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RATIONAL SEQUENCES ON DIFFERENT MODELS OF ELLIPTIC CURVES

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Abstract. Given a set S of elements in a number field k, we discuss the existence of planar algebraic curves over k which possess rational points whose x-coordinates are exactly the elements of S. If the size $|S|$ of S is either $4, 5$, or 6, we exhibit infinite families of (twisted) Edwards curves and (general) Huff curves for which the elements of S are realized as the x-coordinates of rational points on these curves. This generalizes earlier work on progressions of certain types on some algebraic curves.

1. Introduction

An algebraic (affine) plane curve C of degree d over some field k is defined by an equation of the form

$$\{(x, y) \in k^2 : f(x, y) = 0\}$$

where f is a polynomial of degree d. The algebraic affine plane curve C can also be extended to the projective plane by homogenising the polynomial f. If $P = (x, y)$, then we write $x = x(P)$ and $y = y(P)$.

Studying the set of k-rational points on C, $C(k)$, has been subject to extensive research in arithmetic geometry and number theory, especially when k is a number field. For example, if f is a polynomial of degree 2, then one knows that C is of genus 0, and so if C possesses one rational point then it contains infinitely many such points. If f is of degree 3, then C is a genus 1 curve if it is smooth. In this case, if $C(k)$ contains one rational point, then it is an elliptic curve, and according to Mordell-Weil Theorem, $C(k)$ is a finitely

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generated abelian group. In particular, $C(k)$ can be written as $T \times \mathbb{Z}^r$ where T is the subgroup of points of finite order, and $r \geq 0$ is the *rank* of C over k.

In enumerative geometry, one may pose the following question. Given a set of points S in k^2, how many algebraic plane curves C of degree d satisfy that $S \subseteq C(k)$? It turns out that sometimes the answer is straightforward. For example, given 10 points in k^2, in order for a cubic curve to pass through these points, a system of 10 linear equations will be obtained by substituting the points of S in

$$a_1x^3 + a_2x^2y + a_3x^2 + a_4xy^2 + a_5xy + a_6x + a_7y^3 + a_8y^2 + a_9y + a_{10} = 0$$

and solving for $a_1, \cdots, a_{10}$. Therefore, there exists a nontrivial solution to the system if the determinant of the corresponding matrix of coefficients is zero, hence a cubic curve through the points of S. Thus, one needs linear algebra to check the existence of algebraic curves of a certain degree through various specified points in k^2.

In this article, we address the following, relatively harder, question. Given $S \subset k$, are there algebraic curves C of degree d such that for every $x \in S$, $x = x(P)$ for some $P \in C(k)$? In other words, S constitutes the x-coordinates of a subset of $C(k)$. The latter question can be reformulated to involve y-coordinates instead of x-coordinates. It is obvious that linear algebra cannot be utilized to attack the problem as substituting with the x-values of S will not yield linear equations.

Given a set $S = \{x_1, x_2, \cdots, x_n\} \subset k$, if (x_i, y_i), $i = 1, \cdots, n$, are k-rational points on an algebraic curve C, then these rational points are said to be an *S-sequence* of length n. In what follows, we summarize the current state of knowledge for different types of S.

We first describe the state-of-art when the elements of $S \subset \mathbb{Q}$ are chosen to form an *arithmetic progression*, Lee and Vélez ([10]) found infinitely many curves described by $y^2 = x^3 + a$ containing S-sequences of length 4. Bremner ([2]) showed that there are infinitely many elliptic curves with S-sequences of length 7 and 8. Campbell ([5]) gave a different method to produce infinite families of elliptic curves with S-sequences of length 7 and 8. In addition, he described a method for obtaining infinite families of quartic elliptic curves with S-sequences of length 9, and gave an example of a quartic elliptic curve with an S-sequence of length 12. Ulas ([17]) first described a construction method for an infinite family of quartic elliptic curves on which there exists an S-sequence of length 10. Secondly he showed that there is an infinite family of quartics containing S-sequences of length 12. Macleod ([11]) showed that simplifying Ulas' approach may provide a few examples of quartics with S-sequences of length 14. Ulas ([18]) found an infinite family of genus two curves described by $y^2 = f(x)$ where $\deg(f(x)) = 5$ possessing S-sequences of length 11. Alvarado ([1]) showed the existence of an infinite family of such curves with S-sequences of length 12. Moody ([12]) found an infinite number of

Edwards curves with an S-sequence of length 9. He also asked whether any such curve will allow an extension to an S-sequence of length 11. Bremner ([3]) showed that such curves do not exist. Also, Moody ([14]) found an infinite number of Huff curves with S-sequences of length 9, and Choudhry ([6]) extended Moody's result to find several Huff curves with S-sequences of length 11.

Now we consider the case when the elements of S form a *geometric progression*, Bremner and Ulas ([4]) obtained an infinite family of elliptic curves with S-sequences of length 4, and they also pointed out infinitely many elliptic curves with S-sequences of length 5. Ciss and Moody ([13]) found infinite families of twisted Edwards curves with S-sequences of length 5 and Edwards curves with S-sequences of length 4. When the elements of $S \subset \mathbb{Q}$ are *consecutive squares*, Kamel and Sadek ([9]) constructed infinitely many elliptic curves given by the equation $y^2 = ax^3 + bx + c$ with S-sequences of length 5. When the elements of $S \subset \mathbb{Q}$ are *consecutive cubes*, Çelik and Soydan ([7]) found infinitely many elliptic curves of the form $y^2 = ax^3 + bx + c$ with S-sequences of length 5.

In the present work, we consider the following families of elliptic curves due to the symmetry enjoyed by the equations defining them: (twisted) Edwards curves and (general) Huff curves. Given an arbitrary subset S of a number field k, we tackle the general question of the existence of infinitely many such curves with an S-sequence when there is no restriction on the elements of S. We provide explicit examples when the length of the S-sequence is $4, 5$, or 6. This is achieved by studying the existence of rational points on certain quadratic and elliptic surfaces.

2. Edwards curves with S-sequences of length 6

Throughout this work, k will be a number field unless otherwise stated.

An *Edwards curve* over k is defined by

$$E_d : x^2 + y^2 = 1 + dx^2y^2, \tag{2.1}$$

where d is a non-zero element in k. It is clear that the points $(x, y) = (-1, 0), (0, \pm 1), (1, 0) \in E_d(k)$. We show that given any set

$$S = \{s_{-1} = -1, s_0 = 0, s_1 = 1, s_2, s_3, s_4\} \subset k,$$

$s_i \neq s_j$ if $i \neq j$, there are infinitely many Edwards curves E_d that possess rational points whose x-coordinates are s_i, $-1 \leq i \leq 4$, i.e., the set S is realized as x-coordinates in $E_d(k)$. In other words, there are infinitely many Edwards curves that possess an S-sequence.

We start with assuming that s_2 is the x-coordinate of a point in $E_d(k)$, then one must have $y^2 = \dfrac{s_2^2 - 1}{s_2^2 d - 1}$, or $s_2^2 d - 1 = (s_2^2 - 1)p^2$ for some $p = 1/y \in k$.

Similarly, if s_3 is the x-coordinate of a point in $E_d(k)$, then $y^2 = \dfrac{s_3^2-1}{s_3^2 d-1}$, or $s_3^2 d - 1 = (s_3^2-1)q^2$. So

$$d = \frac{(s_2^2-1)p^2+1}{s_2^2} = \frac{(s_3^2-1)q^2+1}{s_3^2}.$$

Thus we have the following quadratic curve

$$s_3^2\left[(s_2^2-1)p^2+1\right] - s_2^2\left[(s_3^2-1)q^2+1\right] = 0$$

on which we have the rational point $(p,q) = (1,1)$. Parametrizing the rational points on the latter quadratic curve yields

$$p = \frac{2ts_2^2 - t^2s_2^2 - s_3^2 + s_2^2s_3^2 - 2ts_2^2s_3^2 + t^2s_2^2s_3^2}{-t^2s_2^2 + s_3^2 - s_2^2s_3^2 + t^2s_2^2s_3^2},$$
$$q = -\frac{(-1+s_2^2)s_3^2 - 2t(-1+s_2^2)s_3^2 + t^2s_2^2(-1+s_3^2)}{-(-1+s_2^2)s_3^2 + t^2s_2^2(-1+s_3^2)}.$$

Therefore, fixing s_2 and s_3 in k, one sees that p and q lie in $k(t)$. Now we obtain the following result.

THEOREM 2.1. *Let $s_{-1} = -1$, $s_0 = 0$, $s_1 = 1$, s_2, s_3 and s_4, $s_i \neq s_j$ if $i \neq j$, be a sequence in $\mathbb{Z}$ such that*

$$h(s_2,s_3) = -3 + 4s_3^2 + s_2^4s_3^4 + s_2^2(4-6s_3^2) \neq 0$$

where either $g_1(s_2,s_3)/h(s_2,s_3)^2$ or $g_2(s_2,s_3)/h(s_2,s_3)^3$ are not integers, g_1 and g_2 are defined in (2.3). There are infinitely many Edwards curves described by

$$E_d : x^2 + y^2 = 1 + dx^2y^2, \quad d \in \mathbb{Q}$$

on which s_i, $-1 \leq i \leq 4$, are the x-coordinates of rational points in $E_d(\mathbb{Q})$. In other words, there are infinitely many Edwards curves that possess an S-sequence where $S = \{s_i : -1 \leq i \leq 4\}$.

PROOF. Substituting the value for p in $d = \dfrac{(s_2^2-1)p^2+1}{s_2^2}$ yields that

$$\begin{aligned}
&(-t^2s_2^2 + s_3^2 - s_2^2s_3^2 + t^2s_2^2s_3^2)^2 d \\
&\quad = (s_3^4 - 2s_2^2s_3^4 + s_2^4s_3^4) + (4s_3^2 - 8s_2^2s_3^2 + 4s_2^4s_3^2 - 4s_3^4 + 8s_2^2s_3^4 - 4s_2^4s_3^4)t \\
&\qquad + (-4s_2^2 + 4s_2^4 - 4s_3^2 + 14s_2^2s_3^2 - 10s_2^4s_3^2 + 4s_3^4 - 10s_2^2s_3^4 + 6s_2^4s_3^4)t^2 \\
&\qquad + (4s_2^2 - 4s_2^4 - 8s_2^2s_3^2 + 8s_2^4s_3^2 + 4s_2^2s_3^4 - 4s_2^4s_3^4)t^3 + (s_2^4 - 2s_2^4s_3^2 + s_2^4s_3^4)t^4.
\end{aligned}$$

Thus, for fixed values of s_2 and s_3, we have $d \in \mathbb{Q}(t)$.

Now we show the existence of infinitely many values of t such that s_4 is the x-coordinate of a rational point on E_d. In fact, we will show that t can be chosen to be the x-coordinate of a rational point on an elliptic curve with positive Mordell-Weil rank, hence the existence of infinitely many such possible values for t. Forcing (s_4, r) to be a point in $E_d(\mathbb{Q})$ for some rational r yields that

$$r^2 = \frac{s_4^2 - 1}{s_4^2 d - 1} = (A_0 + A_1 t + A_2 t^2 + A_3 t^3 + A_4 t^4)/B(t)^2, \tag{2.2}$$

where $A_i \in \mathbb{Z}$ and $B(t) = -t^2 s_2^2 + t^2 s_2^2 s_3^2 + s_3^2 - s_2^2 s_3^2$. This implies that $A_0 + A_1 t + A_2 t^2 + A_3 t^3 + A_4 t^4$ must be a rational square. This yields the elliptic curve C defined by

$$z^2 = A_0 + A_1 t + A_2 t^2 + A_3 t^3 + A_4 t^4,$$

with the following rational point

$$(t, z) = \left(0, s_3^2(s_2^2 - 1)\right).$$

The latter elliptic curve is isomorphic to the elliptic curve described by the Weierstrass equation $E_{I,J} : y^2 = x^3 - 27Ix - 27J$ where

$$\begin{aligned} I &= 12A_0A_4 - 3A_1A_3 + A_2^2 \\ J &= 72A_0A_2A_4 + 9A_1A_2A_3 - 27A_1^2A_4 - 27A_0A_3^2 - 2A_2^3, \end{aligned}$$

see for example [16, §2]. The latter elliptic curve has the following rational point

$$P = \left(-12(-1 + s_2^2)(-1 + s_3^2)(-3 + s_2^2 + s_3^2), -216(-1 + s_2^2)^2(-1 + s_3^2)^2\right).$$

One notices that the coordinates of $3P$ are rational functions. Indeed,

$$3P = \left(\frac{g_1(s_2, s_3)}{h(s_2, s_3)^2}, \frac{g_2(s_2, s_3)}{h(s_2, s_3)^3}\right), \qquad \text{where } g_1, g_2 \in \mathbb{Q}[s_2, s_3] \tag{2.3}$$

and

$$h(s_2, s_3) = -3 + 4s_3^2 + s_2^4 s_3^4 + s_2^2(4 - 6s_3^2).$$

Hence, as long as $h(s_2, s_3) \neq 0$, and $g_1/h^2 \notin \mathbb{Z}$ or $g_2/h^3 \notin \mathbb{Z}$, one sees that $3P$ is a point of infinite order by virtue of Lutz-Nagell Theorem. Thus, P itself is a point of infinite order. It follows that $E_{I,J}$ is of positive Mordell-Weil rank. Since C is isomorphic to $E_{I,J}$, it follows that C is also of positive Mordell-Weil rank. Therefore, there are infinitely many rational points $(t, z) \in C(\mathbb{Q})$, each giving rise to a value for d, by substituting in (2.2), hence an Edwards curve E_d possessing the aforementioned rational points. That infinitely many of these curves are pairwise non-isomorphic over $\mathbb{Q}$ follows, for instance, from [8, Proposition 6.1]. $\square$

3. Twisted Edwards curves with S-sequences of length 4

A *Twisted Edwards curve* over k is given by

$$E_{a,d} : ax^2 + y^2 = 1 + dx^2y^2, \tag{3.1}$$

where a and d are nonzero elements in k. Note that the point $(x, y) = (0, \pm 1) \in E_{a,d}(k)$. Given a set $\{u_0 = 0, u_1, u_2, u_3\} \subset k$, $u_i \neq u_j$ if $i \neq j$, we prove that there are infinitely many twisted Edwards curves $E_{a,d}$ for which S is realized as the x-coordinates of rational points on $E_{a,d}$.

We begin by assuming that u_1 is the x-coordinate of a point in $E_{a,d}(k)$, then one must get $y^2 = \dfrac{au_1^2-1}{u_1^2d-1}$, or $u_1^2d-1=(au_1^2-1)i^2$ for some $i \in k$.

Now, if u_2 is the x-coordinate of a point in $E_{a,d}(k)$, then $y^2 = \dfrac{au_2^2-1}{u_2^2d-1}$ or $u_2^2d-1=(au_2^2-1)j^2$. So

$$d = \frac{(au_1^2-1)i^2+1}{u_1^2} = \frac{(au_2^2-1)j^2+1}{u_2^2}.$$

Hence we obtain the following quadratic surface

$$u_2^2\left[(au_1^2-1)i^2+1\right] - u_1^2\left[(au_2^2-1)j^2+1\right] = 0,$$

on which we have the rational point $(i,j)=(1,1)$. Solving the above quadratic surface gives the following

$$i = \frac{-au_1^2u_2^2+u_2^2+2tau_1^2u_2^2-2tu_1^2-at^2u_1^2u_2^2+u_1^2t^2}{au_1^2u_2^2-u_2^2-at^2u_1^2u_2^2+u_1^2t^2},$$
$$j = \frac{-2atu_1^2u_2^2+2tu_2^2+at^2u_1^2u_2^2-u_1^2t^2+au_1^2u_2^2-u_2^2}{au_1^2u_2^2-u_2^2-at^2u_1^2u_2^2+u_1^2t^2}.$$

Now we get the following result.

Theorem 3.1. *Let $u_0=0$, u_1, u_2 and u_3 , $u_i \neq u_j$ if $i \neq j$, be a sequence in $\mathbb{Z}$ such that $h(u_1,u_2) \neq 0$, and either $g_1(s_2,s_3)/h(s_2,s_3)^2$ or $g_2(s_2,s_3)/h(s_2,s_3)^3$ are not integers, where h, g_1, g_2 are defined in (3.3). There are infinitely many twisted Edwards curves described by*

$$E_{a,d} : ax^2+y^2 = 1+dx^2y^2, \quad d \in \mathbb{Q},\ a \in \mathbb{Q}^\times \text{ is arbitrary}$$

on which u_i, $0 \le i \le 3$, are the x-coordinates of rational points in $E(\mathbb{Q})$. In other words, there are infinitely many twisted Edwards curves that possess an S-sequence where $S=\{u_i : 0 \le i \le 3\}$.

Proof. Substituting the expression for i in $d = \dfrac{(au_1^2-1)i^2+1}{u_1^2}$ gives that

$$\begin{aligned}
&(au_1^2u_2^2-u_2^2-at^2u_1^2u_2^2+u_1^2t^2)^2d \\
&\quad = (u_1^4a^3u_2^4-2u_1^4a^2u_2^2+u_1^4a)t^4+(-8au_1^2u_2^2+4u_1^2+4u_1^2a^2u_2^4-4u_1^4a \\
&\qquad -4u_1^4a^3u_2^4+8u_1^4a^2u_2^2)t^3+(-4u_1^2-10u_1^2a^2u_2^4+14au_1^2u_2^2+6u_1^4a^3u_2^4 \\
&\qquad -4u_2^2-10u_1^4a^2u_2^2+4u_1^4a+4au_2^4)t^2+(4u_2^2+8u_1^2a^2u_2^4-8au_1^2u_2^2 \\
&\qquad +4u_1^4a^2u_2^2-4au_2^4-4u_1^4a^3u_2^4)t+u_1^4a^3u_2^4-2u_1^2a^2u_2^4+au_2^4.
\end{aligned}$$

Then, assuming $(u_3, \ell) \in E(\mathbb{Q})$ yields

$$\ell^2 = \frac{au_3^2 - 1}{du_3^2 - 1} = (C_0 + C_1 t + C_2 t^2 + C_3 t^3 + C_4 t^4)/D(t)^2, \tag{3.2}$$

where $C_i \in \mathbb{Q}$ and $D(t) = au_1^2 u_2^2 - u_2^2 - at^2 u_1^2 u_2^2 + u_1^2 t^2$.

For the latter equation to be satisfied, one needs to find rational points on the elliptic curve C' defined by

$$z^2 = C_0 + C_1 t + C_2 t^2 + C_3 t^3 + C_4 t^4$$

that possesses the rational point

$$(t, z) = \left(0, u_2^2(au_1^2 - 1)\right).$$

The latter elliptic curve is isomorphic to the elliptic curve described by the Weierstrass equation $E_{I,J} : y^2 = x^3 - 27Ix - 27J$ where

$$\begin{aligned} I &= 12C_0C_4 - 3C_1C_3 + C_2^2, \\ J &= 72C_0C_2C_4 + 9C_1C_2C_3 - 27C_1^2C_4 - 27C_0C_3^2 - 2C_2^3, \end{aligned}$$

see for example [16, §2]. The latter elliptic curve has the following rational point

$$Q = \left(-12(-1 + au_2^2)(-1 + au_1^2)(-3 + au_2^2 + u_1^2), -216(-1 + au_2^2)^2(-1 + au_1^2)^2\right).$$

One notices that the coordinates of $3Q$ are rational functions. In fact,

$$3Q = \left(\frac{g_1(u_1, u_2)}{h(u_1, u_2)^2}, \frac{g_2(u_1, u_2)}{h(u_1, u_2)^3}\right), \qquad \text{where } g_1, g_2 \in \mathbb{Q}[u_1, u_2]$$

and

$$\begin{aligned} h(u_1, u_2) = {} & -27 - 72u_1^2 + 36u_1^4 + 18u_1^2u_2^2 - 12u_1^4u_2^2 - 18u_2^4 + 12u_1^2u_2^4 \\ & + u_1^4u_2^4 - 2u_1^2u_2^6 + u_2^8 + a(36u_1^2 - 12u_1^4 - 24u_1^2(-3 + u_1^2) \\ & + 36u_2^2 + 72u_1^2u_2^2 - 24u_1^4u_2^2 - 12u_1^2u_2^4 + 4u_1^4u_2^4 \\ & - 4(-3 + u_1^2)u_2^6) + a^2(-144u_1^2u_2^2 + 36u_1^4u_2^2 + 18u_2^4 \\ & - 36u_1^2u_2^4 + 4u_1^4u_2^4 + 2u_1^2u_2^6 - 2u_2^8) + a^3(36u_1^2u_2^4 \\ & + 4(-3 + u_1^2)u_2^6) + a^4u_2^8. \end{aligned} \tag{3.3}$$

Therefore, as long as $h(u_1, u_2) \neq 0$ and $g_1/h^2 \notin \mathbb{Z}$ or $g_2/h^3 \notin \mathbb{Z}$, one sees that $E_{I,J}$ is of positive Mordell-Weil rank where the point Q is of infinite order. Since C' is isomorphic to $E_{I,J}$, it follows that C' is also of positive Mordell-Weil rank. Hence, there are infinitely many rational points $(t, z) \in C'(\mathbb{Q})$, each giving rise to a value for d, by substituting in (3.2), therefore a twisted Edwards curve $E_{a,d}$ possessing the aforementioned rational points. That infinitely many of these curves are pairwise non-isomorphic over $\mathbb{Q}$ again follows from [8, Proposition 6.1]. ∎

Remark 3.2. Since $(0,-1),(0,1)$ are rational points on any twisted Edwards curve, one can show that if $u_{-1}=-1$, $u_1=1$, u_2, u_3 and u_4, $u_i\neq u_j$ if $i\neq j$, is a sequence in $\mathbb{Z}$, there are infinitely many Edwards curves on which u_i, $i\in\{-1,1,2,3,4\}$, are the y-coordinates of rational points in $E_{a,d}(\mathbb{Q})$.

4. Huff curves with S-sequences of length 5

A *Huff curve* over a number field k is defined by

$$H_{a,b}: ax(y^2-1)=by(x^2-1), \tag{4.1}$$

with $a^2\neq b^2$. Note that the points $(x,y)=(-1,\pm1),(0,0),(1,\pm1)$ are in $H_{a,b}(k)$. We prove that given $s_{-1}=-1$, $s_0=0$, $s_1=1$, s_2, $s_3\in k$, $s_i\neq s_j$ if $i\neq j$, there are infinitely many Huff curves on which these numbers are realized as the x-coordinates of rational points.

Assuming (s_2,p) and (s_3,q) are two points on $H_{a,b}$ yields

$$as_2(p^2-1)=bp(s_2^2-1), \tag{4.2}$$

and

$$as_3(q^2-1)=bq(s_3^2-1), \tag{4.3}$$

respectively. Using (4.2) and (4.3), one obtains

$$\frac{s_2(p^2-1)}{s_3(q^2-1)}=\frac{p(s_2^2-1)}{q(s_3^2-1)},$$

therefore, one needs to consider the curve

$$C': Apq^2-Ap-Bqp^2+Bq=0,$$

where $A=s_3s_2^2-s_2$ and $B=s_2s_3^2-s_2$. Dividing both sides of the above equality by q^3 gives

$$A\frac{p}{q}-A\frac{p}{q}\frac{1}{q^2}-B(\frac{p}{q})^2+B\frac{1}{q^2}=0.$$

Substituting $x=\frac{p}{q}$ and $y=\frac{1}{q^2}$ in the above equation yields the following quadratic curve

$$Ax-Axy-Bx^2+By=0,$$

on which we have the rational point $(x,y)=(1,1)$. Parametrizing the rational points on the latter quadratic curve gives

$$x=\frac{Bt-B}{At+B}, \tag{4.4}$$

$$y=\frac{At(1-t)+B(1-t)^2}{At+B}. \tag{4.5}$$

Now we have the following result.

THEOREM 4.1. *Let* $s_{-1} = -1, s_0 = 0, s_1 = 1, s_2, s_3$, $s_m \neq s_n$ *if* $m \neq n$, *be a sequence in* $\mathbb{Z}$ *such that*

$$h = -4 + A^2 - 3AB + B^2 \neq 0$$

where A *and* B *are defined as above, and either* g_1/h^2 *or* g_2/h^3 *are not integers, where* g_1, g_2 *are defined in (4.6). There are infinitely many Huff curves described by*

$$H_{a,b} : ax(y^2 - 1) = by(x^2 - 1), \qquad a, b \in \mathbb{Q}, \quad a^2 \neq b^2$$

on which s_m, $-1 \leq m \leq 3$, *are the* x*-coordinates of rational points in* $H_{a,b}(\mathbb{Q})$. *In other words, there are infinitely many Huff curves that possess an* S*-sequence where* $S = \{s_i : -1 \leq i \leq 3\}$.

PROOF. Using the equalities (4.4) and (4.5), we obtain the following

$$p^2 = \frac{x^2}{y} = \frac{B^2(-1+t)}{(B(-1+t) - At)(B + At)},$$
$$q^2 = \frac{1}{y} = \frac{(B + At)}{(-1+t)(B(-1+t) - At)}.$$

In both cases we need $(B + At)(-1 + t)(B(-1 + t) - At)$ to be a square or in other words we need t to be the x-coordinate of a rational point on the elliptic curve C'' defined by

$$z^2 = (At + B)(t - 1)(t(B - A) - B),$$

with the following k-rational point $(t, z) = (0, B)$. The latter curve can be described by the following equation

$$Y^2 = X^3 + ((B - A)^2 - AB)X^2 - 2AB(B - A)^2 X + A^2B^2(B - A)^2,$$

where $A(B - A)t = X$ and $A(B - A)z = Y$. This curve has the rational point

$$R = (X, Y) = (0, AB(B - A)).$$

Observing that

$$3R = \left(\frac{g_1(A, B)}{h(A, B)^2}, \frac{g_2(A, B)}{h(A, B)^3}\right) \tag{4.6}$$

where $h(A, B) = -4 + A^2 - 3AB + B^2$, one concludes as in the proof of Theorem 2.1. □

5. General Huff curves with S-sequences of length 4

A *general Huff curve* over a number field k is defined by

$$G_{a,b} : x(ay^2 - 1) = y(bx^2 - 1), \tag{5.1}$$

where $a, b \in k$ and $ab(a - b) \neq 0$. It is clear that the point $(x, y) = (0, 0) \in G_{a,b}(k)$. We show that given $u_0 = 0, u_1, u_2, u_3$ in k, $u_i \neq u_j$ if $i \neq j$, there are infinitely many general Huff curves over which these points are realized as the x-coordinates of rational points.

We start by assuming that if u_1 is the x-coordinates of a point in $G_{a,b}(k)$, then one must have $\dfrac{ay^2-1}{y} = \dfrac{bu_1^2-1}{u_1}$ or $\dfrac{a-i^2}{i} = \dfrac{bu_1^2-1}{u_1}$ for some $i \in k$.

Similarly, if u_2 is the x-coordinate of a point in $G_{a,b}$, then $\dfrac{ay^2-1}{y} = \dfrac{bu_2^2-1}{u_2}$ or $\dfrac{a-j^2}{j} = \dfrac{bu_2^2-1}{u_2}$ for some $j \in k$. Thus, one obtains

$$a = \frac{(bu_1^2-1)i + u_1 i^2}{u_1} = \frac{(bu_2^2-1)j + u_2 j^2}{u_2},$$

which gives the following quadratic curve

$$S : Ai^2 + Bj^2 + Ciz + Djz = 0,$$

where $A = -u_1u_2$, $B = u_1u_2$, $C = -u_1^2u_2b + u_2$, $D = bu_1u_2^2 - u_1$. Then consider the line

$$mP + nQ = (np : nq : m + nr)$$

connecting the rational points $P = (i : j : z) = (0 : 0 : 1)$ and $Q = (p : q : r)$ lying on $S \subset \mathbb{P}^2$. The intersection of S and $mP + nQ$ yields the quadratic equation

$$n^2(Ap^2 + Bq^2 + Cpr + Dqr) + mn(Cp + Dq) = 0.$$

Using P and Q lying on S, one solves this quadratic equation and obtains formulae for the solution $(i : j : z)$ with the following parametrization:

$$\begin{aligned} i &= np = Cp^2 + Dpq, \\ j &= nq = Cpq + Dq^2, \\ z &= m + nr = -Ap^2 - Bq^2. \end{aligned}$$

Now we obtain the following result.

Theorem 5.1. *Let $u_0 = 0$, u_1, u_2 and u_3 , $u_i \neq u_j$ if $i \neq j$, be a sequence in k. There are infinitely many general Huff curves described by*

$$G_{a,b} : x(ay^2 - 1) = y(bx^2 - 1), \qquad a, b \in k, \quad ab(a - b) \neq 0.$$

on which u_i, $0 \leq i \leq 3$, are the x-coordinates of rational points in $G_{a,b}(k)$. In other words, there are infinitely many general Huff curves that possess an S-sequence where $S = \{u_i : 0 \leq i \leq 3\}$.

Proof. Substituting the value for i in $a = \dfrac{(bu_1^2-1)i + u_1 i^2}{u_1}$ yields that

$$\begin{aligned} a =& u_2^2 \left(b{u_1}^2 - 1\right)^2 p^4 - 2\, u_1 u_2 \left(b{u_2}^2 - 1\right)\left(b{u_1}^2 - 1\right) p^3 q \\ &+ {u_1}^2 \left(b{u_2}^2 - 1\right)^2 p^2 q^2 - \frac{u_2 \left(b{u_1}^2 - 1\right)^2}{u_1} p^2 + \left(b{u_2}^2 - 1\right)\left(b{u_1}^2 - 1\right) pq. \end{aligned}$$

Now we assume that $(u_3, \ell) \in G_{a,b}(k)$. This yields that

$$pu_3\left(bp^2{u_1}^3u_2 - bpq{u_1}^2{u_2}^2 - p^2u_1u_2 + pq{u_1}^2 - b{u_1}^2 + 1\right)$$
$$\left(bp{u_1}^2u_2 - bqu_1{u_2}^2 - pu_2 + qu_1\right)\ell^2 - u_1\left(b{u_3}^2 - 1\right)\ell - u_1u_3 = 0.$$

This can be rewritten as

$$\begin{aligned}
&Z^2(b^2p^4u_1^5u_2^2u_3 - 2bp^4u_1^3u_2^2u_3 - b^2p^2u_1^4u_2u_3 + p^4u_1u_2^2u_3 + 2bp^2u_1^2u_2u_3\\
&\quad - p^2u_2u_3) + qZ(-2b^2p^3u_1^4u_2^3u_3 + 2bp^3u_1^4u_2u_3 + 2bp^3u_1^2u_2^3u_3 + b^2pu_1^3u_2^2u_3\\
&\quad - 2p^3u_1^2u_2u_3 - bpu_1^3u_3 - bpu_1u_2^2u_3 + pu_1u_3) + q^2p^2u_1^3u_3(bu_2^2 - 1)^2\\
&\quad - TZu_1\left(b{u_3}^2 - 1\right) - T^2u_1u_3 = 0,
\end{aligned}$$

where $T = 1/\ell$. One sees that the rational point $P = (q : T : Z) = (1 : 0 : u_1(-1+bu_2^2)/pu_2(-1+bu_1^2))$ lies on the quadratic curve above, hence we may parametrize the rational points on the quadratic curve above. This is obtained by considering the intersection of the line $dP + eQ$ where $Q = (q_1 : q_2 : q_3)$ is a point on the quadratic curve. In fact, this yields that

$$\begin{aligned}
d = {}& pu_2(b{u_1}^2 - 1)({q_3}^2b^2p^4{u_1}^5{u_2}^2u_3 - 2{q_3}^2bp^4{u_1}^3{u_2}^2u_3\\
&- {q_3}^2b^2p^2{u_1}^4u_2u_3 + {q_3}^2p^4u_1{u_2}^2u_3 + 2{q_3}^2bp^2{u_1}^2u_2u_3\\
&- {q_3}^2p^2u_2u_3 - u_1q_2q_3b{u_3}^2 + u_1q_2q_3 + p^2{u_1}^3u_3{q_1}^2b^2{u_2}^4\\
&- 2p^2{u_1}^3u_3{q_1}^2b{u_2}^2 + p^2{u_1}^3u_3{q_1}^2 - 2q_1q_3b^2p^3{u_1}^4{u_2}^3u_3\\
&+ 2q_1q_3bp^3{u_1}^4u_2u_3 + 2q_1q_3bp^3{u_1}^2{u_2}^3u_3 + q_1q_3b^2p{u_1}^3{u_2}^2u_3\\
&- 2q_1q_3p^3{u_1}^2u_2u_3 - q_1q_3bp{u_1}^3u_3 - q_1q_3bpu_1{u_2}^2u_3 + q_1q_3pu_1u_3\\
&- u_1u_3{q_2}^2),\\
e = {}& u_1(b{u_2}^2 - 1)(-p{u_1}^3u_3q_1b^2{u_2}^2 + p^2u_3q_3u_2b^2{u_1}^4 + pu_1u_3q_1b{u_2}^2\\
&- 2p^2u_3q_3u_2b{u_1}^2 + p{u_1}^3u_3q_1b + u_1q_2b{u_3}^2 + p^2u_3q_3u_2 - u_1q_2 - pu_1u_3q_1).
\end{aligned}$$

□

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