even}.\\ C(C(n)), & n-\text{odd}.\end{cases} \tag{3}$$

Therefore, in Hypothesis 1 function $C(n)$ can be replaced by function $T(n)$. Literature mostly studies the characteristics of function $T(n)$. For example, the following characteristics are interesting [6]:

- $T^k(2^k-1)=3^k-1$.
- Let $n_0=n_k2^k+m_k$, where $n_k=[n_0/2^k]$ and $m_k=n_0\bmod 2^k$. The general form of $T^k(n_0)$ is $T^k(n_0)=3^{y(k,n_0)}n_k+T^k(m_k)$, where $y(k,n_0)$ is the number of odd steps made, i.e. the number of elements of the set $\{j|0\le j<k,\ T^j(m_k)\equiv 1(\bmod 2)\}$.

The first characteristic shows that in some cases of consecutive iteration of function $T(n)$, the iteration can increase the value, while the other characteristic is the generalization of the first one. As the values in the consecutive iteration change, it is meaningful to make a graphic presentation, which is why we introduce the term trajectory (orbit). The trajectory of the element x for the function $f(\cdot)$ is a series of iterations $(x, f(x), f^2(x), f^3(x),\ldots)$. The trajectory of number 649 for function $T()$ is shown in Fig. 1.

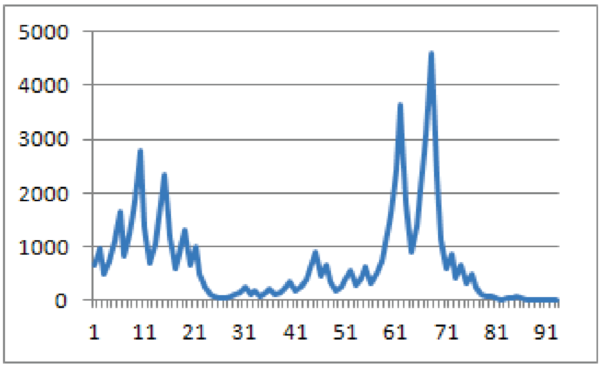

Figure 1 The trajectory of number 649 for function T. The maximum value of the trajectory is 4616.

The trajectory of element x for function f is periodical if there is $k\ge 1$ so that $f^k(x)=x$. So far, 1 and 2 are the only known natural numbers whose trajectories are periodical for function T. Let us note that there are also integers whose trajectories are periodical for function T, such as 0, −1, −5, −17. Therefore, an interesting question is whether 1 and 2 are the only natural numbers whose trajectories are periodical for function T. As for this question, there are interesting results from paper [1]. Let us form $n\times n$ zero-one matrix A_n whose elements are

$$a_{ij}=\begin{cases}1, & T(i)=j,\\ 0, & \text{otherwise}.\end{cases}$$

The claim that 1,2 is the only periodical trajectory for function T among natural numbers is equivalent to $\det(I-xA_n)=1-x^2$ for each $n\ge 1$. It is also demonstrated that $\det(I-xA_n)=\det(I-xA_{n-1})$ for all $n\ne 8(\bmod 18)$. It is concluded in [1] that if there is some other trajectory among natural numbers, then there is $m\equiv 8(\bmod 18)$ such that $n=\dfrac{m}{2}$ is a periodical trajectory. Therefore, it is sufficient to take step 18 in finding other periodical trajectories (if any) using the computer.

In the experimental check of Hypothesis 1 an important consideration is when to stop iterations. This is why the following terms are introduced: the stopping time for number n, referenced as symbol $\sigma(n)$, is the minimum number k for which $T^k(n)<n$, if that number exists, otherwise it is ∞. The total stopping time $\sigma_\infty(n)$ for number n is the minimum number k for which $T^k(n)=1$, if there is such a number, otherwise it is ∞. If Hypothesis 1 is valid for all natural numbers smaller than n, then we would prove that it is also valid for n, where it would be enough to get $T^k(n)<n$. In other words, we need to prove that $\sigma(n)$ is a finite number. If we do not have information on the predecessor of number n, then we have to prove that $\sigma_\infty(n)$ is a finite number, which leads to the conclusion that Hypothesis 1 is to be valid for number n.

If we assume that Hypothesis 1 is valid, the natural approach would be to find a subset $S\subset N$, such that Hypothesis 1 is valid for numbers from S. The hypothesis is then also valid for all natural numbers. It is easy to check if the previous characteristic is valid for the set $S=\{m\in N|m\equiv 3(\bmod 4)\}$. Namely, for other numbers, after a finite number of iterations we will get an element from S or the number is going to decrease every time we change one or two iterations. This is presented in Figure 2.

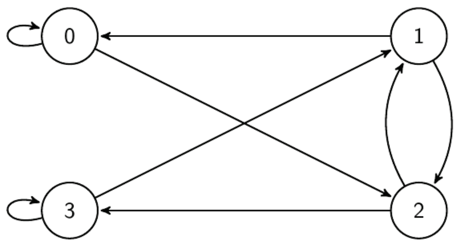

Figure 2 The Dynamics of iterations of function T by module 4.

The paper [8] shows that it is enough to observe the set: $S=\left\{m\in N\middle|m\equiv 3+\dfrac{10}{3}(4^k-1)(\bmod 2^{2k+2})\right\}$ for a random fixed number k.

3 BASIC TRANSFORMATIONS AND PROPOSED ALGORITHM

As it was said in the previous section, function *T(n)* is usually observed instead of function *C(n)*. This is particularly convenient for the implementation of algorithms because it reduces the number of steps. For the binary representation of numbers, the division by power of 2 can be replaced with more efficient operations of shifting bits. Therefore, the basic idea is to have function *T(n)* transformed in such a way that the division by power of 2 gets consolidated.

If n is an even number, then for $k=\max_{j\in N}\left\{j\,\middle|\,n\equiv 0(\mathrm{mod}2^j)\right\}$ is $T^k(n)=\frac{n}{2^k}$. Let us note that in this case $T^k(n)$ is an odd number, so after it we apply an "odd step" (i.e. we apply the formula for odd numbers for calculations). On the other side, if n is an odd number, we can write $T(n)=\frac{3n+1}{2}=\frac{3(n+1)}{2}-1$. Obviously, if $\frac{n+1}{2}$ is an even number, the result $T(n)$ is an odd number. Therefore, we get $T(n)=T(\frac{3(n+1)}{2}-1)=\frac{3^2(n+1)}{2^2}-1$. Using mathematical induction for $l\le k=\max_{j\in N}\left\{j\,\middle|\,(n+1)\equiv 0(\mathrm{mod}2^j)\right\}$, it is easy to prove that $T^l(n)=\frac{3^l(n+1)}{2^l}-1$ is valid. Especially, $T^k(n)=\frac{3^k(n+1)}{2^k}-1$ is an even number and after that the "even step" follows. In other words, all consecutive even (odd) steps can be replaced with one "multiple" even (odd) step. In this way, both even and odd steps will alternate.

Considering previous steps, we can define two functions which map the set of odd natural numbers to the set of odd natural numbers.

$$Heven(n)=\frac{n-1}{2^p}, \text{ where}$$

$$p=\max_{j\in N}\left\{j\,\middle|\,(n-1)\equiv 0(\mathrm{mod}2^j)\right\}$$

on $Hodd(n)=\frac{3^k}{2^k}(n+1)$, where

$$k=\max_{j\in N}\left\{j\,\middle|\,(n+1)\equiv 0(\mathrm{mod}2^j)\right\}.$$

By using simple transformations it is easy to show that the following lemma is valid:

Lemma 1: Let n be an odd number. Further, let us denote other numbers k and p as:

$$k=\max_{j\in N}\left\{j\,\middle|\,(n+1)\equiv 0(\mathrm{mod}2^j)\right\},$$

$$p=\max_{j\in N}\left\{j\,\middle|\,(Hodd(n)-1)\equiv 0(\mathrm{mod}2^j)\right\}.$$

Then the following statement is valid:

$$Heven(Hodd(n))=T^{k+p}(n).$$

The meaning of the previous lemma is as follows: for an odd number n, if there is m such that $T^m(n)=1$, then there is q such that $(Heven\circ Hodd)^q(n)=1$. In this case, we name number q the total stopping time for function $Heven\circ Hodd$ and we write it as $\sigma_h(n)=q$. If we apply function T to an even number n in an infinite number of iterations, the result will be an odd number or one, so we can conclude that Hypothesis 1 is equivalent to the following hypothesis.

Hypothesis 2: For each odd number n there is a natural number q such that $(Heven\circ Hodd)^q(n)=1$.

The trajectory of number 649 for function $Heven\circ Hodd$ is shown in Fig. 3. When we compare the trajectories from Fig. 1 and Fig. 3, we see that the trajectory from Fig. 3 oscillates less.

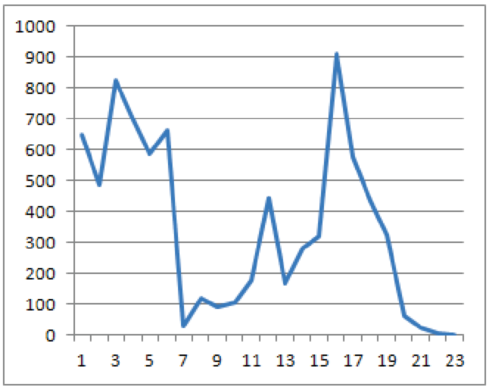

Figure 3 Trajectory of number 649 for function Heven Hood.

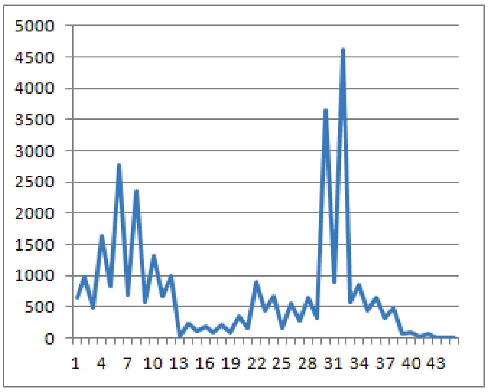

Figure 4 Trajectory of number 649 with inter-step.

However, Fig. 3 hides an inter-step made by the Hodd function. Figure 4 shows the trajectory of number 649 for the function $Heven\circ Hodd$ with its inter-step. Trajectories from Figure 1 and Figure 4 have similarities, which is because they go through same extreme values, save for the fact that the trajectory from Figure 4 is generated in far fewer steps.

Now the task comes down to the validation of Hypothesis 2. The algorithm for this is shown in Tab. 1. Big numbers are represented using base 2, e.g. $2^{16}=65536$. The (Bit vector) array of digits is defined as: array[0] contains the number of digits, array[1] contains the binary digit with weight 0, array[2] contains the digit with weight 1, etc. Such representation of big numbers is chosen because it is directly translated into binary representation of a number; consequently, the division by power of 2 comes down to shifting digits (i.e. bytes) for a corresponding number of positions. The difference between the proposed algorithm with respect to other algorithms [6, 7] is that the former allows for handling big numbers, while the latter ones [6, 7] are optimized for handling numbers which can be put into two 64-bit registers. The improvement *vis-a-vis* of the algorithm in [6, 7] also relates to the Hodd step. In these algorithms there are more consecutive multiplications by number three, after which shifting (i.e. division by two) is realized as one multiplication with an appropriate power of 3 and 1 shifting.

Table 1 Algorithm for the $3n+1$ problem

Algorithm for checking the *3n*+1 hypothesis.
Input: Number *B* from which we start, number of steps *K,* and number S which we increase.
Output: The numbers which require a lot of iterations or become too big through iterations (if there are such numbers) and the numbers which we got through the validation of the hypothesis.

```
Method:
BEGIN
   C=B
   FOR j=1 TO K DO
A=B
      i=1
     WHILE A ≥ C  DO
         i=i+1
Hodd(A)
         Heven(A)
         IF(I > 10^5 or A too big number) THEN
           Print (B)
A=1
END IF
END WHILE
       B=B+S
  END FOR
  Print(B)
END BEGIN.
```

After the iterations for one number, the algorithm goes on to the next one. Numbers requiring a large number of iterations or those which become too big due to applied iterations (for example, the number of digits of the current number is three times bigger than the number of digits of the starting number) are given as the output and they need to be additionally tested. Namely, these numbers are candidates for a periodical trajectory or their trajectory has a sub-sequence with the trend to infinity.

4 EXPERIMENTAL RESULTS

As far as we know, Hypothesis 1 has been computer-tested [6, 7] for all numbers $n \le 20\times2^{58}$. We are not ruling out the possibility that this limit has already been exceeded but to our knowledge the results thereof have not yet been published. In order to preclude any possibility of overlap with results of other authors, the testing was done on big numbers with a specific format. The powers of 3 are particularly important as they appear in the iteration of $3n+1$ functions.

The algorithm described in the previous chapter served as the basis for developing several procedures implemented using C programming language for a number of experiments, i.e. numerical testing. In this section we present the results of some of the testing procedures we conducted.

The Periodical Trajectory Test examines if there is any periodical trajectory. The test started from number $5\times2^{60}+1$, and step 18 (see [1, 7]). Altogether 2^{32} cycles were made (i.e. conclusive with number $5\times2^{60}+9\times2^{33}+1$) and no periodical trajectories were found.

The hypothesis was tested for numbers in the form of $3+\frac{10}{3}(4^k-1)(\bmod 2^{2k+2})$, for $k=29$. The test started from number $3+\frac{10}{3}(4^{29}-1)$ and 4294967296 cycles were made, i.e. conclusive with number $3+2^{92}+\frac{10}{3}(4^{29}-1)$. For all tested numbers, the hypothesis $3n+1$ remains valid.

The hypothesis was tested and proved to be valid for the powers of number 3, up to number 3^{32768}. It is necessary to mention that these are big numbers. Number 3^{32768} has 3247 digits represented in base 2^{16}, which indicates that the number therefore had more than 15500 decimal digits. To check the calculus, we carried out a test and found that $3^{32768} \equiv -1(\bmod 2^{2^4}+1)$ is correct (Peppino's test for Fermat's number 65537). Also, the maximum total stopping time for this testing for function $Heven \circ Hodd$ was 62275.

The hypothesis was tested and proved to be valid for numbers 3^k-1, conclusive with number $3^{32768}-1$. Since numbers in the form 3^k-1 are even numbers, the step $Heven(3^k)$ was completed before the main algorithm started. The testing also included numbers in the format 2^k-1 due to their feature $T^k(2^k-1)=3^k-1$.

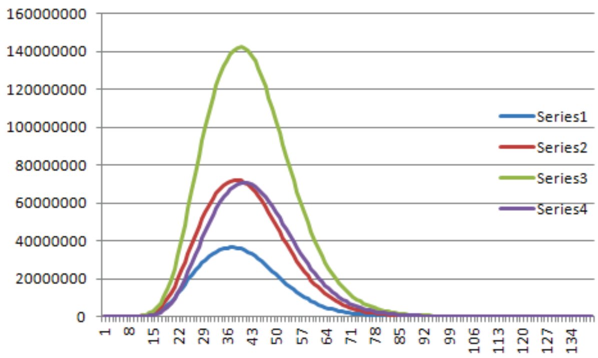

Figure 5 The frequency of number $\sigma_h(n)$ for 2^{30},2^{31}, 2^{32} cycles and frequency difference between 2^{32} and 2^{31} cycles.

The frequency of number $\sigma_h(n)$ was also tested for situations of the increased digit of the highest weight of the initial number*n*. The frequency thereof was found to be normal (similar to normal distribution), which is not the case with $\sigma_\infty(n)$. The tests were done on numbers in the form of $3+\frac{10}{3}(4^k-1)(\bmod 2^{2k+2})$, for $k = 29$ (see Fig. 5) as well as numbers $100\times[\pi\times10^{35}]+1$ and $10^6\times[e\times10^{50}]+1$. The digits of numbers π and e on one hand behave as randomly generated numbers, and on the other hand they can be easily compared to other tests.

5 CONCLUSION

The importance of the solution of the $3n+1$ problem is that new ideas and techniques have to be created to solve it. It could potentially open new horizons and give new techniques in number theory. Functions *Heven* and *Hodd* were introduced and studied instead of function *T*, which has different convergence features. The analysis of experimental results produced through intensive computer-conducted numerical testing indicates that Hypothesis 1 is correct for all 3^n, where $n = 1, 32768$. The frequency trends of number $\sigma_h(n)$ are particularly interesting and they merit further examination to produce mathematical explanation of the phenomenon.

6 REFERENCES

[1] Alves, J. F., Graca, M. M., Sousa Dias, M. E., & Ramos, J. S. (2005). A linear algebra approach to the conjecture of Collatz. *Lin. Alg. Appl.*, 394, 277-289. https://doi.org/10.1016/j.laa.2004.07.008

[2] Bruckman, P. S. (2008). A proof of the Collatz conjecture. *International Journal of Mathematical Education in Science and Technology, 39*(3), 403-407. https://doi.org/10.1080/00207390701691574

[3] Cadogan, C. C. (2006). A Solution to the 3*x*+1 Problem. *Caribbean J. Math. Comp. Sci.*, 13, 1-11.

[4] Collatz, L. (1980). *Letter to Michael E. Mays*.

[5] Collatz, L. (1986). On the motivation and origin of the (3*n*+1) - problem (Chinese). *J. Qufu Normal University, Natural Science Edition [Qufushi fan da xuexuebao], 12*(3), 9-11.

[6] Oliveira e Silva, T. (1999). Maximum excursion and stopping time record-holders for the 3*x*+1 problem: computational results. *Mathematics of Computation, 68*(225), 371-384. https://doi.org/10.1090/S0025-5718-99-01031-5

[7] Oliveira e Silva, T. (2010). Empirical verification of the 3*x* + 1 and related conjectures. *The Ultimate Challenge: The* 3*x*+1 *Problem* (J. Lagarias, ed.), American Mathematical Society, Providence, RI, 189207.

[8] Yang, Z. H. (1998). An Equivalent Set for the 3*x*+1 Conjecture. *Journal of South China Normal University, Natural Science Edition*, 2, 66-68.

[9] Roosendaal, E. (2019). On the 3*x*+1 problem. Retrieved from http://www.ericr.nl/wondrous/.

[10] Rozier, O. (2017). The 3*x*+1 problem: a lower bound hypothesis. *Funct. Approx. Comment. Math.*, 56, 7-23. https://doi.org/10.7169/facm/1583

[11] Wang, S. (2018). An E-sequence approach to the 3*x*+1 problem. Retrieved from https://arxiv.org/abs/1809.02278

[12] Rozier, O. (2018). Parity sequences of the 3*x*+1 map on the 2-adic integers and Euclidean embedding. Retrieved from https://arxiv.org/abs/1805.00133

[13] Daudin, J. J. & Pierre, L. (2018). A simple proof of the Wirsching-Goodwin representation of integers connected to 1 in the 3*x*+1 problem. Retrieved from https://arxiv.org/abs/1801.07678

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